$\boxed{Q1}$ Find any relative or absolute extremum of

$$f(x) = \sqrt{3x}e^{1-x^2}$$

$D_f = [0, +\infty)$

$$f'(x) = \frac{3}{2\sqrt{3x}}e^{1-x^2} - 2x\sqrt{3x}\,e^{1-x^2} = \left(\frac{3}{2\sqrt{3x}} - 2x\sqrt{3x}\right)e^{1-x^2}$$

$$= \left(\frac{3-6x^2}{2\sqrt{3x}}\right)e^{1-x^2}$$

So $f'(x) = 0$ iff $3-12x^2=0$ iff $1-4x^2=0$ iff $x = \frac{1}{2}$

x	0		$\frac{1}{2}$		$+\infty$
$f'(x)$	$\Vert$	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$f(\frac{1}{2})$	$\searrow$	0

$\lim\limits_{x \to +\infty} f(x) = 0$

0 is an absolute minimum

(f is not differentiable at 0)

$\frac{1}{2}$ is an absolute maximum

$\boxed{Q2}$ Find the absolute extrema of the function

$$f(x) = \sqrt[3]{x}(12 - x)$$

on the closed interval $[0, 12]$.

$$f'(x) = \frac{1}{3}x^{-2/3}(12 - x) - \sqrt[3]{x}$$

$$= x^{-2/3}\left[\frac{1}{3}(12 - x) - x\right] = x^{-2/3}\left(4 - \frac{4}{3}x\right)$$

So $f'(x) = 0$ iff $4 - \frac{4}{3}x = 0$ iff $x = 3$.

$f(0) = 0$, $f(12) = 0$, $f(3) = 9\sqrt[3]{3}$

Hence: 3 is the absolute maximum

$0 + 12$ are the absolute minima

$\boxed{Q3}$ Let $f(x) = \dfrac{x^2}{\sqrt{x+2}}$

a) What is the domain of definition of f?

b) Find the intervals of increase or decrease.

a) $D_f = (-2, +\infty)$

b) $$f'(x) = \frac{2x\sqrt{x+2} - \dfrac{x^2}{2\sqrt{x+2}}}{x+2}$$

$$= \frac{4x(x+2) - x^2}{2(x+2)^{3/2}} = \frac{x(3x+8)}{2(x+2)^{3/2}}$$

Thus $f'(x) = 0$ iff $x = 0$ or $\underbrace{x = -\frac{8}{3}}$

↑ to reject because $\notin (-2, +\infty)$

x	-2		0		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	↘	0	↗	$+\infty$

f is ↘ on $(-2, 0)$

f is ↗ on $(0, +\infty)$

$\boxed{Q4}$ Let $f(x) = x^2e^{-x}$. Find the intervals of concavity and the inflection points.

$$f'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)\,e^{-x}$$

$$f''(x) = (2-2x)e^{-x} - (2x-x^2)e^{-x}$$

$$= (2-2x-2x+x^2)\,e^{-x} = (2-4x+x^2)\,e^{-x}$$

Thus $f''(x) = 0$ iff $2-4x+x^2 = 0$ iff $x = 2 \pm \sqrt{2}$

x	$-\infty$		$2-\sqrt{2}$		$2+\sqrt{2}$		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	

f is concave up on $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, +\infty)$

f is concave down on $(2-\sqrt{2}, 2+\sqrt{2})$

$2-\sqrt{2}$ & $2+\sqrt{2}$ are inflection points

$\boxed{Q5}$ Find all the asymptotes of $f(x) = \dfrac{x^4 - 6x^2}{x^3 - 6x^2 + 5x}$.

*) $X^3 - 6X^2 + 5X = X(X^2 - 6X + 5) = X(X-1)(X-5)$

Therefore only $X=1$ and $X=5$ are vertical asymptotes.
$X=0$ is not a vertical asymptote. In fact 0 is also a zero for the numerator. It is easy to see that:

$$\lim_{X\to 0} f(X) = \lim_{X\to 0} \frac{X^2(X^2-6)}{X(X-1)(X-5)} = \lim_{X\to 0} X \frac{X^2-6}{(X-1)(X-5)} = 0 \neq \pm\infty$$

*)
$$\begin{array}{r|l} X^4 - 6X^2 & X^3 - 6X^2 + 5X \\ \hline - \; X^4 - 6X^3 + 5X^2 & X + 6 \\ \hline 6X^3 - 11X^2 & \\ - \; 6X^3 - 36X^2 + 30X & \\ \hline 25X^2 - 30X & \end{array}$$

So: $X^4 - 6X^2 = (X+6)(X^3 - 6X^2 + 5X) + (25X^2 - 30X)$

Thus $\dfrac{X^4 - 6X^2}{X^3 - 6X^2 + 5X} = X + 6 + \dfrac{25X^2 - 30X}{X^3 - 6X^2 + 5X}$.

Therefore $f(X) - (X+6) = \dfrac{25X^2 - 30X}{X^3 - 6X^2 + 5X}$. Hence

$$\lim_{X\to\pm\infty} \left[f(X) - (X+6) \right] = \lim_{X\to\pm\infty} \frac{25X^2 - 30X}{X^3 - 6X^2 + 5X} = \lim_{X\to\pm\infty} \frac{25X^2}{X^3} = \lim_{X\to\pm\infty} \frac{25}{X} = 0$$

Thus: $y = X + 6$ is an oblique asymptote.

$\boxed{Q6}$ Use the Second Derivative Test to find any relative extremum of $f(x) = \dfrac{\ln x}{x^2}$.

