King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 003

September 1980

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ELEMENTS OF THE SOCLE OF A SEMI-SIMPLE BANACH ALGEBRA

by

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The purpose of this paper is to give different characterizations of the elements of socA for a semi-simple Banach algebra A. In [1] J.C. Alexander proved that socA exists and $x \in$ socA if and only if the operator $a \to xax$ has finite rank. That is xAx is finite dimensional. On the other hand A.W. Tullo proved in [6] that the condition "A = socA" is equivalent to the finite dimensionality of A. Other conditions which are equivalent to finite dimensionality of A (and hence to the condition "A = socA") can be found in [2], [3] and [4]. These conditions include (i) Every element of A is algebraic (i.e. satisfies a polynomial identity) ([2] and [3]), (ii) The spectrum of every element in A is finite ([2] and [4]) and (iii) Every closed right ideal in A contains an idempotent and A contains only a finite set of orthogonal idempotents ([2]).

Following the general pattern of proofs in [2] and [4] we will attempt to characterize elements of socA. We will show that imposing conditions similar to (i), (ii) and (iii) above on the right ideal xA gives equivalences to the condition that socA exists and $x \in$ socA. Our results also give an alternative proof to Alexander's result.

Let A be an algebra over the field of complex numbers $\mathbb{C}$. We mean by an idempotent in A a non-zero element $e \in A$ such that $e^2 = e$. Two idempotents e and f are orthogonal if $ef = fe = 0$. The idempotent e is said to be minimal if eAe is a division algebra. We mention that if A is normed then in this case, the Gelfand-Mazur theorem implies that eAe consists of scalar multiples of e. Moreover, it is easy to check that e is a minimal idempotent if and only if eA(Ae) is a minimal right (left) ideal in A. It is also well known that if J is a minimal right ideal in A such that $J^2 \neq 0$ then there exists a minimal idempotent e such that J = eA [5; 2.15]. If A contains minimal right (left) ideals then their sum is called the right (left) socle of A. If the right and left socles exist and are equal, then the resulting two sided ideal is called the socle of A and is denoted by socA. The elements of socA are precisely the finite sums of elements from minimal right (left) ideals. By previous remarks one can easily see that if (0) is the only ideal in A whose square is (0) then socA exists if and only if A contains one-sided minimal ideals (right or left).

Our terminology is consistent with that of [5], and algebras considered are over the field of complex numbers $\mathbb{C}$. We recall that if A is an algebra and $x \in A$ then the spectrum of x in A (denoted by $\sigma(x)$) is the set $\{\lambda \in \mathbb{C} : \lambda - x$ is not invertible$\}$ if A has an identity. If A does not possess an identity then $\sigma(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0$ and $\frac{1}{\lambda} x$ is not quasiregular$\}$.

Let A be a semi-simple normed algebra and suppose that e and f are minimal idempotents in A. If $x \in A$ and $exf \neq 0$ then $exfA \neq 0$ by semi-simplicity. Therefore by minimality of eA we get $exfA = eA$ and hence $exfAf = eAf$. But $fAf = \mathbb{C}f$ by the Gelfand-Mazur theorem and therefore $eAf = \mathbb{C}\, exf$. Therefore eAf is either (0) or 1-dimensional. This observation appears as a lemma in [2] and other parts of the literature. It is used in the proofs of our theorems.

<u>THEOREM 1</u>. If A is a semi-simple Banach algebra and $x \in A$ then the following conditions are equivalent,

a) socA exists and $x \in$ socA.

b) The subalgebra xAx is finite dimensional.

c) The right ideal xA is algebraic of bounded degree.

d) For every closed right ideal J of A either $J \cap xA = (0)$ or $J \cap xA$ contains a nonzero idempotent. Moreover, xA contains only a finite number of orthogonal idempotents.

e) There exists orthogonal minimal idempotents $e_1, \cdots, e_d$ in xA such that $x = (\sum_{i=1}^{d} e_i)x$.

PROOF. If (a) holds there exist minimal idempotents $e_1, \cdots, e_n$; $f_1, \cdots, f_m$ and elements $x_1, \cdots, x_n$; $y_1, \cdots, y_m$ such that $x = \Sigma e_i x_i = \Sigma y_j f_j$. Therefore $xAx \subset \sum_{i,j=1,1}^{n,m} e_i A f_j$. But $e_i A f_j$ is either (0) or 1-dimensional. Therefore, xAx is finite dimensional.

