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D.G. Willmer

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

MULTIPOLE MOMENTS IN GENERAL RELATIVITY

PART II

D.G. WILLMER
Department of Mathematical Sciences
University of Petroleum and Minerals
Dhahran, Saudi Arabia.

ABSTRACT

Certain integral expressions involving the Bondi news function are used, together with multipole solutions related to gravitationally radiating isolated sources, to determine the main contributions to their energy momentum losses.

1. INTRODUCTION

In part I of this paper * two definitions of multipole moment were given and analysed. One definition was specifically related to the source structure, the other was not. It was mentioned that the source dependent moment was too detailed and cumbersome for establishing general features of gravitational radiation but was, by its very nature, the one to be employed when calculations involving given source configurations are required. In this paper we will consider only this source dependent definition.

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The study of energy momentum losses from gravitationally radiating sources is often associated with the energy pseudo tensor. The expression for this quantity in terms of the metric tensor is long and complicated, and calculations using it turn out to be laborious even for simple situations. The methods of part I were based on the spinor and null tetrad formalisms, in which a much more useful quantity for the study of energy momentum losses occurs, namely the news function of Bondi. This quantity will also be used here, and will be combined with the definition of moment given above to obtain the main contributions for energy momentum losses from gravitationally radiating sources. The expressions are simple, and completely general.

The sources will be taken to be spatially compact i.e. of finite extent. We shall also suppose that the multipole moments are constant except for a a finite retarded time interval, a condition described by saying that 'the source is in motion only for a finite period'. However it should be clearly understood that this phraseology refers only to linearised theory. If non linear approximations are taken into account then during the period in which the moments are nonconstant the source will in general recoil due to the linear momentum carried away by the gravitational waves. This recoil continues after the moments have once again assumed constant values.

A specific source is considered at the end of the paper - the rotating rod. This example is used merely to illustrate the previous theory; the results, especially for the energy loss of the rod, are well known and certainly not new.

2. ENERGY MOMENTUM RADIATION FROM COMPACT SOURCES

The paper of Sachs (1962) introduced a certain group - the Bondi - Metzner - Sachs (or B.M.S.) group - as an asymptotic symmetry group of transformations for an asymptotically flat space - time[1]. Although not a Lie transformation group (it is neither locally compact nor finitely generated) we may imitate the methods used in Lie group theory to obtain its infinitesimal generators. Sachs, using a somewhat heuristic quantum argument, proceeded to associate with these generators certain integrals. He then identified some of the integrals with the energy momentum content of the space - time. The rate of energy and linear momentum loss from a radiating source is given as the (retarded) time derivative of the momentum 4-vector $P_\mu = (P_0, P_1, P_2, P_3)$ whose components are related

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[1] This statement is not precisely correct, but is adequate for our purpose. Sachs in fact refers to 'AF spaces', a generalization of asymptotically flat spaces.

to the subgroup of ordinary translations of the B.M.S. group. In general the generator of such translations is $\alpha\, \partial/\partial u$, with[1]

$$\alpha = \varepsilon_0 + \varepsilon_1 \sin\theta\cos\phi + \varepsilon_2 \sin\theta\sin\phi + \varepsilon_3 \cos\theta \tag{2.1}$$

By taking ε_0, ε_1, ε_2, ε_3 equal to 1 in turn and putting all other ε's zero we obtain the integrals representing P_μ:

$$(P_0, P_1, P_2, P_3) = \frac{1}{2} \int \dot{\sigma}^0 \dot{\bar{\sigma}}^0 (1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \tag{2.2}$$

where

$$\int f \equiv \int_{-\infty}^{u} du \int_0^{\pi} d\theta \int_0^{2\pi} f(u,\ \theta,\ \phi) \sin\theta\ d\phi \tag{2.3}$$

Here σ^0 is the asymptotic shear of congruences of null geodesics and the dot denotes a retarded time derivative; $\dot{\sigma}^0$ is Bondi's news function.

These P_μ define a quantity which transforms as a free Lorentz

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[1] The co-ordinates for the space-time are Bondi-Sachs co-ordinates u, r, θ, ϕ in which u is a retarded time, r is a luminosity (distance) parameter, and θ, ϕ label null rays. r, θ, ϕ are analogous to spherical polar coordinates in flat space.

vector under all B.M.S. transformations, and is timelike or zero, vanishing if and only if $\dot{\sigma}^0$ vanishes for all u. The integrals (2.2) have also been derived by a purely relativistic method by Tamburino and Winicour (1966) using 'flux linkages' for the asymptotically flat space-time. Thus their interpretation as components of a momentum 4-vector does indeed seem to be correct.

