1. An expression of the slope of the tangent line of $f(x) = \cos^2 x$ at $x = \frac{\pi}{4}$ as a limit is

(a) $\lim_{x\to\frac{\pi}{4}} \frac{\cos^2 x - 1/2}{x - \pi/4}$

(b) $\lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sqrt{2}/2}{x - \pi/4}$

(c) $\lim_{x\to\frac{\pi}{4}} \frac{\cos^2 x - \sqrt{2}/2}{x - \pi/4}$

(d) $\lim_{x\to\frac{\pi}{4}} \frac{\cos^2 x - \sqrt{2}/2}{x^2 - \pi/4}$

(e) $\lim_{x\to\frac{\pi}{4}} \frac{\cos^2 x - 1/2}{x^2 - \pi^2/16}$

$$\text{slope} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to\pi/4} \frac{\cos^2 x - \left(\frac{\sqrt{2}}{2}\right)^2}{x - \pi/4} = \lim_{x\to\pi/4} \frac{\cos^2 x - 1/2}{x - \pi/4}$$

2. Find the limit

$$\lim_{h\to 0} \frac{(2+h)^{-1} - 2^{-1}}{h}$$

(a) $\frac{-1}{4}$

(b) 0

(c) 2

(d) ∞

(e) $-\infty$

$$\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{2 - (2+h)}{2h(2+h)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{2\cancel{h}(2+h)} = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}$$

3. If a ball is thrown into the air with its height (in meters) given by $y(t) = 10\,t - \frac{49}{10}\,t^2$ (t in seconds) then its velocity when it hits the ground is:

(a) $-10\,m/s$

(b) $10\,m/s$

(c) $-20\,m/s$

(d) $20\,m/s$

(e) $9.8\,m/s$

$y(t)=0$ iff $10t - \frac{49}{10}t^2 = 0$

$\Rightarrow t\left(10 - \frac{49}{10}t\right) = 0$

$t_1 = 0$ OR $t_2 = \frac{100}{49}$

$v(t_2) = \lim_{t\to t_2} \frac{y(t) - y(t_2)}{t - t_2} = \lim_{t\to t_2} \frac{10t - \frac{49}{10}t^2 - 0}{t - \frac{100}{49}}$

$= \lim_{t\to\frac{100}{49}} \frac{t\,\cancel{(10 - \frac{49}{10}t)}}{-\frac{10}{49}\cancel{(10 - \frac{49}{10}t)}} = \lim_{t\to\frac{100}{49}} -\frac{49}{10}\cdot t$

$= -10\ m/s$

4. Find the limit

$$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} =$$

(a) $\frac{3}{2}$

(b) 0

(c) ∞

(d) $\frac{1}{6}$

(e) $\frac{1}{3}$

$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} \times \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$

$= \lim_{x\to 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9} = \lim_{x\to 1} \frac{\cancel{(x-1)}(\sqrt{4x+5}+3)}{4\cancel{(x-1)}}$

$= \lim_{x\to 1} \frac{\sqrt{4x+5}+3}{4} = \frac{6}{4} = 3/2.$

5. If $f(x) = \begin{cases} 2 & ; \quad x \leq 0 \\ 4 - x & ; \quad 0 < x < 3 \\ \dfrac{1}{4-x} & ; \quad x \geq 3 \end{cases}$

then $f'_{-}(3) =$

($f'_{-}(3)$: Left-hand derivative at 3)

(a) -1

(b) 1

(c) 0

(d) 2

(e) 3

$$f'_{-}(3) = \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0^-} \frac{4 - (3+h) - \frac{1}{4-3}}{h}$$

$$= \lim_{h \to 0^-} \frac{1 - h - 1}{h} = -1$$

6. An equation of the tangent line to the hyperbola $y = \dfrac{-2}{x}$ at the point $(2; -1)$ is:

(a) $x - 2y - 4 = 0$

(b) $x - 2y + 4 = 0$

(c) $x - y + 5 = 0$

(d) $x + y - 3 = 0$

(e) $-x + 2y - 4 = 0$

$$\lim_{h \to 0} \frac{-\frac{2}{2+h} + 1}{h} = \lim_{h \to 0} \frac{-2 + 2 + h}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{1}{2+h} = \frac{1}{2}$$