Now suppose that (b) holds. Let d = dimension $(xAx) + 2$. If $y \in A$, then $xyx, (xy)^2x, \cdots, (xy)^{d-1}x$ are linearly dependent, since xAx has dimension $d - 2$. Therefore there exist scalars $\alpha_1, \cdots, \alpha_{d-1}$ not all zeros such that $\sum_{i=1}^{d-1} {}_i(xy)^i x = 0$. Hence multiplying on the right by y we get $\sum_{i=2}^{d} \alpha_{i-1}(xy)^i = 0$. Thus xy satisfies a polynomial identity of degree d. That is xA is algebraic of the bounded degree d.

Next assume that (c) holds. Let d be a positive integer such that every element of xA satisfies a polynomial identity of degree d. Then, by the spectral mapping theorem [5;1.6.10], every element of xA has at most d elements in its spectrum. (If $p(y) = 0$, then $p(\sigma(y)) = \sigma(p(y)) = \sigma(0) = \{0\}$. Therefore every $\lambda \in \sigma(y)$ is a root of p. If p is of degree d then p has at most d distinct roots.) Now, suppose that $J \cap xA \neq (0)$. Then, by semi-simplicity there is $y \in J \cap xA$ such that $\sigma(y) \neq (0)$. Let B be the subalgebra generated by y. Then $B \subset J \cap xA$. Since y is algebraic, then B is finite dimensional, therefore, closed. That is B is a commutative Banach algebra, and there is a 1 - 1 correspondence between the set of maximal modular ideals in B and the set of non-zero elements in $\sigma(y)$ which is finite. Thus B has a finite number of maximal modular ideals. It follows that if R is the Jacobson radical of B, then B/R is the direct sum of finite number of copies of the field of complex numbers. Therefore, B/R contains idempotents which can be lifted to B [5;2.3.9]. Thus B contains idempotents. But $B \subset J \cap xA$. Hence, $J \cap xA$

contains idempotents.

This proves the first assertion of (d). On the other hand if xA contains non-zero orthogonal idempotents $e_1, \ldots, e_n$ with $n > d$, then setting $y = \sum_{k=1}^{n} \frac{1}{k} e_k$ we get $y \in xA$ and $ye_k = \frac{1}{k} e_k$. That is $(\frac{1}{k} - y)e_k = 0$. This implies that $\frac{1}{k} \in \sigma(y)$. Therefore, $\sigma(y)$ contains at least n elements, namely $1, \frac{1}{2}, \ldots, \frac{1}{n}$. This contradicts the fact that $\sigma(y)$ contains at most d elements. Therefore, xA contains at most d orthogonal idempotents, which concludes (d).

Finally, let (d) hold. Let $\{e_1, \ldots, e_d\}$ be a set of pairwise orthogonal idempotents of maximal cardinality in xA. We first observe that e_i is a minimal idempotent for each $i = 1, \ldots, n$. For if this is not the case, say e_1 is not minimal, then e_1A is not a minimal ideal. Hence, e_1A is not a minimal closed ideal [5;2.1.20] and hence it properly contains a closed right ideal $J \neq (0)$. Then $J = J \cap xA$ contains an idempotent f (necessarily different from e_1). Now $f \in e_1A$ implies $f = e_1f$. Moreover, since $f^2 = f \neq 0$ and $e_1A \neq fA$ we have $fe_1 \neq 0 \neq e_1 - fe_1$. Hence $fe_1, e_1 - fe_1, e_2, \ldots, e_d$ is a pairwise orthogonal family of idempotents which contradicts the maximality of d.

Now, let $f = e_1 + \cdots + e_d$. Then f is an idempotent and hence $(1 - f)A$ is a closed ideal in A, (note that no identity is needed to define $(1 - f)A$). We have $(1 - f)A \cap xA = (1 - f)xA$ since $f \in xA$. If $(1 - f)xA \neq (0)$. Then by the assumption of (d) $(1 - f)xA$ contains an idempotent $g \neq 0$. We have $g = (1 - f)xy$ for some $y \in A$. Let $h = g(1 - f) = (1 - f)xy(1 - f)$. Then $h^2 = (1 - f)xy(1 - f)xy(1 - f)$ $= g^2(1 - f) = g(1 - f) = h$. Hence, h is an idempotent. Moreover, h is orthogonal to e_i for $i = 1, \cdots, d$. Therefore, by the maximality of d we get $h = 0$, which implies that $g = g^2 = g(1 - f)xy = hxy = 0$ which is a contradiction. Hence, our assumption that $(1 - f)xA \neq (0)$ is false and we have $(1 - f)xA = (0)$. Therefore, by semi-simplicity, $(1 - f)x = 0$, i.e. $x = fx = (\sum_{i=1}^{d} e_i)x$. This proves (e). Since it is trivial that (e) implies (a), this concludes the proof of the theorem.

