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MULTIPOLE MOMENTS IN GENERAL RELATIVITY
PART I

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ABSTRACT

The Bianchi identities and field equations of general relativity are analysed in the linear approximation. There are two cases to consider. Firstly we examine the case in which the sources for the gravitational field are ignored, and obtain the solutions of the resulting homogeneous equations for the frame components of the Weyl spinor. Secondly we include the sources and obtain solutions of the wave - like equation which occurs. The two solutions - with and without source terms - are compared, especially with reference to the multipole moments which arise from these analyses. The moments arising from the solutions of the homogeneous case are referred to as 'dimensionally mixed', since they are due to contributions from terms with different dimensionalities. The moments of the inhomogeneous case are associated with a single dimension only.

1. INTRODUCTION

Consider the case of the inhomogeneous wave equation

$$\Box \phi \equiv \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 4\pi\rho \qquad (1.1)$$

The quantity ρ gives information concerning the structure of the source (or sources) producing the potential ϕ. If we are at some distance from the source we might set $\rho = 0$ and solve the homogeneous equation, using only general boundary conditions relating to the asymptotic behaviour of the potential and certain of its derivatives. The resulting solution, ϕ_1, will be to an extent arbitrary in that a wide variety of source configurations will give rise to the same <u>form</u> ϕ_1. On the other hand we might wish to consider the source in detail. Assuming that it is contained within a finite space region and is in motion (with respect to its centre of mass or charge) for only finite time intervals we can write down a Kirchhoff solution:-

$$\phi_2(P) = \int_V \frac{[\rho]\,dv}{r^*} \qquad (1.2)$$

where the integral is taken over any volume containing the source, where r^* refers to the distance between a field point P and the source points, and where the square brackets around ρ indicate that ρ must be evaluated at a retarded time $t - r^*$. This solution contains no arbitrariness; ϕ_2 is clearly dependent upon, and fixed by, the source.

Let us turn now to linearised general relativity and suppose that we have some (spatially) compact distribution of energy momentum producing gravitational radiation. We find that the field equations and Bianchi identities in spinor form give rise to equations for certain field variables (frame components of the Weyl tensor) which are similar to (1.1) although, in the first instance, the left hand side of the equations involves only first order derivatives. By equating to zero those terms representing the energy momentum distribution and solving the resulting (first order) homogeneous partial differential equations we arrive at solutions analogous to the ϕ_1 mentioned above. We will refer to these as 'ϕ_1 - like' solutions. The full - inhomogeneous - equations can be converted into wave equations, although it is found convenient to represent their solutions in a form slightly different to the Kirchhoff form (1.2). These solutions will be called 'ϕ_2 - like'.

The 'ϕ_1 - like' solutions have been obtained and discussed by a wide variety of authors (see, for example, Lamb (1966), Couch et. al. (1968), Janis and Newman (1965), and Willmer (1977)). They will be obtained in this paper also, for completeness and because they will be required for comparison with the 'ϕ_2-like' solutions. In the course of deriving the 'ϕ_1-like' solutions certain functions, called multipole moments, arise via coefficients of the spin weighted spherical harmonics (which occur when spherical polar type coordinates are used). They will be referred to as 'ϕ_1-like' moments.

The situation regarding equations of the inhomogeneous type is rather different. They do not appear to have been studied in much detail, and where they have been discussed only partial solutions have been obtained. A certain proportion of the blame for lack of progress in this direction can be attributed to the tensor methods used in the analyses, which turned out to be somewhat cumbersome. The spinor and null tetrad formalisms used in this paper bypass many of the calculational difficulties experienced by earlier authors; in particular, awkward coordinate transformations - an unpleasant feature of earlier work - are obviated.

The way in which the 'ϕ_2-like' analysis is performed leads in a natural way to the definition of multipole moment based upon dimensional behaviour, rather than upon the angular behaviour associated with the 'ϕ_1-like' analysis. We refer to these moments as 'ϕ_2-like'. It turns out that the 'ϕ_1-like' moments are due to contributions from terms with different dimensionalities, a behaviour which I will describe by the term 'dimensional mixing'. Even for the simplest systems this mixing is infinite; an infinite number of terms with different dimensionalities contribute to any 'ϕ_1-like' moment. This is markedly different from the 'ϕ_2-like' moments, which are related to a single dimension.

2. EQUATIONS FOR THE FIELD

In the following and throughout the paper Latin capitals will be used for spinor indices and will take the values 0, 1, and Greek letters will be used for tensor indices and will take the values 0, 1, 2, 3.

Brackets ( ) and [ ] around indices indicate symmetrisation and skew symmetrisation respectively. The metric has signature (+, -, -, -). Some degree of familiarity with spinor calculus is assumed in this and the next two sections although much of the notation will be explained when introduced. More complete accounts of spinor calculus are given in Penrose (1960), Newman and Penrose (1962) or Pirani (1964).

In general space - time the Bianchi identities reduce, in spinor form, to the two sets

$$\nabla^{D}{}_{E'}\Psi_{ABCD} = \nabla_{(C}{}^{H'}\,\Phi_{AB)E'H'} \tag{2.1a}$$

and

$$\nabla^{AG'}\,\Phi_{ABG'H'} = -3\nabla_{BH'}\,\Lambda \tag{2.1b}$$

There are no identities independent of these. The spinor Ψ_{ABCD} (symmetric in all indices) is the Weyl spinor: it is related to the empty space Riemann tensor (Weyl tensor) $C_{\alpha\beta\gamma\delta}$ via

$$C_{\alpha\beta\gamma\delta}\,\sigma^{\alpha}{}_{AA'}\sigma^{\beta}{}_{BB'}\sigma^{\gamma}{}_{CC'}\sigma^{\delta}{}_{DD'} = \Psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\,\epsilon_{AB}\,\epsilon_{CD} \tag{2.2}$$

The $\sigma^{\alpha}{}_{AA'}$ are symbols which provide a translation from tensor indices to spinor indices i.e. a vector V^{μ} and its spinor equavalent $V^{AA'}$ are related by $V^{\mu} = \sigma^{\mu}{}_{AA'}V^{AA'}$. ϵ_{AB}, $\epsilon_{A'B'}$ together with ϵ^{AB}, $\epsilon^{A'B'}$ are Levi-Civita symbols (skew symmetric expressions with

$\epsilon_{01} = \epsilon_{0'1'} = \epsilon^{01} = \epsilon^{0'1'} = 1$). They are used for raising and lowering spinor indices: $k_B = \epsilon_{AB}\, k^A$, $k^A = \epsilon^{AB}\, k_B$.

The two quantities $\phi_{ABA'B'}$ and Λ are related to the Ricci tensor $R_{\alpha\beta}$ and Ricci scalar R via

$$\left.\begin{aligned} \phi_{ABA'B'} &\equiv \phi_{(AB)(A'B')} = \left(\frac{1}{8}\, R g_{\alpha\beta} - \frac{1}{2}\, R_{\alpha\beta}\right) \sigma^{\alpha}{}_{AA'}\, \sigma^{\beta}{}_{BB'} \\ \Lambda &= \frac{1}{24}\, R \end{aligned}\right\} \qquad (2.3)$$

Using the Einstein equations in non empty space we find

$$\left.\begin{aligned} \phi_{ABA'B'} &= 4\pi\, \sigma^{\mu}{}_{AA'}\, \sigma^{\nu}{}_{BB'}\, \tau_{\mu\nu} \\ \Lambda &= \frac{1}{3}\, \pi T \end{aligned}\right\} \qquad (2.4)$$

where $\tau_{\mu\nu}$, T denote the trace free part and trace respectively of the energy momentum tensor $T_{\mu\nu}$. The three sets of quantities Ψ_{ABCD}, $\phi_{ABA'B'}$, Λ comprise the irreducible (with respect to local SL(2,C) transformations) spinor parts of the Riemann tensor.

Finally, $\nabla_{AA'}$ represent covariant differentiation operators; their tensor equivalent ∇_{μ} is given by $\nabla_{AA'} = \sigma^{\mu}{}_{AA'}\, \nabla_{\mu}$.

