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Finitely Cogenerated Modules

Syed M. Fakhruddin

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by

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Introduction

Let R be a ring and M a left R-module. M is said to be finitely cogenerated if it has a finitely generated essential socle. This concept first appeared in [7]. Then Vamos [9] showed that this is dual to the usual notion of finitely generated in many respects. Related ideas were also expounded in [4]. L. Fuchs [3] treats in detail finitely cogenorated abelian groups.

The main observation we make here is that a finitely cogenerated module over a commutative conoetherian ring (defined below) is pure-injective. This fact is then used to prove a structure theorem for pure injective modules over these rings.

We use the customary notations. All rings are associative and possess identity. All modules are left unitary.

§1. Finitely Cogenerated Modules and Conoetherian Rings

Let R be a ring and M a left R-module. M is said to be *finitely cogenerated* if it had a finitely generated essential socle. Hence it follows that in order for a module M to be finitely cogenerated it is necessary and sufficient that M be a submodule of a finite product of injective envelopes of simple R-modules. We can also characterise M by means of a sort of artinian property: namely M is finitely cogenerated precisely when every decreasing chain of its non-zero submodules is bounded below by a non-zero submodule ([4] and [9]).

A ring R is *conoetherian* if every finitely cogenerated R-module is artinian. Clearly then R is conoetherian if and only if the injective envelope of every simple R-module is artinian [4]. Another equivalent condition on R in order that it be conoetherian is that the class of finitely cogenerated R-modules be closed for the formation of quotients. This is due to the fact that a finitely cogenerated module M is artinian exactly if all its quotient modules are finitely cogenerated [9].

An R-module M is *linearly compact* if every filter base of cosets of submodules of M has non-void intersection. Clearly an artinian R-module is linearly compact. Hence we have

Proposition 1. A finitely cogenerated module M over a conoetherian ring is linearly compact.

2. Pure-Exact Sequences and Pure-Injective Modules

A short exact sequence $\mathcal{E}: 0 \to M \to N \to P \to 0$ of R-modules is said to be pure-exact if $\mathcal{E} \otimes X$ is a short exact sequence of abelian groups for any right R-module X. The tensor product is formed over the base ring R. Since the tensor product commutes with inductive limit, the right R-module X above can be assumed to be finitely presented.

A module M is pure-injective if $\mathrm{Hom}(\mathcal{E}, M)$ is a short exact sequence of abelian groups for any pure-exact sequence . The class of pure-injective modules is closed for product and summand. The structure of pure-injective modules over a commutative ring is given in [2]: a module over a commutative ring is pure-injective when and only when it is a factor of a product of duals of finitely presented modules, where the duals are formed with respect to the circle group.

A linearly compact module over a commutative ring is pure-injective

[10]. Hence we have in conjunction with proposition 1,

Proposition 2. A finitely cogenerated module over a commutative conoetherians ring is pure-injective.

§3. Finitely Cogenerated Quotients

We shall consider quotients of an arbitrary module which are finitely cogenerated. It happens that an arbitrary module has sufficient number of quotients which are finitely cogenerated.

Proposition 3. Let R be a ring M an R-module, N a submodule of M and $x_1, \ldots, x_m \in M - N$. Then there exists a submodule P of M containing N and not containing any of the x_i such that M/P is finitely cogenerated.

Proof. Let us choose, by Zorn's lemma, a submodule P of M not containing any of the x_i and maximal among the submodules with these properties. Now any decreasing chain of submodules of M containing P, whose intersection is P must have P as one of the members. Thus M/P is finitely cogenerated.

Thus in particular for every non-zero element x in M, there exists (at least) a submodule P not containing x such that M/P is finitely cogenerated. Clearly M/P is an essential extension of $\frac{P + Rx}{P}$ which is a simple module. Let us call such a module cocyclic. In order that the module M/P be cocyclic, it is necessary and sufficient that P be completely irreducible in M; in other words there are no submodules N and Q of M properly containing P with $N \cap Q = P$. Thus a module is cocyclic exactly when its zero submodule is completely irreducible.