$$(T): (y - y_0) = \frac{1}{2}(x - x_0)$$

$$\Rightarrow y + 1 = \frac{1}{2}(x - 2)$$

$$2y + 2 = x - 2 \Rightarrow \boxed{x - 2y - 4 = 0}$$

7. If $f(x) = \begin{cases} x+2 & \text{if} \quad x<2 \\ ax^2 - bx + 3 & \text{if} \quad 2 \le x < 3 \\ 2x - a + b & \text{if} \quad x \ge 3 \end{cases}$

is continuous everywhere, then $a + b =$

(a) 1

(b) 3

(c) -2

(d) 2

(e) $\frac{1}{2}$

f is continuous at $x=2$ iff

$\lim_{x \to 2^-} x+2 = f(2) = 4a - 2b + 3$

$\Rightarrow 4 = 4a - 2b + 3 \Rightarrow 4a - 2b = 1$ (I)

f is continuous at $x=3$ iff

$\lim_{x \to 3} ax^2 + bx + c = f(3) = 6 - a + b$

$\Rightarrow 9a - 3b + 3 = 6 - a + b$

(II) $10a - 4b = 3$

(I) & (II) : $2a = 1 \to a = \frac{1}{2}$, $2b = 1 \to b = \frac{1}{2}$ $\Big\}$ $a + b = 1$

8. One horizontal asymptote of

$$f(x) = x + \sqrt{x^2 + 2x}$$

is

(a) $y = -1$

(b) $y = 1$

(c) $y = 2$

(d) $y = \sqrt{2}$

(e) $y = -\sqrt{2}$

$\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right) \cdot \frac{x - \sqrt{x^2 + 2x}}{x - \sqrt{x^2 + 2x}}$

$= \lim_{x \to -\infty} \frac{x^2 - (x^2 + 2x)}{x - \sqrt{x^2 + 2x}} = \lim_{x \to -\infty} \frac{-2x}{x - \sqrt{x^2 + 2x}}$

$= \lim_{x \to -\infty} \frac{\left(\frac{1}{\sqrt{x^2}}\right) \cdot (-2x)}{\left(\frac{1}{\sqrt{x^2}}\right)\left(x - \sqrt{x^2 + 2x}\right)} = \lim_{x \to -\infty} \frac{2}{-1 - \sqrt{1 + \frac{2}{x}}}$

$= -1$

N.B: $\lim_{x \to \infty} x + \sqrt{x^2 + 2x} = \infty$

9. Given $f(x) = \sqrt{x-4}$, the largest number $\delta > 0$, such that if $0 < |x - 13| < \delta$, then $|f(x) - 3| < 1$, is

(a) 5

(b) 3

(c) 7

(d) 1

(e) 16

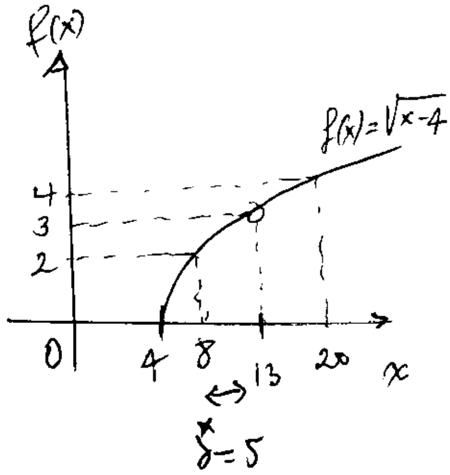
$-1 < \sqrt{x-4} - 3 < 1$

$\Leftrightarrow 2 < \sqrt{x-4} < 4$

$\Rightarrow 4 < x-4 < 16$

$\Rightarrow -5 < x-13 < 7$

$\Rightarrow |x-13| < 5$



10. The number of vertical asymptotes of the function

$$f(x) = \frac{\ln x}{x^2 - 4}$$

is:

(a) 2

(b) 0

(c) 1

(d) 3

(e) 4

$D_f = (0,2) \cup (2,\infty)$

$\lim\limits_{x\to 0^+} \frac{\ln x}{x^2-4} = \frac{-\infty}{-4} = \infty \Rightarrow x=0$ is $\underline{\underline{\text{V.A}}}$

$\lim\limits_{x\to 2^-} \frac{\ln x}{x^2-4} = \lim\limits_{x\to 2^-} \frac{\ln(2)}{0^-} = -\infty$

$\lim\limits_{x\to 2^+} \frac{\ln x}{x^2-4} = \lim\limits_{x\to 2^+} \frac{\ln(2)}{0^+} = \infty$

$\left.\right\}$ $x=2$ is $\underline{\underline{\text{V.A}}}$

<u>Two Vertical asymptotes.</u>

11. Consider the equation

$$\ln(x^2 - 3x + 2) = 1.$$

The equation has at least one root in the interval:
(**Hint**: Use the intermediate value theorem.)