In flat space-time Minkowski co-ordinates X^{μ} may be chosen and $\sigma_{\mu}{}^{AA'}$ may be given constant components, which we will take to be $2^{-\frac{1}{2}}\times$ the Pauli matrices and the unit matrix. Thus, with

$$\sigma_{\mu}{}^{AA'} \equiv \begin{pmatrix} \sigma_{\mu}{}^{00'} & \sigma_{\mu}{}^{01'} \\ \sigma_{\mu}{}^{10'} & \sigma_{\mu}{}^{11'} \end{pmatrix} \quad \text{we have}$$

$$\sigma^{0}{}_{AA'} = \sigma_{0}{}^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma^{1}{}_{AA'} = \sigma_{1}{}^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$-\sigma^{2}{}_{AA'} = \sigma_{2}{}^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^{3}{}_{AA'} = \sigma_{3}{}^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From this we have $\nabla_{AA'} \equiv \partial/\partial X^{AA'} \equiv \partial_{AA'} (\sigma^{\mu}{}_{AA'} X^{AA'} = X^{\mu})$

where

$$\begin{pmatrix} \partial_{00'} & \partial_{01'} \\ \partial_{10'} & \partial_{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_0+\partial_3 & \partial_1+i\partial_2 \\ \partial_1-i\partial_2 & \partial_0-\partial_3 \end{pmatrix} \tag{2.6}$$

with $\partial_{\mu} \equiv \partial/\partial X^{\mu}$.

3. THE 'ϕ_1-LIKE' SOLUTIONS

In empty space we can put $\phi_{ABA'B'} = \Lambda = 0$, and thus (2.1) reduce to

$$\nabla^{D}{}_{E'}\psi_{ABCD} = 0 \tag{3.1}$$

Noting that spinor contractions are essentially skew symmetrisations, (3.1) can be written

$$\nabla_{\varepsilon'[X}\,\psi_{Y]ABC} = 0 \tag{3.2}$$

Introduce now a basis of spinors o^A, ι^A (spin dyad) satisfying

$$\varepsilon_{AB}\, o^A\, \iota^B = 1; \quad \varepsilon^{AB} = 2\, o^{[A}\, \iota^{B]} \tag{3.3}$$

It is useful to have a generic symbol $\zeta_a{}^A$ for both o^A and ι^A. Thus

$$\zeta_0{}^A \equiv o^A \quad , \quad \zeta_1{}^A \equiv \iota^A \tag{3.4}$$

For the complex conjugates we have the symbol $\bar{\zeta}_{a'}{}^{A'}$: $\bar{\zeta}_{0'}{}^{A'} \equiv \overline{o^A} \equiv o^{A'}$, $\bar{\zeta}_{1'}{}^{A'} \equiv \overline{\iota^A} \equiv \iota^{A'}$. Using these we can define the dyad components of a spinor e.g.[1]

$$Y_{ab'c} = Y_{AB'C}\, \zeta_a{}^A \bar{\zeta}_{b'}{}^{B'} \zeta_c{}^C \tag{3.5}$$

Taking dyad components of (3.2) gives

$$\nabla_{xe'}\,\psi_{abcy} + \varepsilon^{zd}\,\psi_{abcd}\,\Gamma_{yzxe'} + \cdots + \varepsilon^{zd}\,\psi_{dbcy}\,\Gamma_{azxe'}$$

$$= \nabla_{ye'}\,\psi_{abcx} + \varepsilon^{zd}\,\psi_{abcd}\,\Gamma_{xzye'} + \cdots + \varepsilon^{zd}\,\psi_{dbcx}\,\Gamma_{azye'} \tag{3.6}$$

[1] Lower case unprimed indices take the values 0,1 in this section. In the remainder of the paper they take the values 1,2,3.

where $\Gamma_{azxe'}$, symmetric in the first two indices, are given by

$$\Gamma_{abcd'} = \zeta_b{}^A \zeta_{aA;\mu}\, \sigma^\mu{}_{CD'} \zeta_c{}^C \bar{\zeta}_{d'}{}^{D'} \tag{3.7}$$

(semi colon denoting covariant differentiation). These quantities are the spin coefficients of Newman and Penrose (1962).

Since we will be working in the linearised theory we can take the covariant operators $\nabla_{AA'}$ (or $\nabla_{aa'}$) to be the operators $\partial_{AA'}$ (or $\partial_{aa'}$) given by (2.6), but with pseudo-Galilean co-ordinates (t, x, y, z) replacing the Minkowski co-ordinates (X^0, X^1, X^2, X^3). We shall find it convenient to work in spherical polar coordinates (r, θ, ϕ) and a retarded time u rather than in the coordinates $(t, x, y, z) = (u+r, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. The dyad which we shall use takes the form

$$o^A = 2^{\frac{1}{2}} \begin{Bmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{Bmatrix} ; \quad \iota^A = 2^{-\frac{1}{2}} \begin{Bmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{Bmatrix} \tag{3.8}$$

Using this dyad the operators $\partial_{aa'}$ are found to be [1]

$$\partial_{\underline{0}\underline{0}'} \equiv o^A o^{A'} \partial_{AA'} = \partial/\partial r$$

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[1] Bars are used underneath the 0 and 1 to stress that we are talking about dyad components rather than spinor components.

$$\partial_{\underline{01}'} \equiv o^A \iota^{A'} \partial_{AA'} = \frac{1}{\sqrt{2}r} (\partial/\partial\theta + \frac{i}{\sin\theta} \partial/\partial\phi) = \overline{\partial_{\underline{10}'}}$$

$$\partial_{\underline{11}'} \equiv \iota^A \iota^{A'} \partial_{AA'} = \partial/\partial u - \tfrac{1}{2} \partial/\partial r \tag{3.9}$$

The spin coefficients (3.7) can now be calculated. Taking the $\sigma^{\mu}{}_{AA'}$ to be given by (2.5), and the differential operators by (3.9), the only non-vanishing coefficients are

$$\Gamma_{\underline{0110}'} = -\Gamma_{\underline{0101}'} = \frac{-\cot\theta}{2\sqrt{2}r}, \quad \Gamma_{\underline{0010}'} = 2\Gamma_{\underline{1101}'} = -\frac{1}{r} \tag{3.10}$$

Since Ψ_{ABCD} is totally symmetric it has only 5 independent (complex) components Ψ_k $(k = 0, 1, 2, 3, 4)$ defined by $\psi_{abcd} = \psi_{(a+b+c+d)}$. Using (3.9) and (3.10) in (3.6) we therefore obtain the following set of equations for the dyad components of ψ_{ABCD},

$$(\partial_r + \frac{5-k}{r}) \psi_k + \frac{1}{\sqrt{2}r} \bar{\eth} \psi_{k-1} = 0 \tag{3.11}$$

$$(k = 1, 2, 3, 4)$$

and

$$\dot{\psi}_{k-1} - \tfrac{1}{2}(\partial_r + \frac{k}{r}) \psi_{k-1} + \frac{1}{\sqrt{2}r} \eth \psi_k = 0 \tag{3.12}$$

where $\partial_r \equiv \partial/\partial r$, $\cdot \equiv \partial/\partial u$ and $\eth$, $\bar{\eth}$ are spin weight raising and lowering operators, given by[1]

$$\left.\begin{aligned} \eth\psi_k &= -(\partial/\partial\theta + \frac{i}{\sin\theta}\,\partial/\partial\phi + (k-2)\cot\theta)\psi_k \\ \bar{\eth}\psi_k &= -(\partial/\partial\theta - \frac{i}{\sin\theta}\,\partial/\partial\phi + (2-k)\cot\theta)\psi_k \end{aligned}\right\} \tag{3.13}$$

At this stage an assumption on the behaviour of ψ_0 at large distance r from the sources of the field must be made. We assume (see Newman and Unti (1962)) that the asymptotic behaviour of ψ_0 is given by[2]

$$\psi_0 = \psi_0^0\, r^{-5} + O(r^{-6}) \;; \quad \frac{\partial\psi_0^0}{\partial r} = 0 \tag{3.14}$$

With this (3.11) are immediately integrated to give

$$\psi_k = \frac{\psi_k^0}{r^{5-k}} - \frac{1}{\sqrt{2}r^{5-k}} \int_\infty^r r'^{(4-k)}\,\bar{\eth}\,\psi_{k-1}\,dr' \tag{3.15}$$

the ψ_k^0 being functions of integration independent of r. The $k = 1$ member of (3.12) becomes

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[1] The definition of $\eth$, $\bar{\eth}$ applied to any spin weight quantity is given in Appendix A, where spin weight is also defined. The ψ_k can be shown to have spin weights $(2-k)$, whence (3.13) follows.

