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ON A QUADRATIC MATRIX INEQUALITY AND THE CORRESPONDING ALGEBRAIC RICCATI EQUATION

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ABSTRACT

The purpose of this paper is to report some new results concerning some structural aspects in stochastic realization theory. In particular, the boundary and extreme points of the set of all state convariance matrices P of stochastic realizations of a stationary continuous - time stochastic process are studied. The set P is also the solution set of the Quadratic Matrix Inequality (QMI) [1], and a certain subset P_0 of P contains the solutions of the corresponding Algebraic Riccati Equation (ARE) [2]. Some new results linking P_0 to P are reported.

1. Introduction

The (wide sense) stochastic realization problem may be stated as follows: Given an m-dimensional stochastic process $\{y(t);\ t \in R\}$, find all linear stochastic systems having the process y as its output process. This problem has applications and connections to many

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fields of study, among which are network theory [3], spectral factorization [1,3], optimal control theory [4,5], stability theory [4] and the smoothing problem [7,8].

It is fairly known [3,4,9-12], that the set of all state covariance matrices $\mathcal{P}$ of stochastic realizations (systems) of y is the solution of a Quadratic Matrix Inequality and contains a set P_0, the elements of which are solutions of the corresponding Algebraic Riccati Equation. In this paper, we present some new results concerning the boundary and extreme points of P. After a complete characterization of the boundary points of $P(\partial P)$, we prove that $P_0 \subset \partial P$. We also show that (under a reasonable assumption) line segments of P_0 are contained in ∂P. Then we show that the elements of P_0 are extreme points of P. This seems to be well-known; however, we have been unable to find a proof of this anywhere in the literature. These results are generalizations of some found in [1,2,4,13]. As a by-product, we obtain some new results concerning the "feedback matrix" corresponding to any element of P_0.

After a brief review of some facts from stochastic realization theory in Section 2, we present the above mentioned results in Section 3.

2. The Stochastic Realization Problem: A Brief Review

In reviewing the continuous-time stochastic realization problem, we shall closely follow the presentation in [9]. Let $\{y(t);\ t \in R\}$ be a mean-square and purely nondeterministic m-dimensional stochastic process with stationary increments and zero mean. Then there exists an orthogonal stochastic measure $d\hat{y}$ such that $y(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega}\, d\hat{y}(\omega)$ and $E\{d\hat{y}(\omega)\, d\hat{y}(\omega)^{+}\} = \Phi(i\omega)d\omega$. (Here + denotes conjugation and trans-position.) The $m \times m$-matrix of real functions Φ is the spectral density satisfying (i) each element of Φ is analytic on the imaginary axis, (ii) $\Phi(s) = \Phi(-s)'$, (iii) $\Phi(i\omega) \geq 0$ for all $\omega \in R$ and (iv) $\Phi(\infty) < \infty$. Furthermore, Φ is assumed to enjoy the additional properties that $R := \Phi(\infty)$ is positive definite (the singular case has been studied in [11]) and that $\Phi(i\omega) > 0$ for all $\omega \in R$. It can be shown that Φ can be written

$$\Phi(s) = Z(s) + Z(-s)' , \tag{2.1}$$

where Z is a positive real function [3]. Let $[F, G, H, R]$ be a minimal realization [6] of Z. Then $\mathrm{Re}\{\lambda(F)\} < 0$, (F, G) is controllable, and (H, F) is observable.

Now, the problem is to find all representations of the type

$$dx = Ax\,dt + Bdw \tag{2.2a}$$

$$dy = Cx\,dt + Ddw\ . \tag{2.2b}$$

such that the output y has spectral density Φ, $n = \dim A$ minimal and $\mathrm{Re}\{\lambda(A)\} < 0$.