(a) $(-1, 0)$

(b) $(1, 2)$

(c) $(2, 3)$

(d) $(0, 1)$

(e) $(-2, -1)$

Let $f(x) = \ln(x^2 - 3x + 2) - 1$

• f is continuous on $(-\infty; 1) \cup (2; \infty)$

$f(-1) = \ln(1+3+2) - 1 = \ln(6) - 1 > 0$

$f(0) = \ln(2) - 1 < 0$

$\Rightarrow f$ is continous on $[-1; 0]$, there is at least one root in $(-1; 0)$.

12. If $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} \frac{x\,f(x) - 6}{x - 2} = 5$, then $\lim_{x\to 2} f(x) =$

(a) 3

(b) 6

(c) 0

(d) 10

(e) 5

$$\lim_{x\to 2} xf(x) - 6 = \lim_{x\to 2} \frac{xf(x) - 6}{x - 2} \cdot (x - 2)$$

$$= \lim_{x\to 2} \frac{xf(x) - 6}{x - 2} \cdot \lim_{x\to 2} (x - 2) = 5 \cdot (0) = 0$$

Therefore, $\lim_{x\to 2} xf(x) - 6 = 0$

$$\Rightarrow \lim_{x\to 2} xf(x) = 6$$

$$\Rightarrow \lim_{x\to 2} f(x) = 3$$

13. The function $f(x) = \begin{cases} \dfrac{x^2 - x}{2x - 2} & ; \text{if} \quad x \neq 1 \\ 2 & ; \text{if} \quad x = 1 \end{cases}$

(a) has a removable discontinuity at $x = 1$.

(b) has a jump discontinuity at $x = 1$.

(c) has an infinite discontinuity at $x = 1$.

(d) is continuous at $x = 1$.

(e) is continuous on $(-\infty, \infty)$.

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x(x-1)}{2(x-1)} = \frac{1}{2}$

$\Rightarrow \lim_{x \to 1} f(x) = \frac{1}{2} \neq f(1)$

f discontinuous at $x = 1$.

the limit at $x = 1$ exists as

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \frac{1}{2}$

$\Rightarrow$ Removable discontinuity.

14. The limit

$$\lim_{x\to\infty} (\pi - \tan^{-1}(x - 2x^3)) =$$

(a) $\dfrac{3\pi}{2}$

(b) $\dfrac{\pi}{2}$

(c) does not exist

(d) π

(e) 0

$\lim_{x \to \infty} \pi - \overbrace{\tan^{-1}(\underbrace{x - 2x^3}_{-\infty})}^{-\pi/2} = \pi + \pi/2$

$= 3\pi/2$

15. The following limit:

$$\lim_{x\to -3^-} \frac{|x^2+3x|}{2x^2+5x-3} =$$

(a) $\frac{3}{7}$

(b) $\frac{-3}{7}$

(c) 0

(d) $-\infty$

(e) $\frac{1}{2}$

$$\lim_{x\to -3^-} \frac{|x^2+3x|}{2x^2+5x-3} = \lim_{x\to -3^-} \frac{|x(x+3)|}{2(x+3)(x-\frac{1}{2})}$$

$$= \lim_{x\to -3^-} \frac{|x|\,|x+3|}{2(x+3)(x-\frac{1}{2})} = \lim_{x\to -3^-} \frac{-|x|\,\cancel{(x+3)}}{2\cancel{(x+3)}(x-\frac{1}{2})}$$

$$= \lim_{x\to -3^-} \frac{-|x|}{2(x-\frac{1}{2})} = \frac{-3}{2(-\frac{7}{2})} = 3/7$$

