Quiz 8

Math 101

Use logarithmic differentiation to find the derivative of the function.

a) $y=(\tan^{-1}x)^{\sqrt{x}}$; b) $y=\left(\dfrac{x^2}{x+1}\right)^{\log_2 x}$

Solutions: a) $\ln y = \sqrt{x}\ln(\tan^{-1}x) \Leftrightarrow \dfrac{d}{dx}\ln y = \dfrac{d}{dx}\sqrt{x}\ln(\tan^{-1}x)$

$$\Leftrightarrow \frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln(\tan^{-1}x) + \frac{\sqrt{x}}{(1+x^2)\tan^{-1}x}$$

$$\Leftrightarrow y' = y\left(\frac{1}{2\sqrt{x}}\ln(\tan^{-1}x) + \frac{\sqrt{x}}{(1+x^2)\tan^{-1}x}\right) = (\tan^{-1}x)^{\sqrt{x}}\left[\frac{1}{2\sqrt{x}}\ln(\tan^{-1}x) + \frac{\sqrt{x}}{(1+x^2)\tan^{-1}x}\right]$$

b) $\ln y = \log_2 x \ln\left(\dfrac{x^2}{x+1}\right) \Leftrightarrow \dfrac{d}{dx}\ln y = \dfrac{d}{dx}\log_2 x \ln\left(\dfrac{x^2}{x+1}\right)$

$$\Leftrightarrow \frac{y'}{y} = \frac{1}{x\ln 2}\ln\left(\frac{x^2}{x+1}\right) + \left(\frac{2}{x} - \frac{1}{x+1}\right)\log_2 x$$

$$\Leftrightarrow y' = y\left[\frac{1}{x\ln 2}\ln\left(\frac{x^2}{x+1}\right) + \left(\frac{2}{x} - \frac{1}{x+1}\right)\log_2 x\right]$$

$$= \left(\frac{x^2}{x+1}\right)^{\log_2 x}\left[\frac{1}{x\ln 2}\ln\left(\frac{x^2}{x+1}\right) + \left(\frac{2}{x} - \frac{1}{x+1}\right)\log_2 x\right].$$

Math 101 Quiz 9

Ex1. Let $y = \sqrt{x}\,\ln(x^2+1)$.

a) Find dy.

b) Evaluate dy for $x = dx = 1$.

Ex2: Use differentials (or, equivalently, a linear approximation) to estimate $(8.01)^{2/3}$.

Ex3. Sketch the graph of $f(x) = \frac{1}{x}$, $0 < x \leq 1$; and use your sketch to find the absolute and local maximum and minimum of f.

Solutions

Ex1: a)
$$dy = \left[\frac{1}{2\sqrt{x}}\ln(x^2+1) + \sqrt{x}\,\frac{2x}{x^2+1}\right]dx$$
$$= \left[\frac{\ln(x^2+1)}{2\sqrt{x}} + \frac{2x^{3/2}}{x^2+1}\right]dx$$

b) For $x = dx = 1$, we have: $dy = \left(\frac{\ln 2}{2} + \frac{2}{2}\right) = 1 + \frac{\ln 2}{2}$.

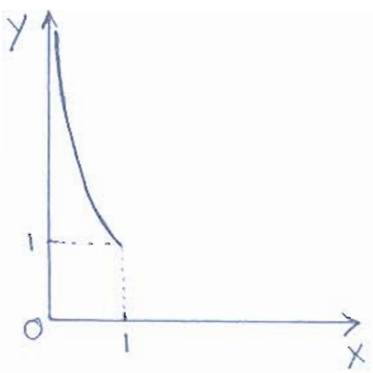
Ex2: Let $f(x) = x^{2/3}$. A linear approximation of f at a point x_0 is given by:
$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$
$$\Leftrightarrow \quad x^{2/3} \approx \frac{2}{3}x_0^{-1/3}(x - x_0) + x_0^{2/3}$$

For $x_0 = 8$; $dx = 0.01$ and $x = x_0 + dx = 8.01$

We obtain:

$$(8.01)^{2/3} \approx \frac{2}{3}(8)^{-1/3}(0.01) + (8)^{2/3}$$
$$\approx \frac{0.01}{3} + 4$$
$$\approx 4.00333\ldots$$

Ex 3:



*) 1 is an absolute minimum. Also 1 is the absolute minimum value.

*) Remark: •) The function has neither a local max nor min because it is strictly decreasing on $(0,1)$.

•) 1 is an absolute minimum but not a local minimum since it is an end point.

•) The function has no absolute maximum since $\lim_{x \to 0^+} f(x) = +\infty$,

So the y-axis is a vertical asymptote at 0.

