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Reciprocity Relations between Transmission Co-Efficients in Love Wave Propagation

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*RECIPROCITY RELATIONS BETWEEN TRANSMISSION CO-EFFICIENTS IN LOVE WAVE PROPAGATION

by

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1. Introduction: In our work (see Kazi 1979) on the problem of the propagation of Love waves normally incident (from either side) upon the vertical plane of discontinuity in a 2-layer model consiting of an infinite strip with a surface step, we described diffration of plane, harmonic monochromatic Love waves by means of a scattering matrix through an integral equation formulation. Approximate expressions for elements of the scattering matrix were obtained through the plane-wave approximation and their variational improvement was sought through the Schwinger-Levine variational principle. Complex reflection and transmission co-efficients were obtained (under both approximations) through a transmission matrix related to the scattering matrix. The formulae for transmission co-efficients for the special cases under both approximations suggested a kind of reciprocity between the transmission co-efficients when the incident wave is travelling from left to right and the transmission co-efficients when the incident wave is travelling from right to left. The purpose of this paper is to prove that this reciprocity is a consequence of the dynamic Betti-Rayleigh theorem (Achenbach 1973) and does not depend upon the nature of the approximations used. It may be mentioned that a useful reciprocity relation for surface waves in a laterally homogeneous half-space has been obtained by Douglas et al., 1971.

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2. EQUATIONS OF MOTION

We consider a surface layer of rigidity μ_1, shear velocity β_1 density ρ_1 and variable thickness h_1, h_2 ($> h_1$), overlying a layer of rigidity $\mu_2 (> \mu_1)$, shear velocity β_2 ($> \beta_1$), density ρ_2 and thickness $H - h_1$. Both the layers are assumed to be homogeneous and isotropic. Co-ordinate axes are chosen in such a way that the z-axis is vertically downward, the interface is given by $z = h_1$, and the step in the surface of the upper layer is taken to lie in the plane $x = 0$ (see fig. 1). The thickness of the upper layer is taken to be h_1 for $x < 0$ and h_2 for $x > 0$, and we write $\delta = h_2 - h_1$.

We consider only the two-dimensional problems of propagation of Love waves, normally incident (from either side) upon the step, and pose the time-dependence e^{-iwt}, w being the angular frequency. Thus the wave motion is entirely SH in character. The displacement fields in domains I $(x < 0)$ and II $(x > 0)$ are denoted by

$$\left.\begin{aligned} e^{-iwt}\, v(x,z) &= e^{-iwt}\, v_1(x,z),\ 0 \le z \le h_1,\ x < 0, \\ &= e^{-iwt}\, v_2(x,z),\ h_1 \le z \le H,\ x < 0 \end{aligned}\right\}$$

and

$$\begin{aligned} e^{-iwt}\, v'(x,z) &= e^{-iwt}\, v_1'(x,z),\ -\delta \le z \le h_1,\ x > 0, \\ &= e^{-iwt}\, v_2'(x,z),\ h_1 \le z \le H,\ \ x > 0, \end{aligned}$$

respectively.

The surfaces $z = 0,\ x < 0,\ z = -\delta = h_1 - h_2,\ x > 0,\ z = H\ (\forall x)$ the vertical surface of the step are stress free. Thus

$$\frac{\partial v_1}{\partial z} = 0 \quad \text{at} \quad z = 0,\ x < 0, \qquad \text{-------} \qquad 1(a)$$

$$\frac{\partial v_1'}{\partial z} = 0 \quad \text{at} \quad z = -\delta,\ x > 0, \qquad \text{-------} \qquad 1(b)$$

$$\frac{\partial v_2}{\partial z} = 0 \quad (x < 0),\ \frac{\partial v_2'}{\partial z} = 0\ (x > 0) \quad \text{at} \quad z = H \ \text{---} \qquad 1(c)$$

and

$$\frac{\partial v_1'}{\partial x} = 0 \quad \text{at} \quad x = 0,\ -\delta \le z < 0\,. \qquad (2)$$

