FINAL EXAM

Q1 $\lim_{x\to 3} \frac{9-x^2}{x^2-2x-3} = \lim_{x\to 3} \frac{(3-x)(3+x)}{(x+1)(x-3)}$

$= \lim_{x\to 3} -\frac{3+x}{x+1} = -\frac{6}{4} = -\frac{3}{2}$

a) $-\frac{3}{2}$

b) 0

c) $+\infty$

d) $-\frac{1}{2}$

e) $-\infty$

Q2 The function $g(x) = \begin{cases} \ln(x^2+1), & \text{if } x \leq 0 \\ 1-e^{-x}, & \text{if } 0 < x \leq 1 \\ x-e, & \text{if } x > 1 \end{cases}$

a) is continuous at 0

b) is continuous at 1

c) is discontinuous at 0

d) is continuous everywhere

e) is continuous at both 0 and 1

*) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \ln(x^2+1) = 0$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} 1-e^{-x} = 0$

Hence $\lim_{x\to 0} g(x) = 0$. Since $g(0)=0$, g is continuous at 0

*) $\lim_{x\to 1^-} g(x) = 1-e^{-1}$

$\lim_{x\to 1^+} g(x) = 1-e$

Since $\lim_{x\to 1^-} g(x) \neq \lim_{x\to 1^+} g(x)$, g is discontinuous at 1.

$\boxed{Q3}$ The slope of the tangent line to the curve

$$y = \frac{1}{\sqrt[4]{x^3}} - x^{-\frac{5}{2}} = x^{-3/4} - x^{-5/2}$$

at $x = 1$ is

$$\frac{dy}{dx} = -\frac{3}{4}x^{-\frac{3}{4}-1} + \frac{5}{2}x^{-\frac{5}{2}-1}$$

$$= -\frac{3}{4}x^{-\frac{7}{4}} + \frac{5}{2}x^{-7/2}$$

$\boxed{\text{a) } \frac{7}{4}}$

b) -1

c) $-\frac{3}{4}$

d) $\frac{5}{4}$

e) $\frac{2}{3}$

Thus: $\left.\frac{dy}{dx}\right|_{x=1} = -\frac{3}{4} + \frac{5}{2} = \frac{7}{4}$

$\boxed{Q4}$ If $y = e^{(u^2-1)} - \frac{1}{\sqrt{u}}$ and $u = \frac{1}{2-x}$, then $\frac{dy}{dx}\ |_{x=1}$ equals

By the Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{du} = 2u\,e^{(u^2-1)} + \frac{1}{2u^{3/2}}$, $\frac{du}{dx} = \frac{1}{(2-x)^2}$

When $x=1$, $u=1$. Thus:

$\left.\frac{dy}{dx}\right|_{x=1} = \left(2 + \frac{1}{2}\right) \cdot 1 = \frac{5}{2}$

$\boxed{\text{a) } \frac{5}{2}}$

b) $-\frac{1}{2}$

c) $\frac{3}{2}$

d) -1

e) 2

Q5 The slope of the tangent line to the curve

$$(x+y)^2 + xe^y = \sqrt{xy+1}$$

at the point $(0,1)$ is given by

Using Implicit differentiation, we have:

$$2(1+y')(x+y) + e^y + xy'e^y = \frac{y+xy'}{2\sqrt{xy+1}}$$

At $(0,1)$, we have:

$$2(1+y') + e = \frac{1}{2}. \text{ Hence } y' = -\frac{3}{4} - \frac{e}{2}$$

a) $\boxed{-\frac{3}{4} - \frac{e}{2}}$

b) $\frac{1}{2} - \frac{e}{2}$

c) $\frac{1}{4} - \frac{e}{2}$

d) $\frac{1}{4} - \frac{e}{4}$

e) $-\frac{e}{4}$

Q6 If $f(x) = x\ln(x^2-2)$, then

$$D_f = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$$

$$f'(x) = \ln(x^2-2) + \frac{2x^2}{x^2-2}$$

$$f''(x) = \frac{2x}{x^2-2} + 2\left[\frac{2x(x^2-2) - 2x^3}{(x^2-2)^2}\right]$$

$$= \frac{2x}{x^2-2}\left[1 - \frac{4}{x^2-2}\right] = \frac{2x(x^2-6)}{(x^2-2)^2}$$