[2] $f(r,u,\theta,\phi) = O(g(r))$ means that $|f(u,r,\theta,\phi)| < g(r).\ F(u,\theta,\phi)$ for some F independent of r and for all sufficiently large r. The asymptotic assumption here is slightly stronger than that of Newman and Penrose (1962): $\psi_0 = O(r^{-5})$.

$$\dot{\psi}_0 - \tfrac{1}{2}(\partial_r + \frac{1}{r})\psi_0 - \frac{1}{2r^5}\int_\infty^r r'^3 \eth\bar{\eth}\psi_0 \, dr' + \frac{1}{\sqrt{2}r^5}\eth\psi_1^0 = 0 \tag{3.16}$$

We also have

$$-\sqrt{2}\,\dot{\psi}_{k-1}^0 = \eth\psi_k^0 \tag{3.17}$$

from (3.12). The component ψ_0 is a spin weight 2 quantity and hence can be written in terms of spin weight 2 spherical harmonics (see Appendix A) :—

$$\psi_0 = \sum_{\ell=2}^{\infty}\sum_{q=-\ell}^{\ell} A_{\ell q}(u,r)\eth^2(P_\ell^{|q|}\; e^{iq\phi}) \tag{3.18}$$

From (3.16) and (3.17) we see that each $A_{\ell q}(u,r)$ satisfies

$$\dot{A}_{\ell q} - \tfrac{1}{2}(\partial_r + \frac{1}{r})\, A_{\ell q} + \frac{(\ell + 2)(\ell - 1)}{2r^5}\int_\infty^r r'^3\, A_{\ell q}(u,r')dr'$$

$$= -\frac{1}{\sqrt{2}r^5}\, B_{\ell q}(u) \tag{3.19}$$

where $$\eth\psi_1^0 = \sum_{\ell=2}^{\infty}\sum_{q=-\ell}^{\ell} B_{\ell q}(u)\eth^2(P_\ell^{|q|}\; e^{iq\phi}) \tag{3.20}$$

Let us now seek a solution for $A_{\ell q}$ throughout the space-time exterior to the sources for the field in the form of a finite series in inverse powers of r. Noting the asymptotic behaviour of (3.14) we try

$$A_{\ell q}(u,r) = \sum_{n=0}^{p} \frac{a_n^{\ell q}(u)}{r^{n+5}} \tag{3.21}$$

with p finite. Substituting this into (3.19) and considering separately the coefficients of the powers of r we obtain

$$\dot{a}_n^{\ell q}(u) = \left[\frac{(\ell - 1)(\ell + 2) - n(n + 3)}{2n}\right] a_{n-1}^{\ell q}(u) \tag{3.22a}$$

$$(n = 1, \cdots, p)$$

and

$$\left[\frac{(\ell - 1)(\ell + 2) - (p + 1)(p + 4)}{2(p + 1)}\right] a_p^{\ell q}(u) = 0 \tag{3.22b}$$

The equation involving $\dot{a}_0^{\ell q}$ is used to obtain $B_{\ell q}(u)$ and is not important here. We see from (3.22) that p can be chosen to be $(\ell - 2)$, for if $p > \ell - 2$ all $a_n^{\ell q}(u)$ automatically vanish for $n \geq \ell - 1$. Further, once $a_{\ell-2}^{\ell q}(u)$ is known, so also are all other $a_n^{\ell q}(u)$ $(n = 0, 1, \cdots, \ell-3)$. Hence by specifying $a_{\ell-2}^{\ell q}(u)$ we obtain a solution for $A_{\ell q}$ of the form (3.21), with $p = \ell - 2$ and $\dot{a}_n^{\ell q}$ $(n \leq \ell - 2)$ satisfying (3.22a) (with (3.22b) automatically satisfied). Now since we are considering linearised theory all field quantities can be considered as being of first order in a parameter m characterising the mass of the radiating system. From the above remarks we accordingly set

$$a^{\ell q}_{\ell-2}(u) = m \overset{(\ell)}{h}_q(u) \tag{3.23}$$

the functions $\overset{(\ell)}{h}_q(u)$ being independent of any mass dimension. These $\overset{(\ell)}{h}_q(u)$ are defined as the 'multipole moments' of the source or, in keeping with the terminology used throughout this paper, the 'ϕ_1-like' moments. The solution for the field corresponding to $\sum_{q=-\ell}^{\ell} \overset{(\ell)}{h}_q(u)$ is called the (ϕ_1-like) 2^{ℓ}-pole (or (1ℓ)) solution.

Using (3.18) and (3.21) → (3.23) we find that the (1ℓ) solutions for ψ_0 (denoted $\overset{(1\ell)}{\psi_0}$) are

$$\left.\begin{aligned} &\overset{(10)}{\psi_0} = \overset{(11)}{\psi_0} = 0 \\ &\overset{(1\ell)}{\psi_0} = \sum_{n=1}^{\ell-1} \sum_{q=-\ell}^{\ell} \frac{\overset{(\ell)}{C}_n \overset{(\ell)}{h}_q{}^{(\ell-n-1)}(u)}{r^{n+4}} \cdot \eth^2(P_\ell^{|q|}\, e^{iq\phi}) \\ &\qquad\qquad (\ell \geq 2) \end{aligned}\right\} \tag{3.24}$$

where

$$\overset{(\ell)}{C}_n = \frac{2^{\ell-n-1}(\ell+n+1)!(\ell-2)!}{(2\ell)!(\ell-n-1)!(n-1)!} \; ; \quad \overset{(\ell)}{h}_q{}^{(k)}(u) \equiv \frac{d^k}{du^k} \overset{(\ell)}{h}_q(u) \tag{3.25}$$

The remainder of the Weyl field i.e. the components ψ_i $(i = 1,2,3,4)$ can now be determined via further consideration of equations (3.11) - (3.12).

and simplified slightly by use of Birkhoff's theorem and certain co-ordinate transformations. The details of these calculations are not of any great value here and have been omitted (see Willmer (1977)). The results have been relegated to Appendix B. Of interest is the solution of $\overset{(1\ell)}{\psi_4}$. $\overset{(1\ell)}{\psi_1}$, $\overset{(1\ell)}{\psi_2}$ and $\overset{(1\ell)}{\psi_3}$ do not contribute to any further discussion.

4. THE 'ϕ_2-LIKE' SOLUTIONS

We now consider (2.1) in full generality i.e. $\phi_{ABA'B'} \neq 0 \neq \Lambda$. To deal with (2.1a) first note the identity

$$\nabla_{FE'}\nabla_D{}^{E'} \equiv \tfrac{1}{2}(\nabla_{FE'}\nabla_D{}^{E'} + \nabla_{DE'}\nabla_F{}^{E'}) + \tfrac{1}{2}\epsilon_{FD}\Box \;,$$

$$\Box \equiv \nabla_{FE'}\nabla^{FE'} \tag{4.1}$$

so that applying the covariant operator $\nabla_{FE'}$ to (2.1a) gives

$$\nabla_{FE'}\nabla_D{}^{E'}\psi_{ABC}{}^D \equiv \tfrac{1}{2}(\nabla_{FE'}\nabla_D{}^{E'} + \nabla_{DE'}\nabla_F{}^{E'})\psi_{ABC}{}^D - \tfrac{1}{2}\Box\psi_{ABCF}$$

$$= -\nabla_{FE'}\nabla_{(C}{}^{H'}\phi_{AB)}{}^{E'}{}_{H'} \tag{4.2}$$

Moreover, the operator in curly brackets obeys the equation

$$\tfrac{1}{2}(\nabla_{FE'}\,\nabla_D{}^{E'} + \nabla_{DE'}\,\nabla_F{}^{E'})\,\xi_A = X_{FDCA}\,\xi^C \tag{4.3}$$

when applied to an arbitrary spinor, where[1]

$$X_{ABCD} = \psi_{ABCD} + 2\Lambda\epsilon_{(A|C|}\,\epsilon_{B)D} \tag{4.4}$$

Therefore we have

$$\tfrac{1}{2}(\nabla_{FE'}\,\nabla_D{}^{E'} + \nabla_{DE'}\,\nabla_F{}^{E'})\,\psi_{ABC}{}^D = 3X_{FDE(A}\,\psi_{BC)}{}^{DE} + X_{FDE}{}^D\psi_{ABC}{}^E \tag{4.5}$$