The generators for angular momentum arise out of the subgroup of B.M.S. transformations relating to conformal transformations on the sphere, and hence to the proper orthochronous Lorentz transformations, with which the group is isomorphic. The six generators $L^{\alpha\beta}(= -L^{\beta\alpha})$ $(\alpha, \beta = 0,1,2,3)$ are given by

$$\left.\begin{aligned} L^{12} &\equiv L_z = \partial/\partial\phi \\ L^{30} &\equiv R_z = \sin\theta\ \partial/\partial\theta + u\cos\theta\ \partial/\partial u \end{aligned}\right\} \quad (2.4a)$$

and, putting $L^{\pm} = L^{13} \pm iL^{23}$, $R^{\pm} = L^{10} \mp iL^{20}$,

$$\left.\begin{aligned} L^{\pm} &= e^{\pm i\phi}(\partial/\partial\theta \pm i\cot\theta\ \partial/\partial\phi) \\ R^{\pm} &= -e^{\pm i\phi}(\cos\theta\ \partial/\partial\theta \pm i\operatorname{cosec}\theta\ \partial/\partial\phi - u\sin\theta\ \partial/\partial u) \end{aligned}\right\} \quad (2.4b)$$

$L^{\alpha\beta}$ satisfy the commutator relationships

$$[L^{\alpha\beta}, L^{\gamma\delta}] = \eta^{\alpha\gamma}L^{\beta\delta} + \eta^{\beta\delta}L^{\alpha\gamma} - \eta^{\alpha\delta}L^{\beta\gamma} - \eta^{\beta\gamma}L^{\alpha\delta} \qquad (2.5)$$

$\eta^{\alpha\beta}$ being the Lorentz metric - diag (1, -1, -1, -1). These are identical to the commutator relationships in flat space for the angular momentum tensor $M^{\alpha\beta} = 2x^{[\alpha}p^{\beta]}$ (x^α are the coordinates of a particle with momentum p^α; [ ] denotes skew symmetrisation). Now in flat space the space components of $M^{\alpha\beta}$ are the components of the 3-dimensional angular momentum vector $\underline{M} = \underline{r} \times \underline{p} = (M_x, M_y, M_z)$ $= (M^{23}, M^{31}, M^{12})$[1]. The analogy between the M's and the L's led Sachs to associate the angular momentum of the system in non-flat space with integrals related to L^{12} and $L^{\pm}$, namely

$$\left.\begin{aligned} I\{L_z\} &= \frac{1}{2}\int \dot{\sigma}^0 L_z \bar{\gamma} \\ I\{L^{\pm}\} &= \frac{1}{2}\int (\dot{\sigma}^0 L^{\pm}\bar{\gamma} \mp 2e^{\pm i\phi}\operatorname{cosec}\theta\,\dot{\sigma}^0\bar{\gamma}) \end{aligned}\right\} \qquad (2.6)$$

where $$\gamma(u,\theta,\phi) \equiv \sigma^0(u,\theta,\phi) - \frac{1}{2}\sigma^0(-\infty,\theta,\phi) - \frac{1}{2}\sigma^0(\infty,\theta,\phi) \qquad (2.7)$$

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[1] The quantities M^{10}, M^{20}, M^{30} do have physical meaning (see Landau and Lifshitz (1962)) but are not important here.

Indeed, under a rigid translation in the (x, y) plane:

$$\theta' = \theta, \quad \phi' = \phi, \quad u' = u + \epsilon_1 \sin\theta\cos\phi + \epsilon_2 \sin\theta\sin\phi \qquad (2.8)$$

$I\{L_z\}$ transforms precisely as we would expect - from experience in flat space - if it is to be a suitable candidate for angular momentum in the z direction[1]:

$$I'\{L_z\} = I\{L_z\} + \epsilon_1 P_2 - \epsilon_2 P_1 \qquad (2.9)$$

P_1, P_2 being components of P_μ. Similar expressions hold for the transformations of $I\{L_z\}$, $I\{L^\pm\}$ under any translation (2.1). Thus we will take P_μ, $I\{L_z\}$ and $I\{L^\pm\}$ as representing the energy momentum of the space-time.

All that is required now to evaluate (2.2) and (2.6) is the expression for σ^0. As mentioned in the introduction we will be considering a definition of multipole moment related to the structure

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[1] Under change from one origin O to another $\hat{O}$, whose co-ordinates are Y^α with respect to O, the 4-momentum and angular momentum in flat space-time transform as

$$\tilde{P}^\mu = P^\mu$$

$$\tilde{M}^{\alpha\beta} = M^{\alpha\beta} - 2Y^{[\alpha}P^{\beta]}$$

of the source producing the gravitational radiation. The multipoles are related to linearised theory. Thus the co-ordinates for the space-time are related from now on to this linear approximation. In part I pseudo-Galilean co-ordinates (t, x, y, z) and related retarded time and spherical polar type co-ordinates, which for the moment we shall call $(\hat{u}, \hat{r}, \hat{\theta}, \hat{\phi})$ (where $(t, x, y, z) = (\hat{u} + \hat{r},\ \hat{r}\sin\hat{\theta}\cos\hat{\phi},\ \hat{r}\sin\hat{\theta}\sin\hat{\phi},\ \hat{r}\cos\hat{\theta})$, were used. Since we are here considering σ^0, which is related to the null tetrad formalism, the $\hat{u}, \hat{r}, \hat{\theta}, \hat{\phi}$ can be considered - from the point of view of the calculations to be performed here - equivalent to the u, r, θ, ϕ used earlier, and hence the carets will be dropped[1]. Further, raising and lowering indices is accomplished via the Lorentz metric.