The complete solution for the displacement $v(x, z)$ in domain I $(x < 0)$ can be expressed in terms of the complete set of eigenfunctions (see Kazi (1976)) associated with the Love wave operator for an infinite strip consisting of a layer of depth $H - h_1$, rigidity μ_2, shear velocity β_2, overlain by a layer of depth h_1, rigidity μ_1 and shear velocity β_1. Likewise, the complete solution for the displacement $v'(x, z)$ in domain II $(x > 0)$ can be expressed in terms of the eigenfunctions for an infinite strip, consisting of a layer of depth $H - h_1$, rigidity μ_2 and shear velocity β_2, overlain by a layer of depth h_2, rigidity μ_1 and shear velocity β_1. Thus in domain I

$$v(x,z) = -\{\sum_{m=1}^{r} (A_n\, e^{-ik_m|x|} + B_m\, e^{ik_m|x|})\, X_m(z)$$
$$+ \sum_{j=1}^{\infty} C(k_j)\, e^{-k_j|x|}\, \psi(z, k)\}\ ,\ \ x < 0 \qquad \text{---}\ (3)$$

and in domain II

$$v'(x,z) = \{\sum_{m=1}^{s} (A'_m e^{-ik'_m x} + B'_m e^{ik'_m x}) X'_m(z)$$

$$+ \sum_{j=1}^{\infty} C'(k'_j) e^{-k'_j x} \psi'(z,k')\}, \; x > 0, \qquad --- \quad (4)$$

where (as in Kazi (1976))

$$X_m(z) = \phi_1^{(m)}(z), \quad 0 \leq z \leq h_1,$$

$$= \phi_2^{(m)}(z), \quad h_1 \leq z \leq H,$$

$$X'_m(z) = \phi_1'^{(m)}(z), \quad -\delta \leq z \leq h_1$$

$$= \phi_2'^{(m)}(z), \quad h_1 \leq z \leq H,$$

$$\phi_1^{(m)}(z) = D_m \frac{\mathrm{Cos}(\sigma_1^{(m)} z)}{\mathrm{Cos}(\sigma_1^{(m)} h_1)} \qquad --- \quad (5)$$

$$\phi_2^{(m)}(z) = D_m \frac{\mathrm{Cosh}\ \{\sigma_2^{(m)}(z-H)\}}{\mathrm{Cosh}\ \{\sigma_2^{(m)}(H-h_1)\}} \qquad --- \quad (6)$$

$$D_m = 2\{\frac{\sigma_2^{(m)}}{\mu_2}\}^{\frac{1}{2}}\{\frac{\beta_1^{-2} - U_m^{-1}\ C_m^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\}^{\frac{1}{2}} \frac{\mathrm{Cosh}\ \{\sigma_2^{(m)}(H-h_1)\}}{\{\mathrm{Sinh} 2\sigma_2^{(m)}(H-h_1) + 2\sigma_2^{(m)}(H-h_1)\}^{\frac{1}{2}}}$$

$$\phi_1'^{(m)}(z) = D'_m \frac{\mathrm{Cos}\ \{\sigma_1'^{(m)}(z+\delta)\}}{\mathrm{Cos}\ \{\sigma_1^{(m)\prime} h_2\}} \qquad --- \quad (7)$$

$$\phi_2'^{(m)}(z) = D'_m \frac{\mathrm{Cosh}\{\sigma_2'^{(m)}(z-H)}{\mathrm{Cosh}\ \{\sigma_2'^{(m)}(H-h_1)\}} \qquad --- \quad (8)$$

$$D_m' = 2\left\{\frac{\sigma_2'^{(m)}}{\mu_2}\right\}^{\frac{1}{2}} \left\{\frac{\beta^{-2} - U_m'^{-1}\, C_m'^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right\}^{\frac{1}{2}} \frac{\mathrm{Cosh}\{\sigma_2'^{(m)}(H-h_1)\}}{\{\mathrm{Sinh}(2\sigma_2'^{(m)}(H-h_1)+2\sigma_2'^{(m)}(H-h_1))\}^{\frac{1}{2}}}$$