Modulo a trivial coordinate transformation in the state space, all solutions to this problem are of the form

$$dx = Fx\,dt + B_1du + B_2dv \tag{2.3a}$$

$$dy = Hxdt + R^{\frac{1}{2}}du\ , \tag{2.3b}$$

where $B = (B_1, B_2)$ and $P = E\{x(t)\,x(t)'\}$ satisfy the Positive Real Lemma Equations

$$FP + PF' + B_1B_1' + B_2B_2' = 0 \tag{2.4a}$$

$$G = PH' + B_1R^{\frac{1}{2}} \tag{2.4b}$$

$$P = P' > 0 \quad (n \times n\text{-matrix})\ . \tag{2.4c}$$

Let $\mathcal{P} = \{P|P$ solves (2.4)$\}$. For each $P \in \mathcal{P}$, define

$$\Lambda(P) = FP + PF' + (G - PH')R^{-1}(G - PH')'\ . \tag{2.5}$$

Then $\mathcal{P} = \{P = P' > 0 \mid \Lambda(P) \leq 0\}$[9]. Let $\mathcal{P}_0 = \{P \in \mathcal{P} \mid \Lambda(P) = 0\}$.

In the following proposition, we collect some facts from [3, 4, 9].

Proposition 2.1. The set $\mathcal{P}$ is convex and compact and there are two elements P_* and P^* in $\mathcal{P}_0$ such that $P_* \leq P \leq P^*$ for all $P \in \mathcal{P}$. Moreover, $\mathcal{P} = \{P \mid \Lambda(P) \leq 0\}$. Finally, $\mathcal{P}_0$ is the set of all solutions of (2.4) for which $B_2 = 0$.

Moreover, this problem is equivalent to the classical spectral factorization problem [1]: Given $\Phi(s)$, find all minimal stable spectral factors of Φ, i.e. all matrices $W(s)$ of proper rational functions of minimal McMillan degree [6] with all poles in the left half plane and satisfying

$$W(s)\, W(-s)' = \Phi(s) \,. \tag{2.6}$$

The minimum P_* and the maximum P^* are of particular interest. The following matrix Riccati equations, given in [4], may be used to calculate them.

Proposition 2.3. Let Π and $\bar{\Pi}$ be the unique solutions of the $n \times n$-matrix differential equations

$$\dot{\Pi}(t) = \Lambda(\Pi(t)) \quad , \quad \Pi(0) = 0 \tag{2.7a}$$

and

$$\dot{\bar{\Pi}}(t) = \bar{\Lambda}(\bar{\Pi}(t)) \quad , \quad \bar{\Pi}(0) = 0 \tag{2.7b}$$

respectively, where Λ is given by (2.5) and $\bar{\Lambda}$ by

$$\bar{\Lambda}(P) = F'P + PF + (H' - PG)R^{-1}(H' - PG)' \ . \tag{2.8}$$

Then $\Lambda(t) \to P_*$ and $\bar{\Pi}(t)^{-1} \to P^*$ as $t \to \infty$.

It should be observed that once the state covariance P of a realization is known, then the whole realization $[F, B, H, (R^{\frac{1}{2}}, 0)]$ is determined upon defining

$$B_1 = (G - PH')R^{-\frac{1}{2}} \tag{2.9a}$$

$$B_2B_2' = -\Lambda(P) \tag{2.9b}$$

3. Structure of the Sets $\mathcal{P}$ and $\mathcal{P}_0$

Since each wide sense stochastic realization is determined by its covariance matrix P, an investigation of the structure of the set $\mathcal{P}$ of all such matrices is deemed necessary. In this section, we shall exploit the role played by a Hamiltonian matrix to be defined below to provide some new links between the solutions of what is known as the Algebraic Riccati Equation (ARE) (the solution set of which is $\mathcal{P}_o$) and those of a Quadratric Matrix Inequality (QMI) with solution set $\mathcal{P}$. The boundary and extreme points of $\mathcal{P}$ will be studied.