With these points in mind, define quantities $h_{\alpha\beta|c_1c_2\cdots c_n}(u)$ by

$$h_{00|c_1c_2\cdots c_n} = \frac{N_{00|c_1c_2\cdots c_n}}{ma^n} \;;$$

$$h_{0c|c_1c_2\cdots c_n} = \frac{N_{0c|c_1c_2\cdots c_n}}{ma^{n+1}} \;;$$

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[1] For further discussion of this point see Willmer (1977).

$$h_{bc|c_1c_2\cdots c_n} = \frac{N_{bc|c_1c_2\cdots c_n}}{ma^{n+2}} \tag{2.10}$$

where

$$N_{\alpha\beta|c_1c_2\cdots c_n}(u) = \int_V T_{\alpha\beta}(u,\xi^a)\xi_{c_1}\xi_{c_2}\cdots\xi_{c_n}\,dv \tag{2.11}$$

$T_{\alpha\beta}(u, \xi^a)$ is the energy momentum tensor of the sources at time u at the source point whose spatial co-ordinates (as functions of (x, y, z)) are $\xi^a \equiv (\xi^1, \xi^2, \xi^3)$ (all lower case Latin indices range and sum over 1, 2 and 3 and Greek indices range over 0,1,2 and 3). V is any volume completely enclosing the sources. It should be noted that the distinction between contra- and covariant components of ξ^a is not absolutely necessary here, and is made merely as a matter of preference. 'm' and 'a' in (2.10) are mass and length parameters respectively for the system. Hence the $h_{\alpha\beta|c_1c_2\cdots c_n}$ are dimensionless i.e. they are not affected by any change in units in 'm' or 'a'. It is shown in part I that the linearised σ^0 is given in terms of these dimensionless h's by (including now a normalisation factor $(2\pi)^{-\frac{1}{2}}$)

$$\sigma^0 = (2\pi)^{-\frac{1}{2}}\, m q^a q^b \{a^2 h_{ab} + a^3 p^{c_1} \dot{h}_{ab|c_1} + \frac{a^4}{2!} p^{c_1} p^{c_2} \ddot{h}_{ab|c_1c_2} + \cdot\cdot\cdot + \frac{a^{t+2}}{t!} p^{c_1} p^{c_2} \cdot\cdot\cdot p^{c_t} \frac{d^t}{du^t} h_{ab|c_1c_2\cdots c_t} + \cdot\cdot\cdot\} \tag{2.12}$$

where

$$\left.\begin{aligned} p^a &= \frac{1}{r}(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ q^a &= (\partial/\partial\theta + \frac{i}{\sin\theta}\partial/\partial\phi) p^a \end{aligned}\right\} \tag{2.13}$$

with dots denoting retarded time derivatives, as previously. To obtain the dominant contributions to the energy momentum losses we require only the terms in ma^2 and ma^3. Denoting the coefficient of ma^2 by $\overset{(12)}{\sigma^0}$ we have

$$\overset{(12)}{\sigma^0} = \frac{1}{\sqrt{2\pi}} q^a q^b h_{ab} = \frac{1}{2\sqrt{2\pi}} q^a q^b \ddot{h}_{00|ab} \tag{2.14}$$

the last equation arising via the relation $h_{ab} = \frac{1}{2}\ddot{h}_{00|ab}$ occurring in the linearised theory. (2.14) can be expressed as a sum of spin weight 2 spherical harmonics ${}_2Y_2^{m_0}$ ($m_0 = -2, -1, 0, 1, 2$) (see Appendix):

$$\overset{(12)}{\sigma^0} = -\frac{1}{\sqrt{3\pi}} \sum_{m_0} \frac{\ddot{H}^{(2)}_{m_0}(u)\ {}_2Y_2^{m_0}}{A_{2m_0}} \tag{2.15}$$

where

$$\left.\begin{aligned} H_0^{(2)} &= \frac{1}{2}\,(K_{11} + K_{22} - 2K_{33}) \\ H_1^{(2)} &= \overline{H_{-1}^{(2)}} = \frac{1}{2}\,(iK_{23} - K_{13}) \\ H_2^{(2)} &= \overline{H_{-2}^{(2)}} = \frac{1}{8}\,(K_{22} - K_{11} + 2iK_{12}) \end{aligned}\right\} \tag{2.16}$$

and

$$K_{ab} = h_{00|ab} \tag{2.17}$$

The coefficients A_{2m_0} are given by (A4).