U_m, U_m' denote the group velocities and C_m, C_m' the phase velocities in the two m-th modes, and

$$\sigma_1(\lambda) = \left(\frac{w^2}{\beta_1^2} - \lambda\right)^{\frac{1}{2}}, \quad \sigma_2(\lambda) = \left(\lambda - \frac{w^2}{\beta_2^2}\right)^{\frac{1}{2}}, \quad \lambda = k^2 ,$$

$$\sigma_1^{(m)} = \sigma_1(\lambda_m), \quad \sigma_2^{(m)} = \sigma_2(\lambda_m),$$

$\lambda_m = k_m^2$, $k_m > 0$, $m = 1, 2, \text{---}\ r$, λ_m being the r positive real eigenvalues satisfying the dispersion equation

$$\mu_1\sigma_1 \tan\sigma_1 h_1 - \mu_2\sigma_2 \tanh \sigma_2(H-h_1) = 0. \qquad \text{---} \quad (9)$$

In $x > 0$,

$$\sigma_1'(\lambda') = \left(\frac{w^2}{\beta_1^2} - \lambda'\right)^{\frac{1}{2}}, \quad \sigma_2'(\lambda') = \left(\lambda' - \frac{w^2}{\beta_2^2}\right)^{\frac{1}{2}}, \quad \lambda' = k'^2,$$

$$\sigma_1'^{(m)} = \sigma_1(\lambda_m'), \quad \sigma_2'^{(m)} = \sigma_2(\lambda_m'),$$

$$\lambda_m' = k_m'^2, \quad k_m' > 0, \quad m = 1, 2, \text{---}\ s,$$

λ_m' being the s real positive eigenvalues satisfying the period equation

$$\mu_1\sigma_1' \tan\sigma_1' h_2 - \mu_2\sigma_2 \tanh\sigma_2'(H-h_1) = 0 \qquad \text{---} \quad (10)$$

In addition to the aforementioned roots, the eigenvalue equations (9) and (10) have infinite, discrete sets $\{\lambda_j\}$, $\{\lambda_j'\}$ of negative real roots, respectively; $\lambda_j = (ik_j)^2$, and $\lambda_j' = (ik_j')^2$, $j = 1,2,\text{---}$, k_j and k_j' being real and positive. The eigenfunctions corresponding to these eigenvalues are given by

$$\psi(z, k_j) = \psi_1(z, k_j), \quad 0 \leq z \leq h_1 ,$$
$$= \psi_2(z, k_j), \quad h_1 \leq z \leq H ,$$

and

$$\left.\begin{aligned} \psi'(z,k_j') &= \psi_1'(z,k_j'), \quad -\delta \leq z \leq h_1 \\ &= \psi_2'(z,k_j'), \quad h_1 \leq z \leq H \end{aligned}\right\}$$

$j = 1, 2, ---$, where $\psi_1(z,k_j)$ and $\psi_2(z,k_j)$ have expressions similar to (5) & (6) with $\sigma_1^{(m)}, \sigma_2^{(m)}$ replaced by $\sigma_1(\lambda_j) = \sigma_1(-k_j^2)$ and $\sigma_2(\lambda_j) = \sigma_2(-k_j^2)$, respectively. The functions $\psi_1'(z,k_j')$ and $\psi_2'(z,k_j')$ are obtained similarly.

The eigenfunctions listed above satisfy the following orthonormality relations (See Kazi (1976))

$$\int_0^H \mu(z)\ X_m(z)\ X_n(z)\ dz = \delta_{mn}\ , \quad 1 \leq m,n \leq r\ , \qquad --- \quad 11(a)$$

$$\int_{-\delta}^H \mu(z)\ X_m'(z)\ X_n'(z)dz = \delta_{mn}\ , \quad 1 \leq m,n \leq s\ , \qquad --- \quad 11(b)$$

$$\int_0^H \mu(z)\ X_m(z)\ \psi(z,k_j)dz = 0\ , \quad 1 \leq m \leq r, \quad 1 \leq j\ , \qquad --- \quad 11(c)$$

$$\int_{-\delta}^H \mu(z)\ X_m'(z)\ \psi'(z,k_j')dz = 0\ , \quad 1 \leq m \leq s, \ 1 \leq j\ , \qquad --- \quad 11(d)$$

$$\int_0^H \mu(z)\, \psi(z,k_i)\, \psi(z,k_j)dz = \delta_{ij},\ i,\ j \geq 1 \qquad \text{---} \qquad 11(e)$$

and

$$\int_{-\delta}^H \mu(z)\, \psi'(z,k_i')\, \psi(z,k_j')dz = \delta_{ij},\ i,\ j \geq 1 \qquad \text{---} \qquad 11(f)$$

where $\mu(z) = \mu_1,\ -\delta \leq z \leq h_1$ in $x > 0$ and $0 \leq z \leq h_1$ in $x < 0$,