a) $\boxed{f \text{ is concave up on } (-\sqrt{6}, -\sqrt{2})}$

b) f has only one inflection point

c) f is concave down on $(\sqrt{6}, +\infty)$

d) f is concave up on $(-\infty, -\sqrt{6})$

e) f is concave down on $(-\infty, -\sqrt{2})$

x	$-\infty$		$-\sqrt{6}$		$-\sqrt{2}$		$\sqrt{2}$		$\sqrt{6}$		$+\infty$
$f''(x)$		$-$	0	$+$	$\Vert$		$\Vert$	$-$	0	$+$	

out of the domain of definition

$\boxed{Q7}$ The y-intercept of the tangent line to the curve

$$y = \left(\frac{x^2}{x+1}\right)^{x+\ln x}$$

at $x = 1$ is given by

$\ln y = (x+\ln x)\left[2\ln x - \ln(x+1)\right]$

$\frac{y'}{y} = \left(1+\frac{1}{x}\right)\left[2\ln x - \ln(x+1)\right] + (x+\ln x)\left(\frac{2}{x} - \frac{1}{x+1}\right)$

$\boxed{\text{a) } -\frac{1}{4} + \ln 2}$

b) $-\frac{1}{2} + \ln 2$

c) $\frac{3}{2} + \ln 2$

d) $\frac{3}{4} + \ln 2$

e) $\frac{3}{2} \ln 2$

<u>Thus</u>: $y'(1) = \frac{3}{4} - \ln 2$. An equation of the tangent line is:

$y = \left(\frac{3}{4} - \ln 2\right)(x-1) + \frac{1}{2}$

The y-intercept is: $\ln 2 - \frac{3}{4} + \frac{1}{2}$

$= \ln 2 - \frac{1}{4}$

$\boxed{Q8}$ If $f(x) = xe^{(1-\sqrt{x})}$, then

$\boxed{\text{a) } f \text{ is increasing on } (0,4)}$

b) f is decreasing on $(0, +\infty)$

c) f has a relative minimum at 4

d) f has a relative maximum at 0

e) f is increasing on $(4, +\infty)$

$D_f = [0, +\infty)$

$f'(x) = e^{1-\sqrt{x}} + x\left(-\frac{1}{2\sqrt{x}}\right)e^{1-\sqrt{x}}$

$= \left(1 - \frac{\sqrt{x}}{2}\right)e^{1-\sqrt{x}}$

x	0		4		$+\infty$
$f'(x)$	‖	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{4}{e}$	$\searrow$	0

0 is a relative min.
4 is a relative max.

$\boxed{Q9}$ If $y'' = \sin(2x) - 3\cos(\frac{x}{2})$, $y'(\pi) = -5$, $y(\pi) = 2\pi$, then $y(0)$ is equal to

$\boxed{\text{a) } 12 + \frac{\pi}{2}}$

b) $6 - \frac{3\pi}{2}$

c) $-2 + \pi$

d) $\frac{1}{4} - \frac{\pi}{2}$

e) $\frac{3\pi}{2}$

• $y' = -\frac{1}{2}\cos(2x) - 6\sin(\frac{x}{2}) + C$

Since $y'(\pi) = -5$, $-\frac{1}{2} - 6 + C = -5$ i.e $C = \frac{3}{2}$

Thus: $y' = -\frac{1}{2}\cos(2x) - 6\sin(\frac{x}{2}) + \frac{3}{2}$

• $y = -\frac{1}{4}\sin(2x) + 12\cos(\frac{x}{2}) + \frac{3}{2}x + C$

Since $y(\pi) = 2\pi$, $\frac{3}{2}\pi + C = 2\pi$ i.e $C = \frac{\pi}{2}$

Thus: $y = -\frac{1}{4}\sin(2x) + 12\cos(\frac{x}{2}) + \frac{3}{2}x + \frac{\pi}{2}$

Hence $y(0) = 12 + \frac{\pi}{2}$

$\boxed{Q10}$ The function $f(x) = \dfrac{x^{70}}{35} - \dfrac{x^{68}}{4} + 213$, defined on $[-1, 1]$, has

$\boxed{\text{a) an absolute maximum at } 0}$

b) an absolute maximum at 1

c) an absolute minimum at 0

d) an absolute maximum at $\sqrt{\dfrac{2}{17}}$

e) an absolute minimum at $-\sqrt{\dfrac{2}{17}}$

• f is continuous on the closed $[-1,1]$.