Likewise the expression for $\overset{(13)}{\sigma^0}$ can be converted to spin weighted spherical harmonics. The result is

$$\begin{aligned} \overset{(13)}{\sigma^0} &= \frac{1}{\sqrt{2\pi}}\, q^a q^b p^c\, \dot{h}_{ab|c} \\ &= \frac{1}{\sqrt{2\pi}}\,\{\frac{1}{6}\, q^a q^b p^c\, \dddot{h}_{00|abc} + \frac{4}{3}\, q^a q^{[b} p^{c]} \ddot{h}_{0c|ba}\} \\ &= -\frac{1}{3\sqrt{15\pi}} \sum_{m_1} \frac{\dddot{H}^{(3)}_{m_1}(u)\ {}_2Y_3^{m_1}}{A_{3m_1}} + \frac{8}{\sqrt{3\pi}} \sum_{m_2} \frac{\ddot{g}_{m_2}(u)\ {}_2Y_2^{m_2}}{A_{2m_2}} \end{aligned} \tag{2.18}$$

with

$$\left.\begin{aligned}
H_0^{(3)} &= \frac{1}{2}\,(3K_{113} + 3K_{223} - 2K_{333}) \\
H_1^{(3)} &= \overline{H_{-1}^{(3)}} = \frac{1}{8}\,(K_{111} + K_{122} - 4K_{133} + i(4K_{233} - K_{112} - K_{222})) \\
H_2^{(3)} &= \overline{H_{-2}^{(3)}} = \frac{1}{8}\,(K_{223} - K_{113} + 2iK_{123}) \\
H_3^{(3)} &= \overline{H_{-3}^{(3)}} = \frac{1}{48}\,(3K_{112} - K_{111} + i(3K_{112} - K_{222}))
\end{aligned}\right\} \tag{2.19}$$

where

$$K_{abc} = h_{00|abc} \tag{2.20}$$

and

$$\left.\begin{aligned}
g_0 &= \frac{i}{6}\,(c_6 - \frac{1}{2}\,(c_4 + c_5)) = -\bar{g}_0 \\
g_1 &= \frac{1}{24}\,(ic_1 + c_3) = -\bar{g}_{-1} \\
g_2 &= \frac{1}{48}\,(i(c_4 - c_5) + c_2) = -\bar{g}_{-2}
\end{aligned}\right\} \tag{2.21}$$

where

$$\left.\begin{aligned}
c_1 &= h_{02|11} + h_{03|23} - h_{01|12} - h_{02|33} \\
c_2 &= h_{01|13} + h_{03|22} - h_{02|23} - h_{03|11} \\
c_3 &= h_{01|33} + h_{02|12} - h_{01|22} - h_{03|13} \\
c_4 &= h_{03|12} - h_{02|13}, \quad c_5 = h_{01|23} - h_{03|12}, \\
c_6 &= h_{02|13} - h_{01|23}
\end{aligned}\right\} \qquad (2.22)$$

Let us return now to (2.2). Differentiating its first component with respect to u gives

$$\begin{aligned}
\frac{dP_0}{du} &= \frac{1}{2}\int_S \dot{\sigma}^0\, \dot{\bar{\sigma}}^0 \, d\Omega \\
&= \frac{1}{2}\int_S \Big(ma^2 \overset{(\dot{12})}{\sigma^0} + ma^3 \overset{(\dot{13})}{\sigma^0} + \cdots\Big)\Big(ma^2 \overset{(\dot{\overline{12}})}{\sigma^0} + ma^3 \overset{(\dot{\overline{13}})}{\sigma^0} + \cdots\Big)\, d\Omega \\
&= \frac{m^2a^4}{2}\int_S \overset{(\dot{12})}{\sigma^0}\, \overset{(\dot{\overline{12}})}{\sigma^0}\, d\Omega + O(m^2a^5) \qquad (2.23)
\end{aligned}$$

the integrals being taken over the sphere, and $O(m^p a^q)$ $(p \geq 1,\ q \geq 0)$

denoting a term of the form $\sum_{r\geq p, s\geq q} m^r a^s \binom{rs}{\alpha}$, each $\binom{rs}{\alpha}$ being independent of m or a. The main contribution to the rate of energy loss is thus

$$\frac{m^2a^4}{2}\int_S \dot{\binom{12}{\sigma^0}}\ \dot{\overline{\binom{12}{\sigma^0}}}\ d\Omega$$

$$= \frac{m^2a^4}{6\pi}\sum_{m_0,m_1}\frac{\dddot{H}^{(2)}_{m_0}}{A_{2m_0}}\ \frac{\overline{\dddot{H}^{(2)}_{m_1}}}{A_{2m_1}}\int_S {}_2Y_2^{m_0}\ {}_2\overline{Y}_2^{m_1}\ d\Omega \qquad (2.24)$$

which, using the orthonormality condition (A5), gives

$$\frac{dP_0}{du} = \frac{2}{15}m^2a^4(\dddot{H}_0^{(2)2} + 12\dddot{H}_1^{(2)}\ \dddot{H}_{-1}^{(2)} + 48\dddot{H}_2^{(2)}\ \dddot{H}_{-2}^{(2)})$$

$$+ O(m^2a^5) \qquad (2.25)$$

Similarly we can find the contributions of order m^2a^4 for the rate of linear momentum loss. We have to find

$$\frac{m^2a^4}{2}\int_S \dot{\binom{12}{\sigma^0}}\ \dot{\overline{\binom{12}{\sigma^0}}}\ (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ d\Omega \qquad (2.26)$$

The results (A10) and (A12) help here. Indeed, we find that all three components vanish, establishing the fact that there exists no linear momentum loss <u>at infinity</u> due to quadrupole-quadrupole interaction of a radiating source[1].