This is a second order expression in the field quantities and hence vanishes in the linear approximation. Using the earlier prescription for changing the operators $\nabla_{AA'}$ to $\partial_{AA'}$, and noting that all raising and lowering of indices is accomplished by using the Lorentz metric - diag (1, -1, -1, -1) - the linearised version of (4.2) becomes

$$\left.\begin{aligned} \Box\psi_{ABCD} &= 2\partial_{DE'}\,\partial_{(C}{}^{H'}\,\phi_{AB)}{}^{E'}{}_{H'} \;, \\ \Box &= \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \end{aligned}\right\} \tag{4.6}$$

(t, x, y, z) being the pseudo-Galilean co-ordinates of §3. A solution of (4.6) is immediately given for outgoing waves:

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[1] (| |) means that the symmetrisation excludes all indices between the bars.

$$\psi_{ABCD}\ (P) = \frac{1}{2\pi}\,\partial_{DE'}\ \partial_{(C}{}^{H'}\int_V \frac{\phi_{AB)}{}^{E'}{}_{H'}}{r^*}\ dV \tag{4.7}$$

where r^* is the distance from the field point P to the source points, where $\phi_{ABA'B'}$ is evaluated at a retarded time $t - r^*$, and where V is any volume which completely encloses all the sources for the field. r^* is related to r (distance of field point from origin 0) via $r^{*2} = r^2 + \xi^2 - 2x^a\xi_a$ $(r^2 = x^a x_a\ ;\ \xi^2 = \xi^a\xi_a)$ where the source and field points have co-ordinates (t, ξ^a) and $(t, x^a) = (t, x, y, z)$ respectively (lower case Latin letters ranging and summing over 1, 2 and 3). The form of (4.7) is slightly different from the Kirchhoff form

$$\psi_{ABCD}(P) = \frac{1}{2\pi}\int_V \frac{\partial_{DE'}\ \partial_{(C}{}^{H'}\ \phi_{AB)}{}^{E'}{}_{H'}}{r^*}\ dV \tag{4.8}$$

and is valid since, in the co-ordinates we are using, $\Box$ and the operators $\partial_{AA'}$ commute.

Using the Levi-Civita alternating symbols $\varepsilon^{A'B'}$, (4.7) can be written as

$$\psi_{ABCD}\ (P) = -\frac{1}{2\pi}\,\varepsilon^{E'G'}\ \varepsilon^{F'H'}\ \partial_{DE'}\ \partial_{(C|F'|}\int_V \frac{\phi_{AB)G'H'}}{r^*}\ dV \tag{4.9}$$

By contracting this equation with suitable combinations of the basis spinors o^A and ι^A we can obtain the dyad components $\psi_0, \cdot \cdot \cdot, \psi_4$. Our interest will centre, however, on ψ_0 and ψ_4. Thus we consider only the following expressions:

$$\left.\begin{aligned} \psi_0 &= -\frac{1}{2\pi}\, o^A o^B o^C o^D\, \varepsilon^{E'G'}\, \varepsilon^{F'H'}\, \partial_{DE'}\partial_{CF'} \int_V \frac{\phi_{ABG'H'}}{r^*}\, dV \\ \psi_4 &= -\frac{1}{2\pi}\, \iota^A \iota^B \iota^C \iota^D\, \varepsilon^{E'G'}\, \varepsilon^{F'H'}\, \partial_{DE'}\partial_{CF'} \int_V \frac{\phi_{ABG'H'}}{r^*}\, dV \end{aligned}\right\} \quad (4.10)$$

Given any spin dyad, a tetrad of null vectors l^μ, m^μ, $\bar{m}^\mu$, n^μ, satisfying pseudo-orthonormality conditions $l^\mu n_\mu = 1 = -m^\mu \bar{m}_\mu$ (with all other inner products vanishing), can be obtained via the transcription

$$l^\mu = \sigma^\mu{}_{AA'}\, o^A o^{A'}, \quad m^\mu = \sigma^\mu{}_{AA'}\, o^A \iota^{A'}, \quad n^\mu = \sigma^\mu{}_{AA'}\, \iota^A \iota^{A'} \qquad (4.11)$$

Using (2.4), (2.6), (3.3) and (4.11), (4.10) can be cast into the alternative forms

$$\psi_0 = 2\,\{\, 2l^\mu m^\nu m^\alpha l^\beta - l^\mu l^\nu m^\alpha m^\beta - m^\mu m^\nu l^\alpha l^\beta \,\} \cdot \partial_\mu \partial_\nu\, \tilde{M}_{\alpha\beta}$$

$$\psi_4 = 2\,\{\,2n^{\mu}\bar{m}^{\nu}\bar{m}^{\alpha}n^{\beta} - n^{\mu}n^{\nu}\bar{m}^{\alpha}\bar{m}^{\beta} - \bar{m}^{\mu}\bar{m}^{\nu}n^{\alpha}n^{\beta}\,\}\cdot\partial_{\mu}\partial_{\nu}\tilde{M}_{\alpha\beta} \tag{4.12}$$

where

$$\tilde{M}_{\alpha\beta} = \int_V \frac{\tau_{\alpha\beta}(t - r^*,\ \xi^a)\,dV}{r^*} \tag{4.13}$$

with $\tau_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{4}\eta_{\alpha\beta}T$, $\eta_{\alpha\beta}$ being the Lorentz metric. The contribution to ψ_0 and ψ_4 from the trace T vanishes since we can bring $\eta_{\alpha\beta}$ through the differential operators of (4.12) and then use the pseudo-orthonomality relations between the vectors of the null tetrad. Thus $\tilde{M}_{\alpha\beta}$ can be replaced in (4.12) by $M_{\alpha\beta}$, with $M_{\alpha\beta}$ being given by the right hand side of (4.13) with $\tau_{\alpha\beta}$ replaced by $T_{\alpha\beta}$.

The null tetrad in the co-ordinates (t, x, y, z) is, using the spin dyad (3.8),

$$l^{\mu} = (1,\ p^a),\quad m^{\mu} = \frac{1}{\sqrt{2}}\,(0,\ q^a),\quad n^{\mu} = \tfrac{1}{2}(1,\ -p^a) \tag{4.14}$$

where

$$\left.\begin{aligned} p^a &= \frac{1}{r}\,(x,\ y,\ z) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \\ q^a &= (\partial/\partial\theta + \frac{i}{\sin\theta}\,\partial/\partial\phi)p^a \end{aligned}\right\} \tag{4.15}$$

Let us perform a co-ordinate transformation $u = t - r$, $x^{a'} = x^a$.