Let $\mathcal{P}_+ = \{P \in \mathcal{P} \mid P > P_*\}$ and $\mathcal{P}_- = \{P \in \mathcal{P} \mid P < P^*\}$, where P_* and P^* are the minimum and maximum elements of $\mathcal{P}$ defined in Section 2. Since $\Phi(i\omega) > 0$ for all real $\omega, P^* - P_* > 0$ [14; p. 360], and consequently $\mathcal{P}_+$ and $\mathcal{P}_-$ are both nonempty. For each $P \in \mathcal{P}$, define the *feedback matrix*

$$\Gamma = F - (G - PH')R^{-1}H \ . \tag{3.1}$$

Let the feedback matrices corresponding to P_* and P^* be denoted Γ_* and Γ^* respectively. It can be shown that $\mathrm{Re}\{\lambda(\Gamma_*)\} < 0$ and $\mathrm{Re}\{\lambda(\Gamma^*)\} > 0$ [14; p. 360], [4 ; p. 53]. Finally, from the given matrices F, G, H and R, construct the $2n \times 2n$-matrix

$$\mathcal{F} = \begin{bmatrix} -(F - GR^{-1}H)' & -H'R^{-1}H \\ GR^{-1}G' & (F - GR^{-1}H) \end{bmatrix} , \tag{3.2}$$

(the significance of which will be clear shortly.) It is trivial to see that $\mathcal{F}$ is a *Hamiltonian matrix* i.e. $\mathcal{F} = \mathcal{I}\mathcal{F}\mathcal{I}'$, where $\mathcal{I} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. Consequently, if λ_i ; $i = 1,2,\ldots,n$ is an eigenvalue of $\mathcal{F}$, so is $-\lambda_i$

[15]. It can also be shown that F must have no purely imaginary eigenvalues [16].

In the following proposition, we collect some facts from Brockett [6], Faurre [4], MacFarlane [16], Mårtensson [17] and Willems [2].

Proposition 3.1. *There is one and only one* $P \in P_o$ *with* $\mathrm{Re}\{\lambda(\Gamma)\} < 0$ ($\mathrm{Re}\{\lambda(\Gamma)\} > 0$), *namely* $P_*(P^*)$. *Moreover, the eigenvalues of the corresponding feedback matrix* $\Gamma_*(\Gamma^*)$ *are the* n *eigenvalues of* F *with negative (positive) real parts.*

The following lemma will be needed in this section.

Lemma 3.2. *Let* P_1 *and* P_2 *be arbitrary elements of* P_o *and let* Γ_1 *and* Γ_2 *be the corresponding feedback matrices* (3.1). *Then*

$$\Gamma_1 \Delta P + \Delta P \Gamma_2{}' = 0 , \tag{3.3}$$

where $\Delta P = P_1 - P_2$.

Proof: Since P_1 and P_2 belong to P_o, $\Lambda(P_1) = 0$ and $\Lambda(P_2) = 0$. Subtracting the second from the first and adding and subtracting the quantity $P_1 H' R^{-1} H P_2$, we obtain (3.3). □

As a first corollary to the above, we can easily prove some of the statements of the previous proposition.

Corollary 3.3. *The feedback matrices* Γ_* *and* $-\Gamma^{*\prime}$ *are similar. (Consequently, if* λ_i ; $i = 1,2,\ldots,n$ *are the eigenvalues of* Γ_*, *then* $-\lambda_i$; $i = 1,2,\ldots,n$ *are the eigenvalues of* Γ^*.)

Proof. Since $\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$, $P^* - P_* > 0$. By the above lemma, $\Gamma_*(P^* - P_*) + (P^* - P_*)\Gamma^{*\prime} = 0$. Hence $\Gamma_* = -(P^* - P_*)\Gamma^{*\prime}(P^* - P_*)^{-1}$. □

Now, we turn our attention to the other solutions of the Algebraic Riccati Equation : $\Lambda(P) = 0$

For an arbitrary $n \times n$-matrix M with n^+ eigenvalues with positive real parts and n^- eigenvalues with negative real parts, let $L^+(M)$ and $L^-(M)$ denote the invariant subspaces spanned by the corresponding (generalized) eigenvectors.