The first non-zero contribution to linear momentum loss comes from the quadrupole-octupole interaction. Written out, we have to find

$$\frac{m^2a^5}{2}\int_S \overset{(12)}{\dot{\sigma}}{}^{0}\;\overset{(\overline{13})}{\dot{\sigma}}{}^{0}\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\,d\Omega$$

$$+ \text{ complex conjugate} \qquad (2.27)$$

Once again results (A10) and (A12) help considerably, and we obtain

$$\left(\frac{dP_1}{du},\frac{dP_2}{du}\right) = \frac{8}{315}\,m^2a^5(\mathrm{Re},\ \mathrm{Im})\{2\dddot{H}{}^{(2)}_{0}\,\overset{(4)}{H}{}^{(3)}_{-1} + 20\dddot{H}{}^{(2)}_{1}\,\overset{(4)}{H}{}^{(3)}_{-2}$$

$$+ 120\dddot{H}{}^{(2)}_{2}\,\overset{(4)}{H}{}^{(3)}_{-3} - \dddot{H}{}^{(2)}_{-1}\,\overset{(4)}{H}{}^{(3)}_{0} - 4\dddot{H}{}^{(2)}_{-2}\,\overset{(4)}{H}{}^{(3)}_{1}\,\}$$

$$+ \frac{64}{15}\,m^2a^5(\mathrm{Re},\mathrm{Im})\{\dddot{H}{}^{(2)}_{0}\,\dddot{g}_{-1} + 4\dddot{H}{}^{(2)}_{-1}\,\dddot{g}_{-2} - \dddot{H}{}^{(2)}_{-1}\,\dddot{g}_{0}$$

$$- 4\dddot{H}{}^{(2)}_{-2}\,\dddot{g}_{1}\} + O(m^2a^6)\ ;\ \overset{(4)}{H}{}^{(3)}_{a} \equiv \frac{d^4}{du^4}H^{(3)}_{a} \qquad (2.28)$$

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[1] Note, however, that quadrupole-quadrupole linear momentum <u>transfer</u> does occur between an emitting source and a radiating source (see Cooperstock and Booth (1969)).

and

$$\frac{dP_3}{du} = \frac{4}{315}\, m^2 a^5 \,\mathrm{Re}\Big(\dddot{H}_0^{(2)} \overset{(4)}{H}{}_0^{(3)} + 16 \dddot{H}_1^{(2)} \overset{(4)}{H}{}_{-1}^{(3)} + 80 \dddot{H}_2^{(2)} \overset{(4)}{H}{}_{-2}^{(3)} \Big)$$

$$+ \frac{128}{15}\, m^2 a^5 \,\mathrm{Re}\big(\dddot{H}_{-1}^{(2)} \dddot{g}_1 + 8 \dddot{H}_{-2}^{(2)} \dddot{g}_2 \big) + O(m^2 a^6) \qquad (2.29)$$

Finally we calculate the integrals corresponding to the rate of angular momentum loss. The fact that the source is in motion only for finite time intervals (in the sense described in the introduction) allows us to set the $\gamma(u, \theta, \phi)$ of (2.7) as $\sigma^0(u, \theta, \phi)$. The rate of loss in the z-direction is

$$\frac{d}{du} I\{L_z\} = \frac{m^2 a^4}{2} \int_S \dot{\overset{(12)}{\sigma^0}}\, L_z\, \overset{(\overline{12})}{\sigma^0}\, d\Omega + O(m^2 a^5)$$

$$= \frac{8}{5}\, m^2 a^4 \,\mathrm{Im}\big(\dddot{H}_1^{(2)} \ddot{H}_{-1}^{(2)} + 8 \dddot{H}_2^{(2)} \ddot{H}_{-2}^{(2)} \big) + O(m^2 a^5) \qquad (2.30)$$

A lengthy calculation gives for the second equation of (2.6)

$$\frac{d}{du} I\{L^{\pm}\} = \frac{8}{15}\, m^2 a^4 \big(9 \ddot{H}_{\pm 1}^{(2)} \dddot{H}_{\mp 2}^{(2)} + \ddot{H}_0^{(2)} \dddot{H}_{\mp 1}^{(2)} - \dddot{H}_0^{(2)} \ddot{H}_{\mp 1}^{(2)}$$

$$- 9 \dddot{H}_{\pm 1}^{(2)} \ddot{H}_{\mp 2}^{(2)} \big) + O(m^2 a^5) \qquad (2.31)$$

3. THE ROTATING ROD

As a check on the results of the previous section we consider a specific radiating system; that of a rotating rod.

