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A Commutativity Theorem for Rings with Involution

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ABSTRACT

Let R be a 2-torsion-free ring with involution which contains a non-zero-divisor central skew element s. Suppose that R satisfies the condition that if $h \in R$ is self adjoint with $hRh = 0$ then $h = 0$. It is shown that if either (i) $[hk, kh] = 0$ for all self adjoint h and k or (ii) $1 - s^2$ is not a zero-divisor and $[xx^*, x^*x] = 0$ for all $x \in R$, then R is commutative.

In the last half century, a good deal of research has been done on commutativity theorems for rings. Many results have confirmed that, under a suitable hypothesis, the imposition of certain commutation

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relations on the elements of a ring forces it to be commutative. Theorems of this type can be found in [2] and [4].

Recently there have been several attempts to transfer these theorems to the setting of rings with involution. The idea is to impose the general commutation relations on the set of self adjoint elements and investigate the effect on the whole ring. Such an imposition does not, usually, yield commutativity of the ring. Nevertheless, it does indeed make the ring special in its structure. A good survey of results of this type can be found in [3].

In this paper we consider a theorem of R. Gupta which states that if in a division ring D the identity $xy^2x = yx^2y$ holds for all $x, y \in D$, then D is commutative [1]. We will show that, under a suitable hypothesis, if the identity above is satisfied by the self adjoint elements of a ring with involution R, then R is commutative. With some added hypothesis the commutativity of R also can be forced by the condition $xx^{*2}x = x^*x^2x^*$ for all $x \in R$. In specific we will prove the following two theorems.

<u>Theorem 1</u>. Let R be a 2-torsion-free ring with involution which

contains a non-zero-divisor central skew element s. Suppose that R satisfies the condition that if $h \in R$ is self adjoint with $hRh = 0$ then $h = 0$. If $hk^2h = kh^2k$ for all self adjoints h and k in R, then R is commutative.

<u>Theorem 2</u>. Let R be a 2-torsion-free ring with involution containing a non-zero-divisor central skew element s such that $1 - s^2$ is not a zero-divisor also. Suppose that if h is self adjoint in R and $hRh = 0$ then $h = 0$. If $xx^{*2}x = x^*x^2x^*$ for all $x \in R$ then R is commutative.

We mention that in the statement of theorem 2, we mean by $1 - s^2$ is not a zero-divisor that if $x - xs^2 = 0$ then $x = 0$. That is we are not assuming the existence of an identity in R.

Before we proceed with the proofs we recall some basic definitions and state some notations.

Let R be a ring. By an involution on R we mean a map $*$ of R onto R which assigns to each element x an element x^* with the properties (i) $x^{**} = x$, (ii) $(x + y)^* = x^* + y^*$ and (iii) $(xy)^* = y^*x^*$ for all $x, y \in R$. The element $x \in R$ is said to be self adjoint if $x = x^*$ and is said to be skew if $x = -x^*$. We will denote the sets of self adjoint elements and skew elements by H and S respectively. If R is a ring with involution we will use the shorter expression "R is a * ring". The standard notation $[x, y]$ will be used oftenly in place of $xy - yx$.

Before proving our theorems we need four lemmas. The first two of these lemmas are well-known and easy. Thus we omit their proofs.

Lemma 1. Given a ring R and elements $x, y, z \in R$ we have $[x + y, z] = [x, z] + [y, z]$.

Lemma 2. If R is a ring and $x, y \in R$ satisfy $[x, [x, y]] = 0$ then $[x, [x, y^2]] = 2[x, y]^2$.

Lemma 3. Let R be a 2-torsion-free * ring. Suppose that there exists in R a non-zero-divisor central $s \in S$. If $hk = kh$ for all $h,k \in H$, then R is commutative.

Proof. Let $x \in R$ and let $h \in H$. We have $x + x^* \in H$ and since $s, x - x^* \in S$ and s is central we have $s(x - x^*) \in H$. Therefore $h(x + x^*) = (x + x^*)h$ and $sh(x - x^*) = s(x - x^*)h$. Since s is not a zero-divisor, we get $h(x - x^*) = (x - x^*)h$. Therefore $2hx = 2xh$ which implies that $hx = xh$ since R is 2-torsion-free.

Now if $x, y \in R$ are arbitrary, then $x + x^*$, $s(x - x^*) \in H$. Therefore, by the above we have y commutes with $x + x^*$ and $s(x - x^*)$ Repeating the above argument with y in place of h we obtain $xy = yx$.

<u>Lemma 4</u>. Let R be a 2-torsion-free * ring with a central non-zero-divisor $s \in S$. Suppose that $[hk, kh] = 0$ for all $h, k \in H$. If $h \in H$ satisfies $h^2 = 0$ then $hRh = 0$.