$= \mu_2,\ h_1 < z \leq H.$

3. RECIPROCITY RELATIONS

We prove the following exact result:

For any value of the frequency let there be r propagated modes in the left hand domain (see fig. 1) and s propagated modes on the right. Let $\vec{B}'_{ij}$, $j = 1, 2, \ldots s$ be the transmission coefficients corresponding to the incident ith mode, $(1 \leq i \leq r)$, travelling from left to right, and $\overleftarrow{B}_{ji}$ be the transmission coefficients corresponding to the incident jth mode $(1 \leq j \leq s)$, from right to left. Then

$$k_i\, \overleftarrow{B}_{ji} = k_j'\, \vec{B}'_{ij} ,$$

where no sum is to be taken over repeated suffixes.

Proof Two elastodynamic states may be given exactly by displacements $v^{(1)}$ and $v^{(2)}$, expressed in terms of propagated and non-propagated modes by:

I

$$v^{(1)}(x,z) = e^{ik_i x}\chi_i(z) + \sum_{n=1}^{r} B_n e^{-ik_n x}\chi_n(z) + \sum_{j=1}^{\infty} C(k_j)e^{k_j x}\psi(z,k_j),\ x<0$$

$$= \sum_{m=1}^{s}(\vec{B}'_{im} e^{ik'_m x})\chi'_m(z) + \sum_{j=1}^{\infty} C'(k'_j)\, e^{-k'_j x}\psi'(z,k'_j),\ x>0$$

--- (12)

and

II

$$v^{(2)}(x,z) = e^{-ik'_j x}\chi'_j(z) + \sum_{m=1}^{s} B'_m\, e^{ik'_m x}\chi'_m(z)$$

$$+ \sum_{j=1}^{\infty} d'(k'_j)\,\psi'(z,k'_j)\, e^{-k'_j x},\ x>0$$

$$= \sum_{n=1}^{r}(\vec{B}_{jn} e^{-ik_n x}\chi_n(z)) + \sum_{j=1}^{\infty} d(k_j)\, e^{k_j x}\psi(z,k_j),\ x<0$$

--- (13)

(These expressions have been written with the help of (3) and (4)).

Applying the Betti-Rayleigh theorem for isotropic regions, under no body forces, to the two elastodynamic states over a region bounded by the contour ABCDEOA ($-b \le x \le 0$, $0 \le z \le H$; $0 \le x \le b$, $-\delta \le z \le H$) in which there is no singularity in displacement (see fig. 2) we obtain

$$-\int_A^B \mu(z) \left\{ \frac{\partial v^{(1)}}{\partial x} v^{(2)} - \frac{\partial v^{(2)}}{\partial x} v^{(1)} \right\} dz$$

$$= \int_C^D \mu(z) \left\{ \frac{\partial v^{(1)}}{\partial x} v^{(2)} - \frac{\partial v^{(2)}}{\partial x} v^{(1)} \right\} dz , \quad \text{--- (14)}$$

the contributions from the sides BC, DE, EO and OA being zero because these surfaces are stress free.