• $f'(x) = 2x^{69} - 17x^{67}$

$= x^{67}(2x^2 - 17)$. Thus

$f'(x) = 0$ iff $x = 0$ ($\pm\sqrt{\frac{17}{2}}$ are not in $[-1,1]$)

• $f(0) = 213$

$f(1) = 213 + \frac{1}{35} - \frac{1}{4}$

$f(-1) = 213 + \frac{1}{35} - \frac{1}{4}$

Hence 0 is the abs. max.

-1 & 1 are the abs. min

12) •) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^4}{x^3} = \lim_{x\to\pm\infty} x = \pm\infty$. Thus f has no horizontal asymptotes.

•) $x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$.

0 & 1 are both zeros for the numerator and the denominator.

*) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^3+2x^2-3x}{(x-1)(x+2)} = 0 \neq \pm\infty$. Thus $x=0$ is not a vertical asymptote

*) $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2(x^2+2x-3)}{x(x-1)(x+2)} = \lim_{x\to 1} \frac{x(x-1)(x+3)}{(x-1)(x+2)}$

$= \lim_{x\to 1} \frac{x(x+3)}{x+2} = \frac{4}{3} \neq \pm\infty$ Hence $x=1$ is not a vertical asymptote

*) -2 is a zero for the denominator but not for the numerator.

Therefore $x=-2$ is a vertical asymptote.

•) The long division of the numerator by the denominator implies:

$$f(x) = x+1 + \frac{2x-2x^2}{x^3+x^2-2x}$$

Since $\lim_{x\to\pm\infty} \left[f(x)-(x+1)\right] = \lim_{x\to\pm\infty} \frac{-2x^2}{x^3} = \lim_{x\to\pm\infty} -\frac{2}{x} = 0$,

the line $y = x+1$ is an oblique asymptote at $\pm\infty$.

$\boxed{Q11}$ The definite integral $\int_0^1 (x+1)3^{x^2+2x-1}\,dx$ is equal to

Let $u = x^2+2x-1$, $du = 2(x+1)dx$
When $x=0$, $u=-1$. When $x=1$, $u=2$

$$I = \frac{1}{2}\int_{-1}^{2} 3^u\,du = \frac{1}{2}\left[\frac{3^u}{\ln 3}\right]_{-1}^{2}$$

$$= \frac{1}{2\ln 3}\left(9 - \frac{1}{3}\right) = \frac{13}{3\ln 3}$$

a) $\frac{13}{3\ln 3}$

b) $\frac{26}{3\ln 3}$

c) $\frac{23}{2\ln 3}$

d) $\frac{3}{2\ln 3}$

e) $\frac{13}{6\ln 3}$

$\boxed{Q12}$ The function $f(x) = \dfrac{x^4+2x^3-3x^2}{x^3+x^2-2x}$ has

a) no horizontal asymptotes and only one vertical asymptote

b) one oblique asymptote and only two vertical asymptotes

c) no horizontal asymptotes and only two vertical asymptotes

d) one horizontal asymptote and one oblique asymptote

e) one horizontal asymptote and only one vertical asymptote

$\boxed{Q13}$ The area of the region bounded by the curves $y = x^3$, $y = 2 - x$ and the x−axis equals

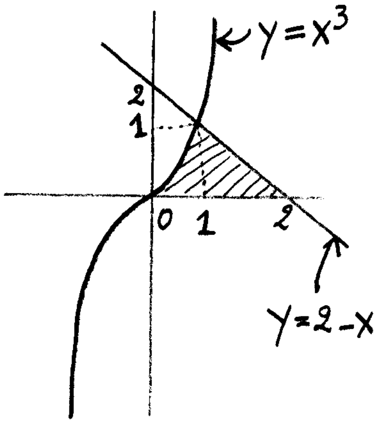
$\boxed{\text{a) } \frac{3}{4}}$

b) $\frac{1}{2}$

c) $\frac{2}{3}$

d) $\frac{1}{4}$

e) $\frac{1}{3}$



$$A = \int_0^1 \left[(2-y) - \sqrt[3]{y}\right] dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{3}{4} y^{4/3}\right]_0^1$$

$$= 2 - \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$$

$\boxed{Q14}$ Let $f(x) = (\sqrt{x} + \ln x)e^{1-x}$. Using differentials to estimate the value of $f(1.003)$, one can show that