Using the above notation we find that ψ_0 and ψ_4 transform to

$$\psi_0 = -q^c q^d \partial_c \partial_d M_{00} + 4q^c q^{[b} p^{d]} \partial_c \partial_d M_{0b}$$

$$+ (2q^a q^c p^b p^d - q^a q^b p^c p^d - p^a p^b q^c q^d)\, \partial_c \partial_d M_{ab} \tag{4.16}$$

and

$$\psi_4 = -\bar{q}^c \bar{q}^d \partial_0 \partial_0 M_{cd} + \bar{q}^c \bar{q}^d \partial_0 \partial_c M_{0d} + 2\, \bar{q}^b \bar{q}^{[c} p^{d]} \partial_0 \partial_d M_{bc}$$

$$- \tfrac{1}{4} \bar{q}^c \bar{q}^d \partial_c \partial_d M_{00} + \bar{q}^c \bar{q}^{[d} p^{b]} \partial_c \partial_d M_{0b} + \tfrac{1}{4}(2 \bar{q}^a \bar{q}^c p^b p^d$$

$$- \bar{q}^a \bar{q}^b p^c p^d - p^a p^b \bar{q}^c \bar{q}^d)\, \partial_c \partial_d M_{ab} \tag{4.17}$$

To proceed further we need to expand $M_{\alpha\beta}$ in terms of u, r, θ and ϕ. It is easy to show that

$$\frac{1}{r^*} = \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\, \xi^{a_1} \cdots \xi^{a_n}\, \partial_{a_1} \cdots \partial_{a_n} \left(\frac{1}{r}\right) \tag{4.18}$$

$$(r^2 > \xi^2)$$

that

$$r - r^* = \frac{x^a \xi_a}{r} + \frac{\tfrac{1}{2}(x^a \xi_a)^2 - \tfrac{1}{2}\xi^2 r^2}{r^3} + \frac{\tfrac{1}{2}(x^a \xi_a)^3 - \tfrac{1}{2}\, \xi^2 r^2 (x^a \xi_a)}{r^5} + \ldots \tag{4.19}$$

$$(r^2 > \xi^2)$$

and that

$$T_{\alpha\beta}(t - r^*, \xi^a) = \sum_{n=0}^{\infty} \frac{1}{n!} (r - r^*)^n \frac{\partial^n}{\partial u^n} T_{\alpha\beta}(u, \xi^a) \tag{4.20}$$

Using these it is not difficult to obtain

$$M_{\alpha\beta} = \int_V \frac{T_{\alpha\beta}(t - r^*, \xi^a)\, dV}{r^*} = \frac{N_{\alpha\beta}}{r} + p^a\left(\frac{\dot{N}_{\alpha\beta|a}}{r} + \frac{N_{\alpha\beta|a}}{r^2}\right)$$

$$+ \tfrac{1}{2}\left(\frac{p^a p^b}{r}\ \ddot{N}_{\alpha\beta|ab} + (3p^a p^b - \delta^{ab})\left(\frac{\dot{N}_{\alpha\beta|ab}}{r^2} + \frac{N_{\alpha\beta|ab}}{r^3}\right)\right.$$

$$+ \frac{1}{6}\left(\frac{p^a p^b p^c}{r}\ \dddot{N}_{\alpha\beta|abc} + 3p^a\left(\frac{2p^b p^c - \delta^{bc}}{r^2}\right) \ddot{N}_{\alpha\beta|abc}\right.$$

$$\left. + 3p^a(5p^b p^c - 3\delta^{bc})\left(\frac{\dot{N}_{\alpha\beta|abc}}{r^3} + \frac{N_{\alpha\beta|abc}}{r^4}\right)\right) + \ldots \tag{4.21}$$

Here, as previously, dots denote differentiation with respect to u.
The quantities $N_{\alpha\beta|c_1c_2\cdots c_n}$ are defined by

$$N_{\alpha\beta|c_1c_2\cdots c_n}(u) = \int_V T_{\alpha\beta}(u,\xi^a)\xi_{c_1}\xi_{c_2}\cdots\xi_{c_n}\, dV \tag{4.22}$$

Let us define quantities $h_{\alpha\beta|c_1c_2\cdots c_n}(u)$ by

$$h_{oo|c_1c_2\cdots c_n} = \frac{N_{oo|c_1c_2\cdots c_n}}{ma^n} \; ; \quad h_{oc|c_1c_2\cdots c_n} = \frac{N_{oc|c_1c_2\cdots c_n}}{ma^{n+1}}$$

$$h_{bc|c_1c_2\cdots c_n} = \frac{N_{bc|c_1c_2\cdots c_n}}{ma^{n+2}} \tag{4.23}$$

'm' being a mass parameter (cf. §3), and 'a' a length parameter for the system. The important point about these quantities (4.23) is that they are dimensionless i.e. they are not affected by any change in units in 'm' or 'a'. We will call them (see also Bonnor (1966)) the moments of mass, momentum and stress respectively. Substituting (4.23) into (4.21) we see that $M_{\alpha\beta}$ can be expanded in singly infinite series in the distance parameter 'a' : $M_{\alpha\beta} = \sum_{i=n}^{\infty} ma^i \overset{(i)}{M}_{\alpha\beta}$ ($\overset{(i)}{M}_{\alpha\beta}$ independent of 'm' or 'a') with $n = 0$ if $\alpha = \beta = 0$, $n = 1$ if $\alpha = 0$, $\beta \neq 0$ and $n = 2$ if $\alpha \neq 0$, $\beta \neq 0$. Thus ψ_0 and ψ_4 can be written as singly infinite series in 'a'. The solutions for ψ_0 and ψ_4 which are the coefficients of ma^ℓ will be called (see §1) the 2^ℓ-pole (or (1ℓ)) 'ϕ_2-like' solutions.

Equations (2.1b), which play an important role in the analysis, have so far not been mentioned. We must discuss them before returning to (4.16) and (4.17). Their linearised version is

$$\epsilon^{AC}\,\epsilon^{G'F'}\,\partial_{CF'}\,\phi_{ABG'H'} = -3\partial_{BH'}\,\Lambda \tag{4.24}$$

These give a set of four equations, obtained by putting $BH' = 00'$, $01'$, $10'$, and $11'$ respectively. The equation for $BH' = 10'$ is, however, simply the conjugate of the equation for $01'$. Adding the equations obtained for $BH' = 00'$ and $BH' = 11'$, then subtracting the equation for $BH' = 11'$ from $BH' = 00'$, and taking the real and imaginary parts of the equation for $BH' = 01'$, a set of four equations is obtained which reduces to the familiar conservation equations $\eta^{\alpha\beta}\,\dfrac{\partial T_{\gamma\alpha}}{\partial\xi^{\beta}} = 0$ $(\xi^{\beta} = (t, \xi^{a}))$. By multiplying them throughout by $\xi_{c_1}\xi_{c_2}\cdots\xi_{c_n}$ and integrating over any volume enclosing the sources several relations between the moments of (4.23) are obtained. For example, we find that

$$\left.\begin{aligned} \dot{h}_{00|c_1c_2\cdots c_n} &= -n\,h_{0(c_1|c_2\cdots c_n)} \\ \dot{h}_{c0|c_1c_2\cdots c_n} &= -n\,h_{c(c_1|c_2\cdots c_n)} \end{aligned}\right\} \tag{4.25}$$

The $n = 1$ version of the second equation of (4.25) leads to

$$\dot{h}_{0[b|c]} = 0 \tag{4.26}$$

which expresses the conservation of angular momentum in the linear theory.

Simply integrating the conservation equations as they stand leads to

$$\dot{h}_{00} = \dot{h}_{c0} = 0 \tag{4.27}$$

expressing the conservation of mass and linear momentum (in the linear theory).

Let us return now to (4.16). Write ψ_0 as[1]

$$\psi_0 = m \sum_{n=0}^{\infty} a^n \overset{(\tilde{1}n)}{\psi_0} \tag{4.28}$$

The monopole solution - $\overset{(\tilde{1}0)}{\psi_0}$ - is, via (4.16) and (4.21) → (4.23),

$$\overset{(\tilde{1}0)}{\psi_0} = - q^b q^c \partial_b \partial_c (\frac{1}{r}) = - \frac{q^b q^c}{r^3} (3p_b p_c - \delta_{bc}) = 0 \tag{4.29}$$

the last step following from $q^c q_c = q^c p_c = 0$. h_{00} has, without loss of generality, been taken to be unity (see (4.27)). Likewise the dipole ($n = 1$) solution can be shown to vanish. The first non-vanishing contribution to ψ_0 comes from $n = 2$. A long but straightforward calculation using (4.25) and the relations $q^c q_c = q^c p_c = 0$ leads to :

---

[1] A tilde above $\overset{(1n)}{\psi_0}$ is used to avoid confusion with earlier notation in §3.

$$\psi_0 = -\frac{3ma^2q^bq^c}{r^5}\,h_{00|bc} + ma^3\Big[8p\frac{[{}^cq^a]q^b}{r^5}\,\dot{h}_{0c|ab}$$

$$-\,5p\frac{{}^aq^bq^c}{r^5}\,\dot{h}_{00|abc} - 15p\frac{{}^aq^bq^c}{r^6}\,h_{00|abc}\Big] + O(ma^4) \qquad (4.30)$$

where $O(m^pa^q)$ $(p \geq 1,\; q \geq 0)$ denotes a term of the form $\sum_{r\geq p,s\geq q} m^ra^s\binom{rs}{\alpha}$, each $\binom{rs}{\alpha}$ being independent of m or a. This notation should not be confused with the order symbol O used in §3.