<u>Proof</u>. Let $k \in H$. Then $hk^2h = [hk, kh] = 0$. We have $s^2 + k \in H$, therefore $2s^2hkh = [h(s^2 + k), (s^2 + k)h] = 0$. Since R is 2-torsion-free and s is not a zero-divisor, this implies that $hkh = 0$. Since $k \in H$ was arbitrary we get $hHh = 0$.

Now let $x \in R$, then $x + x^*$, $s(x - x^*) \in H$. Therefore, $h(x + x^*)h = 0$ and $sh(x - x^*)h = 0$. The latter equality implies that $h(x - x^*)h = 0$. Adding the two equalities we get $2hxh = 0$, and hence $hxh = 0$. Since x was arbitrary we get $hRh = 0$.

<u>Proof of Theorem 1</u>. In view of lemma 3, it is enough to show that

$hk = kh$ for all $h, k \in H$. Moreover, by lemma 4, we may assume that if $h \in H$ satisfies $h^2 = 0$, then $h = 0$. Since $st \in H$ for every $t \in S$ and s is not a zero-divisor, we also have $t^2 = 0$ where $t \in S$ implies $t = 0$.

Let $h, k \in H$. Then by assumption we have,

$$hk^2h = kh^2k \tag{1}$$

Since $h + k \in H$ and (1) holds for all elements of H, we can replace k with $h + k$ in (1) to obtain $h^2kh + hkh^2 = h^3k + kh^3$, that is

$$[h^2, [h, k]] = 0 \tag{2}$$

Since $[h^2, k] = h[h, k] + [h, k]h$ and h^2 commutes with $[h, k]$ by (2), we get $[h^2, [h^2, k]] = 0$. Replacing k with k^2 in the last identity (which holds for all elements of H) we get $[h^2, [h^2, k^2]] = 0$. Therefore, by lemma 2, $2[h^2, k]^2 = [h^2, [h^2, k^2]] = 0$. Since $[h^2, k] \in S$, this implies that

$$[h^2, k] = 0 \tag{3}$$

Now replacing k with $h^2 + k$ in (1) we obtain $[h^3, [h, k]] = 0$

which implies that $[h^3, [h^3, k]] = 0$ since $[h^3, k] = h^2[h, k] + h[h, k]h + [h, k]h^2$. Replacing k with k^2, $[h^3, [h^3, k^2]] = 0$. Hence, by lemma 2, $2[h^3, k]^2 = [h^3, [h^3, k^2]] = 0$. Therefore, since $[h^3, k] \in S$, we have

$$[h^3, k] = 0 \tag{4}$$

Using (3) and (4) we have $(hkh - h^2k)^2 = 0$. But $hkh - h^2k \in H$ (since $h^2k = kh^2$). Hence $hkh - h^2k = 0$, that is $hkh = h^2k = kh^2$. Replacing k with k^2, we have $hk^2h = h^2k^2 = k^2h^2$. Therefore, $(hk - kh)^2 = 0$, and since $hk - kh \in S$, this implies that $hk = kh$. Since $h, k \in H$ were arbitrary, this concludes the proof.

Proof of Theorem 2. Let $h, k \in H$. Let $x = h + sk$. Then $x^* = h - sk$. Let $a = h^2 - s^2k^2$ and $c = hk - kh$. Then $xx^* = a - sc$ and $x^*x = a + sc$. Hence by assumption we have $[a - sc, a + sc] = 0$, which simplifies to $2s[c, a] = 0$. Therefore,

$$[h^2 - s^2k^2, c] = [a, c] = 0 \tag{1}$$

Replacing h with $h + sk$ in (1), we get

$$[h^2 + shk + skh, c] = 0 \tag{2}$$

Exchanging the roles of h and k in (2) we have,

$$[k^2 + shk + skh, c] = 0 \qquad (3)$$

Multiplying (3) by s^2 we get,

$$[s^2k^2 + s^3hk + s^3kh, c] = 0 \qquad (4)$$

Subtracting (4) from (2), applying lemma 1 and using identity (1) we obtain $[shk + skh - s^3hk - s^3kh, c] = 0$. Using lemma 1 and the fact that s is central and not a zero-divisor, this simplifies to $[hk + kh, c] - s^2[hk, kh, c] = 0$. Therefore, since $1 - s^2$ is not a zero-divisor, we have $[hk + kh, hk - kh] = [hk + kh, c] = 0$, which simplifies to $2[hk, kh] = 0$. Hence $[hk, kh] = 0$. Since $h, k \in H$ were arbitrary the conclusion follows by theorem 1.

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