The rod will be supposed to rotate for a finite period, outside of which it is stationary. A mechanism for starting and stopping the motion has been described by Rotenberg (1972). We suppose that we have a uniform circular ring passing through two holes in the rod, so situated that the centre of the ring coincides with the centre of the rod, the common centre of mass being the origin of a rectangular co-ordinate system $0\,xyz$. Initially the ring alone rotates in the xy-plane about 0 with constant angular velocity ω_0, friction between ring and rod being negligible so that the rod practically remains in a fixed position. Shortly before retarded time $u = 0$ friction is set up between ring and rod by a clamping device, smoothly starting the rod in motion so that the entire system rotates about 0 with constant angular velocity $\omega < \omega_0$ from time $u = 0$, when the rod assumes the position chosen as the x-axis, to time $u = U$. Finally the rod is brought smoothly to rest, with the ring rotating at its former angular velocity, during a short interval; this is achieved by a separating recoil device.

In the above, when we say that the system rotates about its centre of mass with constant angular velocity we mean with respect to the linear approximation, in which conservation of mass, linear momentum and angular momentum is observed. In actual fact we will establish that the system does not rotate about its centre of mass and does not rotate with constant angular velocity when we deal with non-linear - $O(m^2)$ - terms.

The rod $A_1\,O\,A_2$ will be taken to be of mass m, length a, and small uniform cross section $\tilde{s}$. Since the rod coincides with the x-axis at $u = 0$ we have $\angle x\,O\,A_2 = \omega u$ at time u. Write $|OA_i| = aK_i$ $(i = 1, 2)$ and consider any point Q on the rod such that $OQ = a\zeta$ $(-K_1 \leq \zeta \leq K_2)$. Let $\rho(\zeta)$ be the volume density at Q. From elementary dynamics the stress at Q at time u is

$$p = ma\omega^2\tilde{s}^{-1}\,J(\zeta) \tag{3.1}$$

where

$$\left.\begin{aligned} J(\zeta) &= \int_{-K_1}^{\zeta} \eta\sigma(\eta)d\eta \\ \sigma(\zeta) &= \frac{a\tilde{s}}{m}\,\rho(\zeta) = \frac{\rho(\zeta)}{\int_{-K_1}^{K_2}\rho(\eta)d\eta} \end{aligned}\right\} \tag{3.2}$$

are dimensionless quantities independent of m or a. Using the formula (Eddington (1924))

$$\left.\begin{aligned} T^{\alpha\beta} &= t^{\alpha\beta} + \rho u^{\alpha} u^{\beta} \;; \\ t^{ik} &= p^{ik} \,, \quad t^{io} = t^{00} = 0 \end{aligned}\right\} \qquad (3.3)$$

we have the energy momentum tensor for any cohesive source expressed in the linear approximation and in Galilean co-ordinates in terms of the stress p^{ik}, volume density ρ and 4-velocity $u^{\alpha} = (1, u^{i})$ of that particle of matter which passes through the field point x^{i} at time u. The dimensionless moments (up to $h_{ab|cde}$) have been given by Rotenberg (1968) for this case. The moments we require are

$$\left.\begin{aligned} &h_{00|11} = \overset{(2)}{J} c^2, \quad h_{00|12} = \overset{(2)}{J} sc, \quad h_{00|22} = \overset{(2)}{J} s^2 \\ &h_{01|11} = \omega \overset{(3)}{J} sc^2, \quad h_{01|12} = \omega \overset{(3)}{J} s^2 c, \quad h_{01|22} = \omega \overset{(3)}{J} s^3 \\ &h_{02|11} = -\omega \overset{(3)}{J} c^3, \quad h_{02|12} = -\omega \overset{(3)}{J} sc^2, \quad h_{02|22} = -\omega \overset{(3)}{J} s^2 c \\ &h_{00|111} = \overset{(3)}{J} c^3, \quad h_{00|112} = \overset{(3)}{J} sc^2, \quad h_{00|122} = \overset{(3)}{J} s^2 c \\ &h_{00|222} = \overset{(3)}{J} s^3 \end{aligned}\right\} \qquad (3.4)$$

all other $h_{00|ab}$, $h_{0a|bc}$, $h_{00|abc}$ vanishing, where

$$s = \sin \omega u \quad , \qquad c = \cos \omega u \tag{3.5}$$

and where[1]

$$\overset{(n)}{J}(u) = \int_{-K_1}^{K_2} \eta^n \,\sigma(\eta)\, d\eta \equiv \frac{\overset{(n)}{I}(u)}{ma^n} \tag{3.6}$$

(3.4) are calculated for the period of constant spin of the rod i.e. for $0 \leq u \leq U$. We therefore find from (2.16), (2.19) and (2.21)-(2.22) that

$$\left.\begin{aligned}
&H_0^{(2)} = \tfrac{1}{2}\overset{(2)}{J} = \text{constant}, \quad H_1^{(2)} = 0 \\
&H_2^{(2)} = \tfrac{1}{8}\overset{(2)}{J}(s + ic)^2 , \\
&H_0^{(3)} = H_2^{(3)} = 0, \quad H_1^{(3)} = \tfrac{1}{8}\overset{(3)}{J}(c - is) \\
&H_3^{(3)} = \tfrac{1}{48}\overset{(3)}{J}\{c(3s^2 - c^2) + is(3c^2 - s^2)\} \\
&g_0 = g_2 = 0, \quad g_1 = -\tfrac{1}{24}\,\omega\overset{(3)}{J}(s + ic)
\end{aligned}\right\} \tag{3.7}$$

---

[1] $\overset{(n)}{I}(u)$ are the n^{th} moments of the rod about its centre of mass.

