Math 101 Quiz 7 1/12/2013

Ex1: Find the absolute extrema of $f(x) = \frac{1}{x} + \ln x$ on $\left[\frac{1}{2}, 4\right]$.

Ex2: Find the extreme values (absolute and local) of $f(x) = e^x + e^{-x}$.

Solution

Ex1: f is continuous on the closed interval $\left[\frac{1}{2}, 4\right]$. Then by the Extreme Value Theorem, f has an absolute maximum and an absolute minimum.

f is differentiable on $\left(\frac{1}{2}, 4\right)$ and for every $x \in \left(\frac{1}{2}, 4\right)$, we have:

$$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{1}{x}\left(1 - \frac{1}{x}\right).$$ Thus

$f'(x) = 0$ iff $1 - \frac{1}{x} = 0$ iff $x = 1$. Hence f has only one critical point $x = 1$ in $\left(\frac{1}{2}, 4\right)$.

Now: $f(4) = \frac{1}{4} + \ln 4$, $f(1) = 1$, $f\left(\frac{1}{2}\right) = 2 - \ln 2$

Therefore 4 is the absolute maximum and 1 is the absolute minimum. (to see it, use the fact that $\sqrt{e} < 2 < e$).

Ex2: Since $\lim_{x \to +\infty} f(x) = +\infty$, f has no absolute maximum. Since $f(-x) = f(x)$ i.e. f is even, it suffices to study the function on $[0, +\infty)$.
f is differentiable everywhere, and:
$f'(x) = e^x - e^{-x}$. Thus the critical points of f are the points where the derivative vanishes:

$$f'(x) = 0 \text{ iff } e^x = e^{-x} \text{ iff } e^{2x} = 1 \text{ iff } x = 0$$

Hence 0 is a local extremum. In fact we can easily prove that 0 is an absolute minimum, and $f(0) = 2$ is the absolute minimum value: If $x > 0$, we have

$$e^x > e > 2. \text{ Thus } f(x) = e^x + \underbrace{e^{-x}}_{>0} > 2 = f(0)$$

Thus $f(x) > f(0)$ for all $x > 0$.
Since f is even, we obtain also that $f(x) > f(0)$ for all $x < 0$. Hence: $f(x) \geqslant f(0)$ for all x in $\mathbb{R}$, which means 0 is an absolute minimum.