16. If $x^2 - 1 \le g(x) \le \frac{3}{2}x + \sin(\pi x)$, then $\lim_{x\to 2} g(x) =$

(a) 3

(b) -4

(c) does not exist

(d) 0

(e) 1

$$\lim_{x\to 2} x^2 - 1 = 3$$

and

$$\lim_{x\to 2} \frac{3}{2}x + \sin(\pi x) = 3 + \sin(2\pi) = 3$$

$\Rightarrow$ By the squeeze theorem

$$\lim_{x\to 2} g(x) = 3.$$

17. If $f(x) = [|x|] + [|-x|]$, then $\lim_{x\to 5} f(x) =$
([| · |]: Greatest Integer.)

(a) -1

(b) 0

(c) does not exist

(d) 1

(e) -2

$$\lim_{x\to 5^-} [\![x]\!] + [\![-x]\!] = 4 - 5 = -1$$

$$\lim_{x\to 5^+} [\![x]\!] + [\![-x]\!] = 5 - 6 = -1$$

$$\Rightarrow \lim_{x\to 5} f(x) = -1$$

18. The following limit

$$\lim_{x\to -\infty} [\ln(2 + x^2) - \ln(1 + 2x^2)] =$$

(a) $-\ln(2)$

(b) 0

(c) $-\infty$

(d) $2\ln(2)$

(e) $\ln(2)$

$$\lim_{x\to -\infty} \ln\left(\frac{2+x^2}{1+2x^2}\right)$$

$$= \lim_{x\to -\infty} \ln\left[\frac{\frac{1}{x^2}(2+x^2)}{\frac{1}{x^2}(1+2x^2)}\right]$$

$$= \lim_{x\to -\infty} \ln\left[\frac{\frac{2}{x^2}+1}{\frac{1}{x^2}+2}\right]$$

$$= \ln\left(\frac{1}{2}\right) = -\ln(2).$$

19. The following limit

$$\lim_{\theta\to 0} \sin^{-1}\left(\frac{\sqrt{2+\cos(\pi\theta)}}{e^{\tan(\theta)}+\cos(\theta)}\right) =$$

$$\lim_{\theta\to 0} \sin^{-1}\left(\frac{\sqrt{2+\cos\pi\theta}}{e^{\tan\theta}+\cos\theta}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/3.$$

(a) $\frac{\pi}{3}$

(b) $\frac{\pi}{6}$

(c) $\frac{\pi}{4}$

(d) 0

(e) $\frac{\pi}{2}$

20. Let a and b be two real numbers such that $a > b$ and $a > 0$. If $f(x) = a\sqrt{x-b}$, for which values of (a, b) the slope of the tangent line to the graph of f at the point $(a, f(a))$ is equal to a?

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a\sqrt{(x+h)-b} - a\sqrt{x-b}}{h}$$

$$= \lim_{h\to 0} \frac{a^2(x+h-b) - a^2(x-b)}{h\left(a\sqrt{(x+h)-b} + a\sqrt{x-b}\right)}$$

$$= \lim_{h\to 0} \frac{a^2(x+h-b-x+b)}{ha\left(\sqrt{x+h-b}+\sqrt{x-b}\right)} = \frac{a}{2\sqrt{x-b}}$$

(a) $\left(a, a-\frac{1}{4}\right)$

(b) $(a, a-1)$

(c) $\left(a, a+\frac{1}{4}\right)$

(d) $\left(a, a+\frac{1}{2}\right)$

(e) $\left(a, a-\frac{1}{2}\right)$

At $x=a$: $\frac{a}{2\sqrt{a-b}} = a \Rightarrow 2\sqrt{a-b} = 1 \quad (a \neq 0)$

$\Rightarrow \sqrt{a-b} = \frac{1}{2} \Rightarrow a-b = \frac{1}{4}$

$\boxed{b = a - 1/4}$

Q	MM	V1	V2	V3	V4
1	a	a	b	a	b
2	a	b	b	d	d
3	a	a	e	c	a
4	a	d	a	b	d
5	a	a	e	a	c
6	a	d	d	d	a
7	a	d	e	a	e
8	a	b	d	e	d
9	a	d	e	e	d
10	a	c	e	d	c
11	a	b	c	a	e
12	a	b	b	b	b
13	a	a	b	a	d
14	a	e	a	d	c
15	a	a	a	a	b
16	a	e	d	c	b
17	a	b	b	e	c
18	a	e	d	a	d
19	a	b	d	c	a
20	a	c	e	e	c