Moreover, if M is an arbitrary non-cocyclic module and $\{P_\alpha\}$ the family of completely irreducible submodule of M, then $P_\alpha \neq 0$, however $\cap P_\alpha = (0)$.

If $f : M \to N$ is an epimorphism then there is a one-one inclusion preserving correspondence between the completely irreducible submodules of M containing the kernel of f and the family of completely irreducible submodules of N.

Hereafter by an arbitrary module M, we shall mean a module which is not cocyclic.

The above discussion together with the usual properties of localisation and completion lead us to the direct verification of the following.

<u>Corollary 5</u>. (1) Let R be commutative ring M an arbitrary module, β an arbitrary prime ideal of R and let $\{P_\alpha\}$ denote the family of non-zero completely irreducible submodules of M. Then $\cap\{(P_\alpha)_\beta \mid (P_\alpha)_\beta \neq (0)\}$ is the family of nonzero completely irreducible submodules of M_β.

(2) Now we suppose R is a commutative local ring with a maximal ideal and $(\hat{\ })$ denote the m-adic completion process. Then $\cap\{\hat{P}_\alpha \mid \hat{P}_\alpha \neq (0)\}$ is the family of non-zero completely irreducible submodules of $\hat{M}$.

Let us note also that in the second case we do not need the restriction $\hat{P}_\alpha \neq 0$, since the functor $(\hat{\ })$ is faithfully flat.

§4. A Structure Theorem for Pure-Injective Modules

The following theorem is crucial to the proof of our final theorem.

<u>Theorem 1</u>. Let R be a commutative conoetherian ring, M an arbitrary R-module $\{P_\alpha\}$ be the family of non-zero completely irreducible submodules of M and N a finitely generated R-module.

Let P^*_α denote the image of $P_\alpha \otimes N$ in $M \otimes N$. Then $\cap\{P^*_\alpha \mid P^*_\alpha \neq (0)$ in $M \otimes N\} = (0)$

<u>Proof</u>. By Corollary 5 above we may suppose that R is a complete local ring and M is a complete R-module in its m-adic topology. A localisation of a commutative conoetherian ring is noetherian (theorem 2 in [9]).

Now N is a finitely generated R-module over a noetherian local ring. So we may suppose that theorem is true for all proper homomorphic images of N.

Given a non-zero element x of $M \otimes N$, in order to prove the theorem, it is enough to find a submodule P_α of the family such that $x \notin P^*_\alpha \subseteq M \otimes N$ and $P^*_\alpha \neq (0)$.

There exists a submodule Q of N such that the image of x is non-zero in $M \otimes N/Q$. If not, the image of x would be zero in the module $M \otimes N/m^n N$ for every n and consequently in the module $\varprojlim M \otimes N/m^n N$. But $\varprojlim M \otimes N/m^n N$ is the completion of $M \otimes N$ ([1] Chap.3 exercise §3.28(d)) for the m-adic topology. Since M is a complete module and N is finitely generated $M \otimes N$ is itself complete in the m-adic topology. But then $x = 0$ in $M \otimes N = M \otimes N$ which is impossible.

Hence let Q be the submodule of N such that the image $\bar{x}$ of x is not zero in the module $M \otimes N/Q$.

The theorem is true for the module N/Q, by hypothesis, there exists a submodule P_α of the family such that the image of $P_\alpha \otimes N/Q$ is not zero in $M \otimes N/Q$ and $\bar{x}$ is not an element of this image.