The rate of energy momentum loss is now easy to compute. Using (2.25) and (2.28) - (2.31) we find easily that

$$\frac{dP_u}{du} = \frac{32}{5}\overset{(2)}{I}{}^2\omega^6 + O(m^2a^5) \qquad (3.8a)$$

$$\frac{dP_a}{du} = -\frac{464}{105}\overset{(2)}{I}\,\overset{(3)}{I}\,\omega^7(\sin\omega u, -\cos\omega u, 0) + O(m^2a^6) \qquad (3.8b)$$

$$\frac{d}{du}\, I\{L_z\} = -\frac{32}{5}\overset{(2)}{I}{}^2\omega^5 + O(m^2a^5) \qquad (3.8c)$$

$$\frac{d}{du}\, I\{L^{\pm}\} = O(m^2a^5) \qquad (3.8d)$$

Result (3.8a) is very well known (Einstein (1916, 1918), Eddington (1924), Clark (1947), Weber (1961), Rotenberg (1968)), and (3.8b) has been obtained by Rotenberg (1968) using the pseudo tensor[1].

What (3.8b) tells us is that the centre of rotation of the rod

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[1] Rotenberg also obtained a different answer for linear momentum loss using a method originally devised by Synge (1960, Chapter IV). He took this second answer to be the correct one. However, as has been pointed out, the integrals P_a, which lead to (3.8b), have been obtained by widely differing methods by different authors so that there is good reason to believe that (3.8b) do indeed give the rate of linear momentum loss.

does not coincide with its centre of mass but is a fixed point O distant $\frac{464}{105}\frac{I^{(2)} I^{(3)} \omega^5}{m}$ from the axis of symmetry of the rod on the line perpendicular to the rod through its centre of mass. Also, (3.8c) could have been derived from (3.8a) (and, vice versa, (3.8a) could have been derived from (3.8c)) in this special case of the rod using ordinary mechanics.

APPENDIX

The spin-s spherical harmonics have been discussed in Appendix A, Part I. We define

$$ {}_sY_\ell^m = \begin{cases} \sqrt{\frac{(\ell - s)!}{(\ell + s)!}}\, \eth^s\, Y_\ell^m, & 0 \le s \le \ell \\ \\ (-1)^s \sqrt{\frac{(\ell + s)!}{(\ell - s)!}}\, \bar{\eth}^{-s}\, Y_\ell^m, & -\ell \le s \le 0 \end{cases} \tag{A1} $$

for general integral s, where $\eth$, $\bar{\eth}$ are spin weight raising and lowering operators defined by

$$ \left.\begin{aligned} \eth\eta &= -(\sin\theta)^s(\partial/\partial\theta + \frac{i}{\sin\theta}\,\partial/\partial\phi)\{(\sin\theta)^{-s}\eta\} \\ \\ \bar{\eth}\eta &= -(\sin\theta)^{-s}(\partial/\partial\theta - \frac{i}{\sin\theta}\,\partial/\partial\phi)\{(\sin\theta)^{s}\eta\} \end{aligned}\right\} \tag{A2} $$

with η being of spin weight s (see Appendix A, part I for definition of spin weight). Y_ℓ^m are ordinary spherical harmonics:

$$ Y_\ell^m(\theta,\ \phi) = A_{\ell m}\, P_\ell^{|m|}(\cos\theta)\, e^{im\phi} \tag{A3} $$

where

$$A_{\ell m} = \begin{cases} (-1)^m \left\{ \dfrac{(2\ell + 1)(\ell - |m|)!}{4\pi(\ell + |m|)!} \right\}^{\frac{1}{2}}, & m \geq 0 \\[2ex] \left\{ \dfrac{(2\ell + 1)(\ell - |m|)!}{4\pi(\ell + |m|)!} \right\}^{\frac{1}{2}}, & m < 0 \end{cases} \tag{A4}$$

and where $P_\ell^{|m|}(\cos\theta)$ are associated Legendre functions. The spin-s spherical harmonics obey the orthonormality condition

$$\int_S {}_sY_\ell^m \, {}_s\overline{Y}_{\ell'}^{m'} \, d\Omega = \delta_{\ell\ell'} \, \delta_{mm'} \tag{A5}$$