To compare (4.30) with the previous 'ϕ_1-like' solutions (3.24) - (3.25) we write $\overset{(\tilde{12})}{\psi_0}$ and $\overset{(\tilde{13})}{\psi_0}$ in terms of spin weight 2 spherical harmonics. Thus

$$\overset{(\tilde{12})}{\psi_0} = \sum_q \frac{H_q^{(2)}(u)}{r^5}\,\eth^2(P_2^{|q|}\,e^{iq\phi}) \qquad (4.31)$$

with

$$\left.\begin{aligned} H_0^{(2)} &= \tfrac{1}{2}\,(K_{11} + K_{22} - 2K_{33}) \\ H_1^{(2)} &= \overline{H_{-1}^{(2)}} = \tfrac{1}{2}\,(i\,K_{23} - K_{13}) \\ H_2^{(2)} &= \overline{H_{-2}^{(2)}} = \tfrac{1}{8}\,(K_{22} - K_{11} + 2iK_{12}) \end{aligned}\right\} \qquad (4.32)$$

and

$$\overset{(\tilde{13})}{\psi_0} = \sum_q\left\{\frac{\dot{H}_q^{(3)}(u)}{3r^5} + \frac{H_q^{(3)}(u)}{r^6}\right\}\eth^2(P_3^{|q|}e^{iq\phi}) + \sum_q a_q\,\eth^2(P_2^{|q|}e^{iq\phi}) \qquad (4.33)$$

with

$$\left.\begin{aligned}
H_0^{(3)} &= \tfrac{1}{2}\,(3K_{113} + 3K_{223} - 2K_{333}) \\
H_1^{(3)} &= \overline{H_{-1}^{(3)}} = \tfrac{1}{8}\,(K_{111} + K_{122} - 4K_{133} + i(4K_{233} - K_{112} - K_{222})) \\
H_2^{(3)} &= \overline{H_{-2}^{(3)}} = \tfrac{1}{8}\,(K_{223} - K_{113} + 2i\;K_{123}) \\
H_3^{(3)} &= \overline{H_{-3}^{(3)}} = \tfrac{1}{48}\,(3K_{112} - K_{111} + i(3K_{112} - K_{222}))
\end{aligned}\right\} \qquad (4.34)$$

where $K_{ab} = h_{00|ab}$, $K_{abc} = h_{00|abc}$, and a_q are a set of quantities which in general will not all be zero. The solutions $\overset{(12)}{\psi_0}$ and $\overset{(13)}{\psi_0}$ of (3.24) - (3.25) are

$$\overset{(12)}{\psi_0} = \sum_q \frac{\overset{(2)}{h}_q(u)}{r^5}\;\eth^2(P_2^{|q|}e^{iq\phi}) \qquad (4.35a)$$

$$\overset{(13)}{\psi_0} = \sum_q \left\{\frac{\dot{\overset{(3)}{h}}_q(u)}{3r^5} + \frac{\overset{(3)}{h}_q(u)}{r^6}\right\}\;\eth^2(P_3^{|q|}e^{iq\phi}) \qquad (4.35b)$$

Comparison of (4.31) and (4.33) with (4.35) allows the following observation: the term $\sum_q a_q\, \eth^2(P_2^{|q|}\, e^{iq\phi})$ of (4.33) is part of the solution (4.35a) (when multiplied by a^3). We might expect that were we to calculate $a^4\tilde{\overset{(14)}{\psi_0}}$ we would find a contribution from that solution to

$\overset{(12)}{\psi_0}$ also. In fact, we might expect that all (or, in particular cases, an infinite subset of) $a^n \overset{(\tilde{1n})}{\psi_0}$ $(n \geq 2)$ contribute to $\overset{(12)}{\psi_0}$. Thus it seems that $\overset{(12)}{\psi_0}$ does not in general have unique dimension, containing contributions from terms of dimensionalities $ma^n (n \geq 2)$. Likewise it seems likely that $\overset{(13)}{\psi_0}$ contains contributions from $a^n \overset{(\tilde{1n})}{\psi_0}$ $(n \geq 3)$, and so on. If this is the case it would seem appropriate to describe the $\overset{(1\ell)}{\psi_0}$ of (3.24), and in particular the 'ϕ_1-like' moments $\overset{(n)}{h_q}(u)$, as being 'dimensionally mixed'.

The above statements are, of course, merely conjecture at this stage. The calculations required to obtain $\overset{(\tilde{12})}{\psi_0}$ and $\overset{(\tilde{13})}{\psi_0}$ in the forms (4.31) and (4.33) are long and involved. Further solutions would be even more complicated. Thus in order to study dimensional mixing further, and to show that the conjectures are correct, we turn to a study of the Weyl component ψ_4.

Considering (4.17) and (4.21) it should be clear that the only term of (4.17) which will give rise to terms of $O(r^{-1})$ in an expansion of ψ_4 in inverse powers of r is the first term on the right hand side of the equation; all the other terms involve space derivatives of M_{ab} where $M_{ab} = O(r^{-1})$. Let us call this r^{-1} coefficient $-\ddot{\bar{\sigma}}^0$. Some explanation of this choice of notation is required. In the full gravitational theory the quantity (spin coefficient) σ occurs which measures the shear of congruences of null geodesics. At large distances r, σ can be expanded as $\sigma = \sigma^0 r^{-2} + O(r^{-4})$ i.e. σ^0 is the asymptotic shear of null geodesic congruences. It is found, in both linearised and exact theories, that the relation between σ^0 and ψ_4 is

$$- \ddot{\bar{\sigma}}^0 = \text{coefficient of } r^{-1} \text{ in } \psi_4 \tag{4.36}$$

$\sigma^0(u, \theta, \phi)$ is of primary importance, since its u-dependence governs the dynamic evolution of the whole field; $\dot{\sigma}^0$ is Bondi's 'news function'[1] (Bondi, van der Burg and Metzner (1962)).

From (4.17), (4.21) → (4.23) and (4.36) we find without too much trouble that[2]

$$\sigma^0 = m \sum_{n=2}^{\infty} a^n \overset{(n)}{\tilde{\sigma}}{}^0 \equiv m q^a q^b \{a^2 h_{ab} + a^3 p^{c_1} \dot{h}_{ab|c_1}$$

$$+ \frac{a^4}{2!} p^{c_1} p^{c_2} \ddot{h}_{ab|c_1 c_2} + \cdots + \frac{a^{t+2}}{t!} p^{c_1} p^{c_2} \cdots p^{c_t} \frac{d^t}{du^t} h_{ab|c_1 c_2 \cdots c_t} + \cdots\} \tag{4.37}$$

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[1] Bondi uses the letter c instead of σ^0.

[2] Since we are assuming that the source is initially stationary, the functions of integration which arise when we integrate (4.36) twice with respect to u vanish.