Then $x \notin \text{image}(P_\alpha \otimes N)$ in $M \otimes N$ and image $(P_\alpha \otimes N)$ is not zero in $M \otimes N$, because of the following commutative diagram.

$$\begin{array}{ccc} P_\alpha \otimes N & \longrightarrow & P_\alpha \otimes N/Q \\ \downarrow & & \downarrow \\ M \otimes N & \longrightarrow & M \otimes N/Q \end{array}$$

<u>Theorem 2</u>. Let R be a commutative conoetherian ring, M an arbitrary module. Then M is a pure submodule of a product of cocyclic modules.

<u>Proof</u>. Let $\{M \xrightarrow{f_\alpha} M_\alpha\}$ be the family of all epimorphisms such that P_α = kernel f_α are completely irreducible in M. Notice since M is arbitrary $P_\alpha \neq (0)$ for any α. But $\cap P_\alpha = (0)$, so that $M \xrightarrow{\Pi f_\alpha} \Pi M_\alpha$ is a monomorphism.

Let P be a finitely presented module. Consider the following diagram, which is commutative as can be verified.

$$\begin{array}{ccc}
P \otimes M & \xrightarrow{\quad id_P \otimes \Pi f_\alpha \quad} & P \otimes (\Pi M_\alpha) \\
\Big\| & & \Big\downarrow \Pi(id_P \otimes pr_\alpha) \\
P \otimes M & \xrightarrow{\quad \Pi(id_P \otimes f_\alpha) \quad} & \Pi(P \otimes M_\alpha)
\end{array}$$

where pr_α is the canonical projection $\Pi M_\alpha \longrightarrow M_\alpha$. By theorem 1, the map below is a monomorphism. By a theorem of Lenzing ([5]), since P is finitely presented, the vertical arrow is an isomorphism. Hence the top arrow is a monomorphism, which proves that $M \longrightarrow \Pi M_\alpha$ is a pure monomorphism.

<u>Theorem 3.</u> (The Structure Theorem). Let R be a commutative conoetherian ring and M an R-module. M is pure-injective exactly if M is a summand of a product of cocyclic modules.

Proof follows by proposition 2 and theorem 2 and the fact that products and summands of pure-injective modules are pure-injective.

§5. Conclusion

Theorem 3 is suggested by and is a generalization of theorem 30.4 in [2] for abelian groups.

The proposition 2 (hence theorem 3) is the best possible. For there exists a cocyclic module on a non-noetherian valuation ring (hence a non-conoetherian ring) which is not pure-injective.

Let k be a field and $\underset{\sim}{Q}$ the field of rationals. Let us consider $K = k(X^{\underset{\sim}{Q}})$, the field of Puiseux series in an indeterminate X, whose exponents are rational numbers and whose support is a well-ordered set in the natural order of $\underset{\sim}{Q}$.

Addition on K is coordinatewise, whereas multiplication is by convolution.

It is well known that K is an almost-maximal field with respect to the valuation $v(f) = \inf\{supp(f)\}$ for $f \in K$. Let (R, m) be the corresponding valuation ring. By theorem 4 of [6] we have $K/m \simeq E(R/m) = E(k)$ where E stands for injective envelope. K/m can be viewed upto isomorphism as the module of "truncated" Puiseux series with well ordered support having only non-negative natural exponents.

In (R, m) every finitely generated ideal is principal. Similarly in K/m every finitely generated submodule of K/m is cyclic. Also by theorem 2[8] every quotient of K/m is absolutely pure, since R is semiheriditary.

Let us consider a quotient M of K/m by a non-cyclic submodule. Then M can be viewed upto isomorphism as the module of "truncated" Puiseux series with well ordered support having rational exponents $\leq b$. This is a cocyclic module with socle generated by X^b and isomorphic to k.

This module cannot be injective, if it were it would be isomorphic to K/m, both being injective envelope of the only simple module k. Further such an isomorphism should take socle into socle. There is only one such map, which takes $\bar{T}$ to X^b, yet this is only additive and not R-linear.

Hence M cannot be pure-injective, since pure-injective absolutely pure modules are injective.

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University of Petroleum & Minerals
Dhahran, Saudi Arabia.