Differentiating equations (12) and (13) with respect to x, we obtain

$$\frac{\partial v^{(1)}}{\partial x} = ik_i \, e^{ik_i x} \, \chi_i(z) - \sum_{n=1}^{r} B_n \, ik_n \, e^{-ik_n x} \chi_n(z)$$

$$+ \sum_{j=1}^{\infty} C(k_j) \, k_j \, e^{k_j x} \, \psi(z,k_j), \; x < 0$$

$$= \sum_{m=1}^{s} (ik_m') \, \vec{B}_{im}' e^{ik_m' x} \, \chi_m'(z) - \sum_{j=1}^{\infty} k_j' \, C(k_j') \, e^{-k_j' x} \, \psi'(z,k_j'), \; x > 0 \quad \text{--- (15)}$$

and

$$\frac{\partial v^{(2)}}{\partial x} = -ik_j' \, e^{-ik_j' x} \, \chi_j'(z) + \sum_{m=1}^{s} (ik_m') \, B_m' \, e^{ik_m' x} \, \chi_m'(z)$$

$$- \sum_{j=1}^{\infty} d'(k_j') \, k_j' \, \psi'(z,k_j') \, e^{-k_j' x}, \; x > 0,$$

$$= -\sum_{n=1}^{r} [ik_n \overleftarrow{B}_{jn} e^{-ik_n x} \chi_n(z)] + \sum_{j=1}^{\infty} k_j\, d(k_j)\, e^{k_j x}\, \psi(z,k_j), \quad x < 0 \qquad --- \quad (16)$$

Substituting (12), (13), (15) and (16) into equation (14) and using the orthonormality relations (11 a -f), we obtain:

$$\text{L.H.S.} = \int_0^H \mu(z)\, [\{ik_i\, e^{-ik_i b} \chi_i(z) - \sum_{n=1}^{r} B_n\, ik_n\, e^{ik_n b} \chi_n(z) + \sum_{j=1}^{\infty} C(k_j)\, k_j\, e^{-k_j b}\, \psi(z,k_j)\} \times$$

$$\{\sum_{n=1}^{r} \overleftarrow{B}_{jn} e^{ik_n b} \chi_n(z) + \sum_{j=1}^{\infty} d(k_j)\, e^{-k_j b}\, \psi(z,k_j)\}$$

$$+ \{\sum_{n=1}^{r} ik_n \overleftarrow{B}_{jn} e^{ik_n b} \chi_n(z) - \sum_{j=1}^{\infty} k_j\, d(k_j)\, e^{-k_j b}\, \psi(z,k_j)\}$$

$$\times \{e^{-ik_i b} \chi_i(z) + \sum_{n=1}^{r} B_n\, e^{ik_n b} \chi_n(z) + \sum_{j=1}^{\infty} C(k_j)\, e^{-k_j b}\, \psi(z,k_j)\}]\, dz,$$

$$= -[-ik_i \overleftarrow{B}_{ji} + \sum_{n=1}^{r} i\, \overleftarrow{B}_{jn} B_n\, k_n\, e^{2ik_n b} - \sum_{j=1}^{\infty} C(k_j)\, k_j\, d(k_j)\, e^{-2k_j b}$$

$$- ik_i \overleftarrow{B}_{ji} - \sum_{n=1}^{r} i\, \overleftarrow{B}_{jn} k_n B_n\, e^{2ik_n b} + \sum_{j=1}^{\infty} C(k_j) k_j\, d(k_j)\, e^{-2k_j b}]$$

$$= 2ik_i \overleftarrow{B}_{ji} \qquad --- \quad (17)$$

and

$$\text{R.H.S.} = \int_{\delta}^{H} \mu(z) \Big[\Big\{ e^{-ik_j'b} \chi_j'(z) + \sum_{m=1}^{s} B_m' e^{ik_m'b} \chi_m'(z) + \sum_{j=1}^{\infty} d'(k_j') \psi'(z,k_j') e^{-k_j'b} \Big\} \times$$

$$\Big\{ \sum_{m=1}^{s} (ik_m') \vec{B}_{im}' e^{ik_m'b} \chi_m'(z) - \sum_{j=1}^{\infty} k_j' C'(k_j') e^{-k_j'b} \psi'(z,k_j') \Big\}$$

$$- \Big\{ \sum_{m=1}^{s} (ik_m') B_m' e^{ik_m'b} \chi_m'(z) - ik_j' e^{-ik_j'b} \chi_j'(z) - \sum_{j=1}^{\infty} d'(k_j') k_j' e^{-k_j'b} \psi'(z, k_j') \Big\} \cdot$$

$$\Big\{ \sum_{m=1}^{s} \vec{B}_{im}' e^{ik_m'b} \chi_m(z) + \sum_{j=1}^{\infty} C'(k_j') e^{-k_j'b} \psi'(z,k_j') \Big\} \Big] dz,$$