Math 101 Quiz 10

Ex 1. Find the absolute maximum and the absolute minimum values of $f(x) = \sin x + \cos x$ on the closed interval $[0, \frac{\pi}{3}]$.

Ex 2. Consider the polynomial $P(x) = 1 + 2x + x^3 + 4x^5$.
a) Use the IVT to show that P has a root.
b) Use Rolle's theorem to show that P has exactly one root.

Solutions

Ex 1. •) f is continuous on $[0, \frac{\pi}{3}]$ as the sum of two continuous functions sine and cosine.
•) f is differentiable on $(0, \frac{\pi}{3})$ and $f'(x) = \cos x - \sin x$.
Setting $f'(x) = 0$ implies $\cos x - \sin x = 0$ which is equivalent to $\cos x = \sin x$ with x in $(0, \frac{\pi}{3})$.
Hence $x = \frac{\pi}{4}$. Therefore f has only one critical point on $(0, \frac{\pi}{3})$ at $x = \frac{\pi}{4}$
•) $f(0) = 1$, $f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ and $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{1}{2}$
•) The largest value is $\sqrt{2}$. Therefore $\sqrt{2}$ is the absolute maximum value of f on $[0, \frac{\pi}{3}]$.
The smallest value is 1. Hence 1 is the absolute minimum value of f on $[0, \frac{\pi}{3}]$.

Ex 2. a) •) P is continuous on $[-1, 0]$
•) $P(-1) = -6 < 0$ and $P(0) = 1 > 0$
Hence by the IVT there exists a point c in $(-1, 0)$ such that $P(c) = 0$
b) Assume that P has a second root denoted c'.
We will apply Rolle's theorem to P on $[c, c']$:

.) P is continuous on $[c, c']$, since P is a polynomial.

.) P is differentiable on (c, c') also because P is a polynomial.

.) Since c and c' are both roots of P, then:

$$P(c) = 0 = P(c')$$

Therefore by Rolle's theorem there must be a point c'' in (c, c') such that $P'(c'') = 0$.

But $P'(x) = 2 + 3x^2 + 20x^4 > 0$ for all x in $\mathbb{R}$, and hence the point c'' cannot exist. Therefore this contradiction implies that P cannot have more than one root.

Math 101

Quiz 11

Consider the function $f(x) = x^2 e^{-x}$.

1) Find the domain of f
2) Find the derivative $f'(x)$ of f
3) Solve $f'(x) = 0$ to find the critical points of f.
4) Find $\lim_{x \to -\infty} f(x)$ & $\lim_{x \to +\infty} f(x)$

5) Draw the table of signs of f
6) Find the intervals of increase or decrease.
7) Find the local maximum and minimum of f.
8°) Find the second derivative $f''(x)$ of f.
9°) Solve $f''(x) = 0$
10°) Find the inflection points of f.
11) Find the interval of concavity.

Solution

1°) The domain of f is $\mathbb{R}$.

2°) f is differentiable on $\mathbb{R}$ and:

$$f'(x) = 2x e^{-x} - x^2 e^{-x} = x(2-x)e^{-x}.$$

3°) $f'(x) = 0 \Leftrightarrow x(2-x) = 0 \Leftrightarrow x = 0$ or $x = 2$.

since $e^{-x} > 0$

Thus f has two critical points at $x = 0$ and at $x = 2$.

4°) •) $\lim_{x \to -\infty} -x = +\infty \Rightarrow \lim_{x \to -\infty} e^{-x} = +\infty$

$$\Rightarrow \lim_{x \to -\infty} x^2 e^{-x} = +\infty$$

•) $\lim_{x \to +\infty} x^2 e^{-x} = 0.$

5°)

x	$-\infty$		0		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$\frac{4}{e^2}$	$\searrow$	0

6°) f is decreasing on $(-\infty,0)\cup(2,+\infty)$
f is increasing on $(0,2)$

7°) 0 is a local minimum and 2 is a local maximum.

8°) $f''(x) = (2-x)e^{-x} - x e^{-x} - x(2-x)e^{-x}$
$= (2-x-x-2x+x^2)e^{-x}$
$= (x^2-4x+2)e^{-x}$

9°) $f''(x)=0 \iff x^2-4x+2=0 \iff x = 2-\sqrt{2}$ or $2+\sqrt{2}$.

10°)

x	$-\infty$		$2-\sqrt{2}$		$2+\sqrt{2}$		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	

f has 2 inflection points namely $2-\sqrt{2}$ and $2+\sqrt{2}$.

11°) f is concave up on $(-\infty, 2-\sqrt{2})\cup(2+\sqrt{2}, +\infty)$
f is concave down on $(2-\sqrt{2}, 2+\sqrt{2})$.