Let us restrict ourselves, for the sake of simplicity, to axisymmetric radiating systems. A very long and involved calculation based on (4.37) yields

$$\overset{\sim}{\sigma^0}^{(1n)} = \frac{1}{(n-2)!}\,\frac{d^{n-2}}{du^{n-2}}\,X_{n-2} \quad (n \geq 2) \tag{4.38}$$

with

$$X_n(u,\theta,\phi) = \sum_{\substack{C,D \\ C+D=n \\ D\text{ even}}} \binom{n}{C} \cos^C\theta \sin^D\theta \Big[\frac{(D\cos^2\theta - \sin^2\theta)}{D+1}\,\alpha_{CD}^{(11,2,1)}$$

$$- \frac{(D + \sin^2\theta)}{D+1}\,\alpha_{CD}^{(22,2,-1)} + \sin^2\theta\,\alpha_{CD}^{(33,0,1)} + \frac{2iD\cos\theta}{D+1}\,\alpha_{CD}^{(12,2,0)}\Big]$$

$$-2\sum_{\substack{C,D \\ C+D=n \\ D\text{ odd}}} \binom{n}{C} \cos^C\theta \sin^{D+1}\theta\,[\cos\theta\,\alpha_{CD}^{(13,1,1)} + i\,\alpha_{CD}^{(23,1,0)}] \qquad (n \geq 0) \tag{4.39}$$

where

$$\alpha_{CD}^{(ij,k,l)}(u) = \frac{1}{ma^{n+2}} \int_0^{2\pi}\int_S T'_{ij}(u,R,z)\cos^{D+k}\phi\, Z^C R^{D+l}\, d\phi\, dS \tag{4.40}$$

In (4.40), the coordinates R, ϕ, z are cylindrical polars, T'_{ij} are the space components of the energy momentum tensor in these coordinates, and S is any surface $g(R, z) = 0$ enclosing the sources $(dS = dR\,dz)$.

A simple but somewhat artificial source model shows quite clearly the dimensional mixing referred to earlier. This model has already been considered by Bonnor (1959) and Rotenberg (1964). We have two particles A, B of masses m_1, m_2 moving in a straight line AB about their centre of mass O, taken to be the origin of a rectangular system of co-ordinates $Oxyz$. The coordinates of A, B at time u are taken to be $(0, 0, \zeta_1(u))$ and $(0, 0, -\zeta_2(u))$ respectively, so that $m_1\zeta_1 = m_2\zeta_2$. In this system the only non-vanishing components of the energy momentum tensor are T_{33}, T_{30} and T_{00}. This implies that the only non-vanishing $\alpha_{CD}^{(ij,\,k,\,\ell)}$ in (4.39) are $\alpha_{n0}^{(33,0,1)}$, which are now given by

$$\alpha_{n0}^{(33,0,1)}(u) = \int_{-\infty}^{\infty} \frac{z^n\, T_{33}(z,\, u)dz}{ma^{n+2}} \qquad (4.41)$$

(the prime above T_{33} is no longer necessary). By (4.22) and (4.23) these $\alpha_{n0}^{(33,0,1)}$ are equivalent to $h_{33|\overbrace{33\ldots\ldots3}^{n\ \text{times}}}$. Further, the conservation equations (4.25) yield the relations

$$\ddot{h}_{00|c_1c_2\cdots c_n} = \frac{n!}{(n-2)!}\, h_{(c_1c_2|c_3\cdots c_n)} \qquad (4.42)$$

between mass and stress moments. Putting

$$H_{(n)}(u) \equiv h_{00|\overbrace{33.....3}^{n\ \text{times}}} = \int_{-\infty}^{\infty} \frac{z^n\ T_{00}(z,\ u)dz}{ma^n} \tag{4.43}$$

we see from (4.38), (4.39) and (4.42) that

$$\overset{(1\tilde{n})}{\sigma^0} = \frac{1}{n!} \sin^2\theta\cos^{n-2}\theta\ \frac{d^n}{du^n} H_{(n)}(u) \quad (n \geq 2) \tag{4.44}$$

Remembering (4.36), we see from (B. 1d) of Appendix B that the 'ϕ_1-like' 2^n-pole solution for σ^0 for this two particle case is

$$\overset{(1n)}{\sigma^0} = \frac{-2^n(n-2)!}{(2n)!} \frac{d^n}{du^n} \overset{(n)}{h}(u) \quad P_n^2(\cos\theta); \tag{4.45}$$

$$(n \geq 2) \quad \overset{(n)}{h}(u) \equiv \overset{(n)}{h_0}(u)$$

Using

$$\sin^2\theta\cos^m\theta = \begin{cases} C_{2,m}\ P_2^2 + C_{4,m}\ P_4^2 + \cdots + C_{m+2,m}\ P_{m+2}^2 \\ \qquad\qquad m \text{ even} \\ C_{3,m}\ P_3^2 + C_{5,m}\ P_5^2 + \cdots + C_{m+2,m}\ P_{m+2}^2 \\ \qquad\qquad m \text{ odd} \end{cases} \tag{4.46}$$

where the coefficients $C_{r,m}$ are non-zero constants[1], we see easily that each $a^n \overset{(1n)}{\tilde{\sigma}^0}$ (n even) contributes to $\overset{(1n)}{\sigma^0}$, $\overset{(1,n-2)}{\sigma^0}$. $\overset{(1,n-4)}{\sigma^0}$, ..., $\overset{(12)}{\sigma^0}$ with similar behaviour for the case n odd. Thus the moments $\overset{(n)}{h}(u)$ are due to an infinite number of contributions, each contribution having different dimensionality. Specifically we have

$$\overset{(n)}{h}(u) = \frac{-(2n)!}{2^n(n-2)!}\sum_{r=0}^{\infty} \frac{a^{n+2r}}{(n+2r)!} C_{n,n+2r-2} \cdot \frac{d^{2r}}{du^{2r}} H_{(n+2r)}(u) \qquad (4.47)$$

the $C_{n,n+2r-2}$ being the coefficients of (4.46).

As mentioned previously this two particle case is somewhat artificial in that the masses considered are singularities. However, it should be obvious that in more realistic situations, in which most or all of the terms of (4.39) would be brought into play (as opposed to the single term used for the two particle case) dimensional mixing is assured for all except the most special systems. Indeed it seems possible that all

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[1] The $C_{r,m}$ $(m \geq 0,\ 2 \leq r \leq m+2)$ are given by

$$C_{r,m} = \begin{cases} \dfrac{(2r+1)\cdot 2^r \cdot m!\ (\frac{1}{2}m + \frac{1}{2}r + 1)!}{(\frac{1}{2}m - \frac{1}{2}r + 1)!\ (m + r + 3)!}, & \text{if } m + r \text{ is even} \\ 0, & \text{if } m + r \text{ is odd} \end{cases}$$

radiating systems[1] give rise to the mixing, although this conjecture might be somewhat difficult to test, vitiated as it is by complicated calculations.

CONCLUSION

The previous study has been that of multipole moments in the theory of spin 2 fields in the region exterior to the sources producing the fields. We find that there are two different types of moments to define. The first relates to a somewhat arbitrary solution of a set of first order differential equations subject only to general boundary conditions. The second is related to the solution of a (wave) equation containing terms describing the sources. The difference between the two moments is very marked in respect of dimensional considerations. As far as establishing the general features of gravitational radiation the second definition is too detailed and cumbersome. Features like wave tails, and the existence of mass and momentum losses from radiating bodies can be (and have been) perfectly well established using the first definition only. However, by its very nature, the second definition is the one which should be employed whenever specific calculations related to given source configurations are required. This holds in particular for energy momentum losses. If the pseudo tensor of energy is used for these calculations they turn out to be

[1] Although the class of radiating systems for which mixing takes place is very wide, it seems that static systems do not suffer the mixing.

very long and involved. This difficulty can be removed by using the news function instead. The formulae for the main contributions to the energy momentum losses are quite easy to obtain, even for the most general radiating systems, and are given in part 2 of this paper.

Thus we see that it is useful, and arguably necessary, to have two 'moment' definitions in (linearised) general relativity. It does not seem in the literature on gravitational radiation that a clear distinction between the two types of moment has ever been made. Indeed, there does not seem to be any reference to a moment of the second type. This lack of awareness of the second type of moment appears to have led to some incorrect results: in one approximation method [1] used in recent years in gravitational radiation certain solutions were obtained which seem incorrect, essentially because the form of the angular-dimensional coupling, and hence dimensional mixing, seems incorrect.

The work described in this paper forms a part of a thesis submitted in 1977 to the University of London for the degree of Ph.D.

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[1] Namely, the double parameter approximation method developed by Bonnor (1959) and Bonnor and Rotenberg (1966). For further discussion regarding this point see Willmer (1977).