$\boxed{\text{a) } f(1.003) \approx 1.0015}$

b) $f(1.003) \approx 1.005$

c) $f(1.003) \approx 1.001$

d) $f(1.003) \approx 1.015$

e) $f(1.003) \approx 1.05$

$f(1.003) = f(1+0.003) \approx f(1) + f'(1)(0.003)$

• $f(1) = 1$

• $f'(x) = \left(\frac{1}{2\sqrt{x}} + \frac{1}{x}\right)e^{1-x} - (\sqrt{x} + \ln x)e^{1-x}$

<u>Thus</u>: $f'(1) = \frac{1}{2} + 1 - 1 = \frac{1}{2}$

Therefore:

$f(1.003) \approx 1 + \frac{1}{2}(0.003)$

≈ 1.0015

$\boxed{Q15}$ The definite integral $\int_0^{\frac{\pi}{4}} (x - \frac{\pi}{4}) \tan^2 x \, dx$ equals

$\boxed{\text{a) } \frac{\pi^2}{32} + \ln\left(\frac{\sqrt{2}}{2}\right)}$

b) $\frac{\pi^2}{8} - \ln 2$

c) $\frac{\pi^2}{4} + \ln\left(\frac{1}{\sqrt{2}}\right)$

d) $\frac{\pi^2}{32} - \ln\left(\frac{1}{\sqrt{2}}\right)$

e) $\frac{1}{4}\left(\ln 2 - \frac{\pi^2}{16}\right)$

Let $u(x) = x - \frac{\pi}{4} \Rightarrow u'(x) = 1$

$v'(x) = \tan^2 x \Rightarrow v(x) = -x + \tan x$

$$I = \left[(x - \tfrac{\pi}{4})(-x + \tan x)\right]_0^{\pi/4} - \int_0^{\pi/4} (-x + \tan x)\, dx$$

$$= \int_0^{\pi/4} (x - \underbrace{\tan x}_{= -\frac{\sin x}{\cos x}})\, dx = \frac{x^2}{2} + \ln|\cos x| \Big|_0^{\pi/4}$$

$$= \frac{\pi^2}{32} + \ln \frac{\sqrt{2}}{2}$$

$\boxed{Q16}$ The definite integral $\int_0^{\frac{\pi}{3}} \sin x \cos(2x)\, dx$ is equal to

a) $\frac{1}{12}$

b) $\frac{2}{3}$

c) $\frac{1}{6}$

d) $\frac{3}{4}$

$\boxed{\text{e) } \frac{5}{12}}$

$\cos(2x) = 2\cos^2 x - 1$. Thus

$$I = \int_0^{\pi/3} \sin x\,(2\cos^2 x - 1)\, dx = 2\int_0^{\pi/3} \sin x \cos^2 x\, dx - \int_0^{\pi/3} \sin x\, dx$$

$$= -\frac{2}{3}\cos^3 x \Big|_0^{\pi/3} + \left[\cos x\right]_0^{\pi/3}$$

$$= -\frac{2}{3}\left(\frac{1}{8} - 1\right) + \left(\frac{1}{2} - 1\right) = \frac{5}{12}$$

$\boxed{Q17}$ The area of the region bounded by the parabolas $y = 3 - 2x^2$ and $y = x^2 - 4x + 4$ is equal to

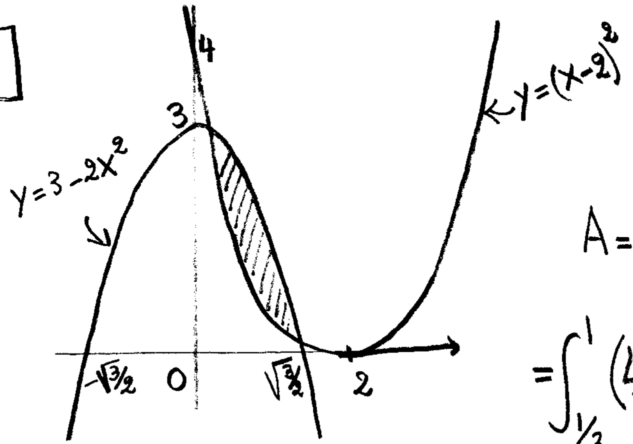
$\boxed{\text{a) } \frac{4}{27}}$

b) $\frac{10}{27}$

c) $\frac{5}{27}$

d) $\frac{8}{27}$

e) $\frac{7}{27}$



The x-coordinates of the intersection points are given by:

$x^2 - 4x + 4 = 3 - 2x^2$

$\Rightarrow 3x^2 - 4x + 1 = 0$

$\Rightarrow x = 1$ & $x = \frac{1}{3}$

$$A = \int_{1/3}^{1} \left[(3 - 2x^2) - (x^2 - 4x + 4)\right] dx$$

$$= \int_{1/3}^{1} (4x - 3x^2 - 1)\,dx = \left[2x^2 - x^3 - x\right]_{1/3}^{1}$$

$$= -\frac{2}{9} + \frac{1}{27} + \frac{1}{3} = \frac{4}{27}$$

$\boxed{Q18}$ The definite integral $\displaystyle\int_0^1 \frac{-2x^4 + x^3 + 7x^2 - 9x + 2}{x^3 - 3x + 3}\,dx$ is equal to

Using a long division of the numerator by the denominator, we show that:

$$I = \int_0^1 \left(-2x + 1 + \frac{x^2 - 1}{x^3 - 3x + 3}\right) dx$$

$$= \left[-x^2 + x + \frac{1}{3}\ln|x^3 - 3x + 3|\right]_0^1 = -\frac{\ln 3}{3}$$

$\boxed{\text{a) } -\frac{\ln 3}{3}}$

b) $1 - \frac{\ln 3}{3}$

c) $-1 + \frac{\ln 3}{3}$

d) $1 + \frac{\ln 3}{3}$

e) $\frac{\ln 3}{3}$

19) •) $f_x(x,y) = x^2+y$, $f_y(x,y) = y^2+x$

Setting $f_x(x,y) = 0 = f_y(x,y)$ implies $\begin{cases} x^2+y=0 \\ \& \\ y^2+x=0 \end{cases}$

$\Rightarrow \begin{cases} y=-x^2 \\ \& \\ (-x^2)^2+x=0 \end{cases} \Rightarrow \begin{cases} y=-x^2 \\ \& \\ x^4+x=0 \end{cases} \Rightarrow \begin{cases} x=0 \text{ or } -1 \\ \& \\ y=-x^2 \end{cases}$

Hence f has only two critical points $(0,0)$ & $(-1,-1)$

•) $f_{xx}(x,y) = 2x$, $f_{yy}(x,y) = 2y$, $f_{xy}(x,y) = 1$

•) $D(0,0) = -1 < 0$ Hence $(0,0)$ is a saddle point

•) $D(-1,-1) = 4-1 = 3 > 0$. Since $f_{xx}(-1,-1) = -2 < 0$, $(-1,-1)$ is a relative max

$\boxed{Q19}$ The function $f(x,y) = \frac{1}{3}\left(x^3 + y^3\right) + xy$ has

$\boxed{\text{a) one saddle point and one relative maximum}}$

b) one relative maximum and one relative minimum

c) two saddle points only

d) one relative maximum only

e) one relative minimum only

$\boxed{Q20}$ If $f(x,y,z) = x^2 e^{x+y^2 z}$, then $f_{xyz}(-1,-1,1)$ equals

$\boxed{\text{a) 4}}$

b) -6

c) 8

d) 2

e) -2

• $f_x(x,y,z) = 2x\,e^{x+y^2z} + x^2 e^{x+y^2z}$
$= x(2+x)e^{x+y^2z}$

• $f_{xy}(x,y,z) = 2xyz(2+x)e^{x+y^2z}$

• $f_{xyz}(x,y,z) = 2(2+x)\left[xy\,e^{x+y^2z} + xy^3z\,e^{x+y^2z}\right]$
$= 2xy(2+x)(1+y^2z)\,e^{x+y^2z}$

<u>Hence</u>: $f_{xyz}(-1,-1,1) = 4$