$$= \Big[ik_j' \vec{B}_{ij}' + \sum_{m=1}^{s} ik_m' B_m' \vec{B}_{im}' e^{2ik_m'b} - \sum_{j=1}^{\infty} k_j' d'(k_j') C'(k_j') e^{-2k_j'b}$$

$$- \sum_{m=1}^{s} ik_m' B_m' \vec{B}_{im}' e^{2ik_m b}$$

$$+ ik_j' \vec{B}_{ij}' + \sum_{j=1}^{\infty} k_j' C'(k_j') d'(k_j') e^{-2k_j'b} \Big]$$

$$= 2ik_j' \vec{B}_{ij}' \qquad \text{---} \quad (18)$$

whence from (17) and (18) we conclude that

$$k_i \overleftarrow{B}_{ji} = k_j' \vec{B}_{ij}'$$

It may be mentioned that similar reciprocity relations can be established for Love wave transmission problems in welded layered half-strips. In another paper we shall investigate the validity of relations of this type when the substratum is a half-space.

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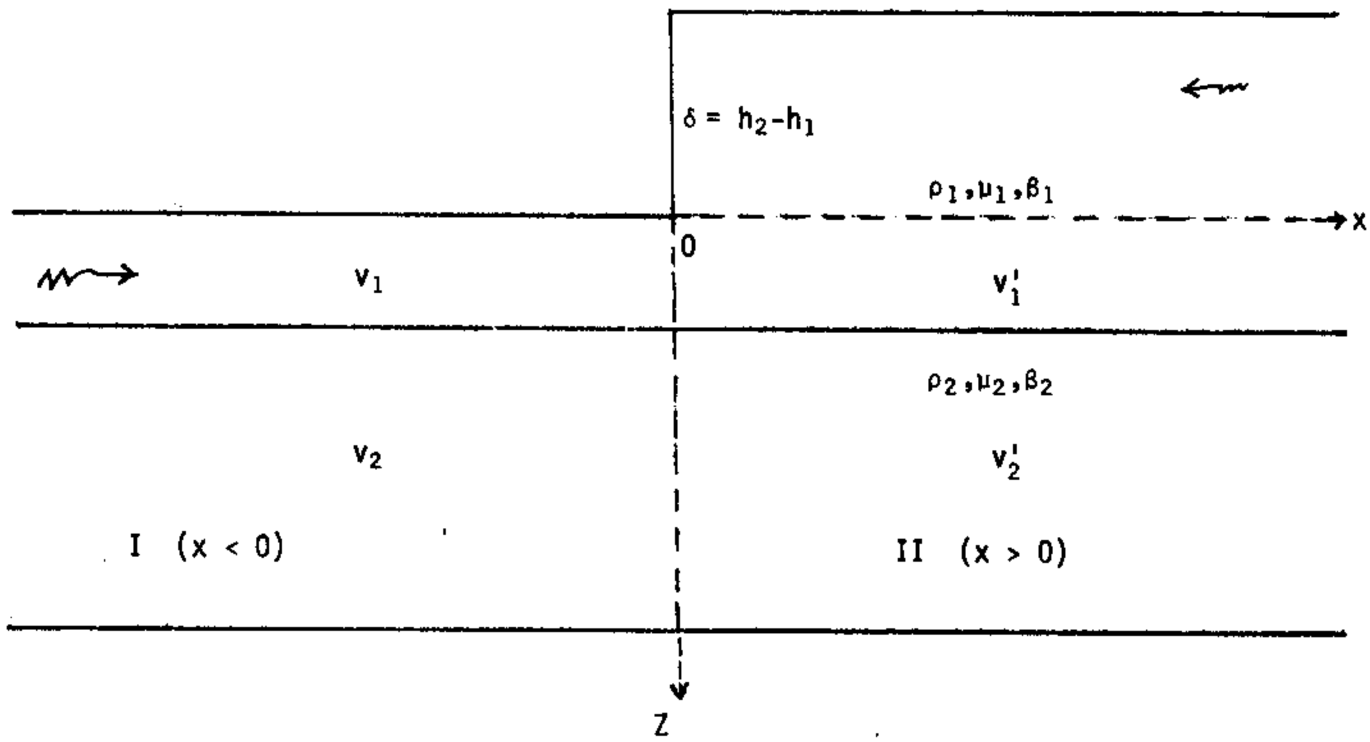

FIG. 1 The Geometry of the Problem.