• $f'(x) = \dfrac{\frac{1}{x}x^2 - 2x\ln x}{x^4} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3}$

$f'(x) = 0$ iff $1-2\ln x = 0$ iff $\ln x = \frac{1}{2}$ iff $x = \sqrt{e}$

f has only on critical point $x = \sqrt{e}$.

• $f''(x) = \dfrac{-\frac{2}{x}x^3 - 3x^2(1-2\ln x)}{x^6} = \dfrac{-2x^2 - 3x^2(1-2\ln x)}{x^6}$

$= \dfrac{-2-3(1-2\ln x)}{x^4}$.

Then $f''(\sqrt{e}) = \dfrac{-2-3(1-2\cdot\frac{1}{2})}{(\sqrt{e})^4} = -\dfrac{2}{e^2} < 0$

Hence $\sqrt{e}$ is a relative maximum.

$\boxed{Q7}$ Let $f(x) = \dfrac{e^{2x}-1}{x+1}$. Use differentials to approximate $f(0.01)$.

$$f'(x) = \frac{2e^{2x}(x+1) - (e^{2x}-1)}{(x+1)^2} = \frac{2xe^{2x} + e^{2x} + 1}{(x+1)^2}$$

$$0.01 = 0 + 0.01$$

$$f(0.01) \approx f(0) + f'(0)(0.01)$$

$$f(0) = 0, \quad f'(0) = 2$$

Therefore $f(0.01) \approx 0.02$

$\boxed{Q8}$ Find the following indefinite integral

$$\int \sqrt[3]{x^2}\left(\frac{3x^{\frac{3}{2}} - x^{-\frac{7}{6}}}{\sqrt{x}}\right) dx$$

$$= \int x^{2/3} x^{-1/2} \left(3x^{3/2} - x^{-7/6}\right) dx$$

$$= \int x^{2/3} \left(3x - x^{-5/3}\right) dx$$

$$= \int \left(3x^{5/3} - x^{-1}\right) dx$$

$$= 3\,\frac{x^{\frac{5}{3}+1}}{\frac{5}{3}+1} - \ln|x| + C$$

$$= \frac{9}{8} x^{8/3} - \ln|x| + C$$

Q9 Find y subject to the following conditions :

$$y'' = \frac{3}{(x+1)^2} + e^{-2x}, \quad y'(0) = -2, \quad y(0) = 0$$

•) $y' = -\frac{3}{x+1} - \frac{1}{2}e^{-2x} + C$

Since $y'(0) = -2$, $-3 - \frac{1}{2} + C = -2$ i.e. $C = \frac{3}{2}$

Thus $y' = -\frac{3}{x+1} - \frac{1}{2}e^{-2x} + \frac{3}{2}$

•) $y = -3\ln|x+1| + \frac{1}{4}e^{-2x} + \frac{3}{2}x + C$

Since $y(0) = 0$, $\frac{1}{4} + C = 0$ i.e. $C = -1/4$

Hence $y = -3\ln|x+1| + \frac{1}{4}e^{-2x} + \frac{3}{2}x - \frac{1}{4}$

$\boxed{Q10}$ Find the following indefinite integrals

a) $I = \displaystyle\int \frac{3}{x^2} e^{1+\frac{2}{x}}\, dx$

b) $J = \displaystyle\int \frac{\ln(5+2\sqrt{x})}{5\sqrt{x}+2x}\, dx$

a) Let $u = 1 + \frac{2}{x}$, $du = -\frac{2}{x^2}dx$ ie $\frac{1}{x^2}dx = -\frac{1}{2}du$

$I = \int -\frac{3}{2} e^u du = -\frac{3}{2} e^u + C = -\frac{3}{2} e^{1+\frac{2}{x}} + C$

b) Let $u = \ln(5+2\sqrt{x})$, $du = \dfrac{\frac{1}{\sqrt{x}}}{5+2\sqrt{x}}dx = \dfrac{1}{5\sqrt{x}+2x}dx$

Hence $J = \int u\, du = \frac{u^2}{2} + C = \ln^2(5+2\sqrt{x}) + C$

$\boxed{Q11}$ Detremine the following indefinite integral

$$I = \int \frac{\ln x - 1}{2x^2} 2^{\frac{\ln x}{x}}\, dx$$

Let $u = \dfrac{\ln x}{x}$ $\quad du = \dfrac{1 - \ln x}{x^2}\, dx$

So $\dfrac{\ln x - 1}{2x^2}\, dx = -\dfrac{1}{2}\, du$. Thus

$$I = \int -\frac{1}{2} 2^u\, du = -\frac{2^u}{2\ln 2} + C$$

$$= -\frac{2^{\frac{\ln x}{x}}}{2\ln 2} + C$$

$\boxed{Q12}$ Find the area of the region bounded by the curve $y = xe^{x^2}$, the $x-$axis and the vertical line $x = 3$.

$$A = \int_0^3 xe^{x^2}dx = \frac{1}{2}e^{x^2}\Big|_0^3 = \frac{1}{2}(e^9 - 1)$$

Rmk: •) $y' = e^{x^2} + 2x^2e^{x^2} = (1+2x^2)e^{x^2} \geq 0$ for all x in $\mathbb{R}$

x	$-\infty$	$+\infty$
y'	$+$	
y	$-\infty \nearrow$	$+\infty$

•) $y'' = 4xe^{x^2} + 2x(1+2x^2)e^{x^2}$
$= 2xe^{x^2}(3+2x^2)$

x	$-\infty$	0	$+\infty$
y''	$-$	0	$+$

So 0 is an inflection point