<u>REMARKS</u>. 1) Let H be a separable Hilbert space and $B(H)$ the algebra of bounded operators on H. Then $\mathrm{soc}B(H)$ consists of the operators on H with finite rank. We observe that condition (e) of theorem 1 in this case says that if $T \in \mathrm{soc}B(H)$ then T can be

represented as an infinite matrix with only a finite number of non-zero rows.

2) In part (b) of theorem 1 the condition about xAx can be relaxed by replacing it with the condition that there is a positive integer n such that $(xA)^n x$ is finite dimensional. This is evident in the proof of "(b) implies (c)" as we can take d in this case to be dimension $(xA)^n x + 2$. Then for $y \in A$ we have $(xy)^n x$, $(xy)^{n+1} x, \cdots$, $(xy)^{n+d-2}$ are linearly dependent. The rest of the proof now proceeds as before. Nevertheless, theorem 1 does not hold if the ideal xA was replaced by the subalgebra xAx in the statement of the theorem. For example if H is a separable Hilbert space with orthonormal basis $\{x_1, x_2, \cdots\}$ and $A = B(H)$. Let $T \in A$ be defined by $Tx_{2i-1} = x_{2i}$ and $Tx_{2i} = 0$. Then $T^2 = 0$, so TAT satisfies the identity $\lambda^2 = 0$. Therefore, TAT satisfies the condition stated about xA in theorem 1 (c). Nevertheless, as T is not of finite rank it does not belong to socA.

We now use condition (c) of the theorem to show that $x \in \text{soc}A$ implies that xA is closed. This helps to relax some of the conditions in theorem 1.

THEOREM 2. If A is a semi-simple Banach algebra and $x \in A$ then the following conditions are equivalent

a) $\text{soc}A$ exists and $x \in \text{soc}A$.

b) The right ideal xA is closed and algebraic.

c) The right ideal xA is closed and $\sigma(y)$ is finite for every $y \in xA$.

PROOF. Let (a) hold. Then, by condition (c) of theorem 1, there exist orthogonal minimal idempotents $e_1, \cdots, e_n$ in xA such that $x = (\sum_{i=1}^{n} e_i)x$. Now let (xy_n) be a sequence in xA converging to y. Then, $xy_n = (\Sigma e_i)xy_n$. Therefore, $(\Sigma e_i)y = (\Sigma e_i)\lim_n xy_n = \lim_n (\Sigma e_i)xy_n = \lim_n xy_n = y$. But $e_i \in xA$ and therefore $y = (\Sigma e_i)y \in xA$. This says that xA is closed. Condition (c) of theorem 1 implies that xA is algebraic, and hence (b) follows.

If (b) holds, then as in the proof of theorem 1, the spectral mapping theorem implies that every element of xA has a finite spectrum. Therefore (c) holds.

Next, let (c) hold. We will show that (d) of theorem 1 holds which is equivalent to (a). Let J be a closed right ideal. If $J \cap xA \neq (0)$, then $J \cap xA$ is a non-zero closed right ideal, and by semi-simplicity it contains an element y with non-zero spectrum. Moreover, by condition (c), $\sigma(y)$ is finite. If B is the closed subalgebra generated by y, then $B \subset J \cap xA$ since $J \cap xA$ is closed. Now, the same argument as in the proof of "(c) implies (d)" in theorem 1 applies and we conclude that $J \cap xA$ contains an idempotent. Moreover, if $\{e_1, e_2, \cdots\}$ is an infinite family of pairwise orthogonal idempotents in xA, we can choose $\{\lambda_i\}_{i=1}^{\infty}$ distinct numbers such that $|\lambda_i| \, \|e_i\| < 2^{-i}$. Then, since xA is closed, $\Sigma\lambda_i e_i$ converges to an element $y \in xA$. Since $ye_i = \lambda_i e_i$ for each i we have $\lambda_i \in \sigma(y)$ for $i = 1, 2, \cdots$. This contradicts the assumption that $\sigma(y)$ is finite. Hence xA contains only a finite number of orthogonal idempotents. Therefore, (d) of theorem 1 holds which condlues the proof.

We mention that in the example of remark (2) following the proof of theorem 1, TAT is closed (the reader can varify that). Therefore TAT

satisfies condition (b) and (c) which are stated about xA in theorem 2. Nevertheless, as seen before, $T \not\subset \mathrm{soc}B(H)$.

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