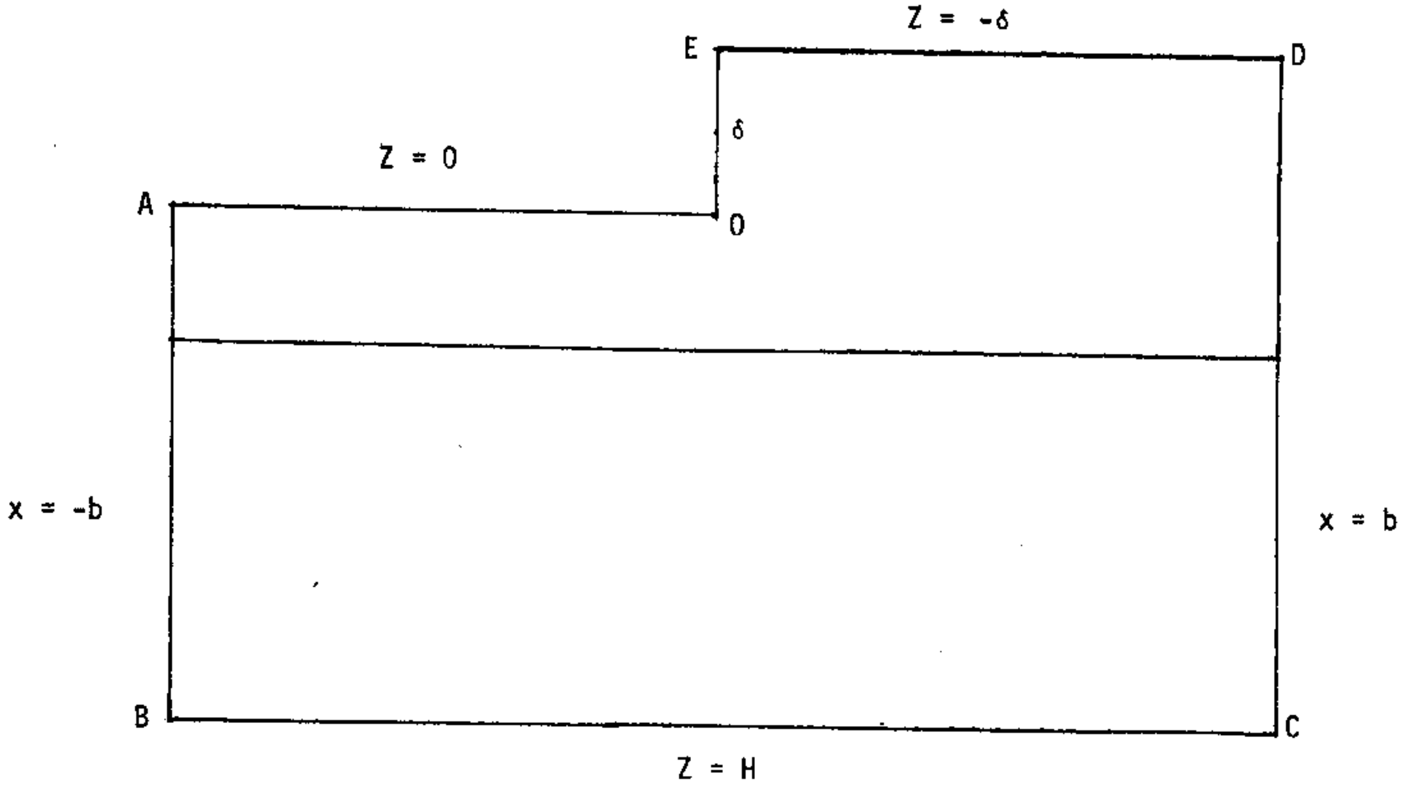

FIG. 2 The Contour of Integration