APPENDIX A

Consider ordinary three dimensional flat space, into which we introduce spherical polars r, θ, ϕ. At each point of each sphere r = const. introduce an orthonormal triad of vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, where $\underline{a}$, $\underline{b}$ are tangent, and $\underline{c}$ normal, to the sphere. $\underline{a}$ and $\underline{b}$ are defined up to a rotation about $\underline{c}$ through an angle χ. Such a rotation is conveniently described by the transformation

$$\underline{m}' = e^{i\chi}\,\underline{m} \tag{A.1}$$

where $\underline{m} = 2^{-\frac{1}{2}}(\underline{a} + i\underline{b})$. A quantity is said to be of spin weight s if it transforms under (I.1) by

$$\eta' = e^{is\chi}\,\eta \tag{A.2}$$

Now specialise $\underline{m}$ so that $\underline{a}$, $\underline{b}$ are respectively tangent to the curves ϕ = const., θ = const. on each sphere. Then any rotation of the coordinate system about the origin must be accompanied by the transformation (I.1). The operators $\eth$, $\bar{\eth}$ defined by

$$\left.\begin{aligned} \eth\eta &= -(\sin\theta)^{s}\left(\partial/\partial\theta + \frac{i}{\sin\theta}\,\partial/\partial\phi\right)\{(\sin\theta)^{-s}\eta\} \\ \bar{\eth}\eta &= -(\sin\theta)^{-s}\left(\partial/\partial\theta - \frac{i}{\sin\theta}\,\partial/\partial\phi\right)\{(\sin\theta)^{s}\eta\} \end{aligned}\right\} \tag{A.3}$$

(η of spin weight s) have the properties of being of spin weights 1, -1 respectively under (I.1), i.e. (with η as in (I.2))

$$\eth' \eta' = e^{i(s+1)\chi}\, \eth\eta \quad , \quad \bar{\eth}' \eta' = e^{i(s-1)\chi}\, \bar{\eth}\eta \tag{A.4}$$

In particular we can consider $\eth$, $\bar{\eth}$ acting on the ordinary spherical harmonics. Write

$$Y_\ell^m(\theta, \phi) = A_{\ell m}\, P_\ell^{|m|}(\cos\theta)\, e^{im\phi} \tag{A.5}$$

where

$$A_{\ell m} = \begin{cases} (-1)^m \left\{\dfrac{(2\ell+1)(\ell-|m|)!}{4\pi\,(\ell+|m|)!}\right\}^{\frac{1}{2}} , & m \geq 0 \\[2ex] \left\{\dfrac{(2\ell+1)(\ell-|m|)!}{4\pi\,(\ell+|m|)!}\right\}^{\frac{1}{2}} , & m < 0 \end{cases} \tag{A.6}$$

and where $P_\ell^{|m|}(\cos\theta)$ are the associated Legendre functions. Operating with $\eth$, $\eth^2$ on (I.5) we obtain

$$\eth Y_\ell^m = A_{\ell m}\left(\frac{m}{\sin\theta} P_\ell^{|m|} - \frac{d}{d\theta}(P_\ell^{|m|})\right) e^{im\phi}$$

$$\eth^2 Y_\ell^m = A_{\ell m}\left[\left(\frac{m^2}{\sin^2\theta} + \frac{2m\cos\theta}{\sin^2\theta}\right)P_\ell^{|m|} - \left(\frac{2m}{\sin\theta} + \cot\theta\right)\frac{d}{d\theta}(P_\ell^{|m|}) + \frac{d^2}{d\theta^2}(P_\ell^{|m|})\right]e^{im\phi} \tag{A.7}$$

These quantities are proportional to the spin - s $(s = 1, 2)$ spherical harmonics defined by

$$
{}_sY_\ell^m = \begin{cases} \sqrt{\dfrac{(\ell - s)!}{(\ell + s)!}}\, \eth^s Y_\ell^m, & 0 \leq s \leq \ell \\ \\ (-1)^s \sqrt{\dfrac{(\ell + s)!}{(\ell - s)!}}\, \bar{\eth}^{-s} Y_\ell^m, & -\ell \leq s \leq 0 \end{cases} \tag{A.8}
$$

for general integral s.

APPENDIX B

The (1ℓ) 'ϕ_1-like' solutions for the Weyl components ψ_1, ψ_2, ψ_3, ψ_4 are given below for $\ell \geq 2$. The only non-vanishing (10) solution is $\overset{(10)}{\psi_2} = -r^{-3}$, and the (11) (or dipole) solution can be made to vanish by a coordinate transformation. These (1ℓ) solutions, together with the $\overset{(1\ell)}{\psi_0}$ solutions of §3, complete the linearised 'ϕ_1-like' Weyl field.

$$
\overset{(1\ell)}{\psi_1} = -\Big[\sum_{q=-\ell}^{\ell} \frac{\sqrt{2}\, \overset{(\ell)}{C_1} \overset{(\ell)}{h_q}{}^{(\ell-1)}}{r^4} + \frac{(\ell - 1)(\ell + 2)}{\sqrt{2}} \sum_{n=1}^{\ell-1} \sum_{q=-\ell}^{\ell} \frac{\overset{(\ell)}{C_n} \overset{(\ell)}{h_q}{}^{(\ell-n-1)}}{n r^{n+4}} \Big]
$$

$$
\times\, \eth (P_\ell^{|q|}\, e^{iq\phi}) \tag{B.1a}
$$

$$\overset{(1\ell)}{\psi_2} = [\sum_{q=-\ell}^{\ell}\{\frac{2\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell)}}{r^3} + \frac{\ell(\ell+1)\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell-1)}}{r^4}\}$$

$$+ \frac{(\ell+2)!}{2(\ell-2)!}\sum_{n=1}^{\ell-1}\sum_{q=-\ell}^{\ell}\frac{\overset{(\ell)}{C_n}\overset{(\ell)}{h_q}{}^{(\ell-n-1)}}{n(n+1)r^{n+4}}].P_\ell^{|q|}\ e^{iq\phi} \qquad \text{(B.1b)}$$

$$\overset{(1\ell)}{\psi_3} = [\sum_{q=-\ell}^{\ell}\{\frac{2\sqrt{2}\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell+1)}}{\ell(\ell+1)r^2} + \frac{\sqrt{2}\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell)}}{r^3} + \frac{\ell(\ell+1)\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell-1)}}{2\sqrt{2}r^4}\}$$

$$+ \frac{(\ell+2)!}{2\sqrt{2}(\ell-2)!}\sum_{n=1}^{\ell-1}\sum_{q=-\ell}^{\ell}\frac{\overset{(\ell)}{C_n}\overset{(\ell)}{h_q}{}^{(\ell-n-1)}}{n(n+1)(n+2)r^{n+4}}]\cdot\bar{\eth}(P_\ell^{|q|}\ e^{iq\phi}) \qquad \text{(B.1c)}$$

$$\overset{(1\ell)}{\psi_4} = [\sum_{q=-\ell}^{\ell}\{\frac{4(\ell-2)!}{(\ell+2)!}\frac{\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell+2)}}{r} + \frac{2\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell+1)}}{\ell(\ell+1)\ r^2}$$

$$+ \frac{\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell)}}{2r^3} + \frac{\ell(\ell+1)\overset{(\ell)}{C_1}\overset{(\ell)}{h_q}{}^{(\ell-1)}}{12r^4}\} +$$

$$+ \frac{(\ell+2)!}{4(\ell-2)!}\sum_{n=1}^{\ell-1}\sum_{q=-\ell}^{\ell}\frac{\overset{(\ell)}{C_n}\ \overset{(\ell)}{h_q}{}^{(\ell-n-1)}}{n(n+1)(n+2)(n+3)r^{n+4}}]\ \bar{\eth}^2(P_\ell^{|q|}\ e^{iq\phi}) \qquad \text{(B.1d)}$$

$\overset{(\ell)}{C_n}$ and $\overset{(\ell)}{h_q}{}^{(\kappa)} \equiv \overset{(\ell)}{h_q}{}^{(\kappa)}(u)$ being given by (3.25).

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