(integration being taken over the sphere), which helps in the evaluation of the integrals

$$\int_S {}_2Y_\ell^m \, {}_2\overline{Y}_{\ell'}^{m'} \, (\sin\theta\cos\phi, \ \sin\theta\sin\phi, \ \cos\theta) \, d\Omega \tag{A6}$$

which arise in (2.26) and (2.27). To calculate (A6) we need the expression for the product of two spherical harmonics of spin weights s and t. This is given by

$${}_sY_\ell^m \cdot {}_tY_k^n = \sum_{j=|\ell-k|}^{\ell+k} \left\{ \frac{(2\ell + 1)(2k + 1)}{4\pi(2j + 1)} \right\}^{\frac{1}{2}} \times \langle kn \ \ell m | k\ell j, \ m + n \rangle$$

$$\langle k - t \ \ \ell - s | k\ell j, \ -s \ -t \rangle \, {}_{s+t}Y_j^{m+n} \tag{A7}$$

where

$$\langle k\beta\ell\alpha | k\ell j, \alpha + \beta\rangle = \sqrt{\frac{(\ell+k-j)!(j+k-\ell)!(j+\ell-k)!(2j+1)}{(j+k+\ell+1)!}} \times$$

$$\sum_{P} \frac{(-1)^{P} \sqrt{(k+\beta)!(k-\beta)!(\ell+\alpha)!(\ell-\alpha)!(j+\alpha+\beta)!(j-\alpha-\beta)!}}{P!(k+\ell-j-P)!(k-\beta-P)!(\ell+\alpha-P)!(j-\ell+\beta+P)!(j-k-\alpha+P)!} \qquad \text{(A8)}$$

are real Clebsch-Gordan coefficients. The conditions to be satisfied in (A8) are $\ell+k \geq j$, $\ell+j \geq k$, $k+j \geq \ell$; otherwise the coefficients are zero. The summation is over those values of P for which the contents of all the factorials are greater than or equal to zero.

Using (A7), (A8), and $\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_1^0$ allows a calculation of ${}_2Y_\ell^m \cos\theta$:

$$ {}_2Y_\ell^m \cos\theta = \frac{1}{\ell} \sqrt{\frac{(\ell+m)(\ell-m)(\ell+2)(\ell-2)}{(2\ell+1)(2\ell-1)}}\; {}_2Y_{\ell-1}^m - \frac{2m}{\ell(\ell+1)}\, {}_2Y_\ell^m$$

$$ + \frac{1}{\ell+1} \sqrt{\frac{(\ell+m+1)(\ell-m+1)(\ell+3)(\ell-1)}{(2\ell+3)(2\ell+1)}}\; {}_2Y_{\ell+1}^m \qquad \text{(A9)}$$

whence

$$\int_S {}_2Y_\ell^m \cdot {}_2\bar{Y}_{\ell'}^{m'} \cos\theta d\Omega = \frac{1}{\ell} \sqrt{\frac{(\ell+m)(\ell-m)(\ell+2)(\ell-2)}{(2\ell+1)(2\ell-1)}} \cdot \delta_{\ell-1,\ell'} \delta_{mm'}$$

$$- \frac{2m}{\ell(\ell+1)} \cdot \delta_{\ell\ell'} \delta_{mm'} + \frac{1}{\ell+1} \sqrt{\frac{(\ell+m+1)(\ell-m+1)(\ell+3)(\ell-1)}{(2\ell+3)(2\ell+1)}} \cdot \delta_{\ell+1,\ell'} \delta_{mm'}$$

(A10)

follows directly on application of (A5). The remaining two integrals of (A6) can be similarly calculated using

$$\sin\theta(\cos\phi, \sin\phi) = \sqrt{\frac{2\pi}{3}} ([Y_1^{-1} - Y_1^1], i[Y_1^{-1} + Y_1^1]). \qquad \text{(A11)}$$

The result is

$$\int_S {}_2Y_\ell^m \cdot {}_2\bar{Y}_{\ell'}^{m'} \sin\theta(\cos\phi, \sin\phi) d\Omega = \frac{1}{2\ell} \sqrt{\frac{(\ell+2)(\ell-2)}{(2\ell+1)(2\ell-1)}}$$

$$[\sqrt{\{(\ell-m)(\ell-m-1)\}}(1, -i)\ \delta_{\ell-1,\ell'}\ \delta_{m+1,m'}$$

$$-\sqrt{\{(\ell+m)(\ell+m-1)\}}(1, i)\ \delta_{\ell-1,\ell'}\ \delta_{m-1,m'}]$$

$$+ \frac{1}{\ell(\ell+1)} [\sqrt{\{(\ell-m)(\ell+m+1)\}}(-1, i) \delta_{\ell\ell'}\ \delta_{m+1,m'}$$

$$- \sqrt{\{(\ell+m)(\ell-m+1)\}}(1,\ i)\ \delta_{\ell\ell'}\ \delta_{m-1,m'}\}$$

$$+ \frac{1}{2(\ell+1)} \sqrt{\frac{(\ell+3)(\ell-1)}{(2\ell+3)(2\ell+1)}} [\sqrt{\{(\ell+m+1)(\ell+m+2)\}}(-1,i)\delta_{\ell+1,\ell'}\delta_{m+1,m'}$$

$$+ \sqrt{\{(\ell-m+1)(\ell-m+2)\}}(1,\ i)\ \delta_{\ell+1,\ell'}\ \delta_{m-1,m'}] \qquad (A12)$$

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