Math 201 07/12/2010

Quiz 2

Consider the following Polar curve:

$$r = 2\sin 6\theta$$

a) Show that the curve is symmetric about the origin.

b) Find an equation of the tangent line to the cure at $\theta = \frac{\pi}{6}$

c) Find the area of the region enclosed by one loop of the curve

d) Set up an integral that represents the length of one loop of the curve.

Solution:

a) When we replace θ by $\pi+\theta$, we obtain:

$$2\sin[6(\pi+\theta)] = 2\sin(6\pi+6\theta) = 2\sin(6\theta + 3.2\pi)$$
$$= 2\sin 6\theta, \text{ because the function Sine is } 2\pi\text{-periodic.}$$

Therefore there is no change in the polar equation when replacing θ by $\pi+\theta$. Hence the curve is symmetric about the origin.

b) $$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Now: $x = r\cos\theta = 2\sin 6\theta . \cos\theta$

$y = r\sin\theta = 2\sin 6\theta . \sin\theta$

$$\frac{dx}{d\theta} = 2\left[6\cos 6\theta.\cos\theta - \sin 6\theta \sin\theta\right]$$

$$\frac{dy}{d\theta} = 2\left[6\cos 6\theta \sin\theta + \sin 6\theta \cos\theta\right]$$ Thus:

$$\frac{dy}{dx} = \frac{6\cos 6\theta \sin\theta + \sin 6\theta\cos\theta}{6\cos 6\theta\cos\theta - \sin 6\theta\sin\theta}.$$ The slope of the tangent line is given by: $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{\sqrt{3}}$.

When $\theta = \frac{\pi}{6}$, $r = 0$. Thus:

$$x = 0 \text{ and } y = 0 \text{ at } \theta = \frac{\pi}{6}.$$

Hence an equation of the tangent line is: $y = \frac{x}{\sqrt{3}}$

c) $A = \int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/6} 4 \sin^2 6\theta \, d\theta$

$= 2 \int_0^{\pi/6} \sin^2 6\theta \, d\theta = 2 \int_0^{\pi/6} \frac{1 - \cos 12\theta}{2} \, d\theta$

$= \int_0^{\pi/6} (1 - \cos 12\theta) \, d\theta = \left[\theta - \frac{1}{12} \sin 12\theta \right]_0^{\pi/6}$

$= \frac{\pi}{6}$.

d) $L = \int_0^{\pi/6} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

$= \int_0^{\pi/6} \sqrt{4 \sin^2 6\theta + (12 \cos 6\theta)^2} \, d\theta$

$= 2 \int_0^{\pi/6} \sqrt{\sin^2 6\theta + 36 \cos^2 6\theta} \, d\theta$