Lemma 3.4 (J. C. Willems [2]). *Let* $P \in P_o$ *and* Γ *the corresponding feedback matrix* (3.1). *Then*

$$\Gamma a = \Gamma_* a \quad , \quad \text{for } a \in L^-(\Gamma) \tag{3.4a}$$

and

$$\Gamma b = \Gamma^* b \quad , \quad \text{for } b \in L^+(\Gamma) \tag{3.4b}$$

The following corollary is a trivial consequence of the above lemma.

Corollary 3.5 *Let* P *and* Γ *be as in* Lemma 3.4. *Then the eigenvalues of* Γ *are among those of* Γ_* *and* Γ^* *(i.e. among those of* F.*) (In particular, the feedback matrix corresponding to any solution* $P \in P_o$ *has no purely imaginary eigenvalues.)*

The above corollary is in agreement with the well known result of Potter [18] (generalized in [17] to the case of nondistinct eigenvalues) that all solutions of the (ARE) may be obtained from the eigenvectors corresponding to the eigenvalues of the Hamiltonian matrix F.

The next corollary, which holds under a natural and standard assumption that F can be transformed to Jordan form, provides some more information about the feedback matrix Γ corresponding to any solution $P \in P_o$.

Corollary 3.6. *Assume the Hamiltonian matrix F has distinct eigenvalues. Let λ be an eigenvalue of the feedback matrix Γ corresponding to an arbitrary element P of P_o. Then $-\lambda$ cannot be an eigenvalue of Γ.*

Proof. Let $[F, B, H, R^{\frac{1}{2}}]$ be the (unique) realization corresponding to P (since $P \in P_o$, $B_2 = 0$), which gives rise to the spectral factor

$$W(s)^{-1} = -R^{-\frac{1}{2}}H(sI - \Gamma)^{-1} B_1 R^{-\frac{1}{2}} + R^{-\frac{1}{2}} \quad ,$$

which can be written $\frac{1}{\chi_\Gamma(s)}M(s)$, where χ_Γ is the characteristic polynomial of Γ and M is a matrix polynomial. Therefore, by Cramer's rule, χ_Γ equals the numerator of det W, and consequently, in view of (2.6) $\chi_\Gamma(s)\, \chi_\Gamma(-s) = \phi(s)$, where ϕ is the numerator polynomial of det Φ. But, since, in particular, this relation holds for $\Gamma = \Gamma_*$ and since $\chi_{\Gamma_*}(-s) = \chi_{\Gamma^*}(s)$ (Corollarly 3.3), ϕ must be the characteristic polynomial χ_F of F (Proposition 3.1), i.e.

$$\chi_\Gamma(s)\, \chi_\Gamma(-s) = \chi_F(s).$$

Now suppose λ and $-\lambda$ are eigenvalues of Γ. Then $(s - \lambda)(s + \lambda)$ is a factor of both $\chi_\Gamma(s)$ and $\chi_\Gamma(-s)$. Consequently, $(s - \lambda)^2(s + \lambda)^2$ is a factor of $\chi_F(s)$, which is clearly a contradiction to the assumption that F has distinct eigenvalues. □

In particular, we are able to see that the sets $\mathcal{P}_o^+ = \{P \in \mathcal{P}_o \mid P > P_*\}$ and $\mathcal{P}_o^- = \{P \in \mathcal{P}_o \mid P < P^*\}$ are singletons.

Corollary 3.7. *An $n \times n$ symmetric matrix P belongs to $\mathcal{P}_o^+$ ($\mathcal{P}_o^-$) if and only if the feedback matrix Γ corresponding to P is similar to $-\Gamma_*'$ (Γ_*).*

Proof. Let $P \in \mathcal{P}_o^+$. Then $P > P_*$. Hence, by (3.3) $\Gamma = -(P - P_*)\Gamma_*' (P - P_*)^{-1}$. Conversely, if Γ is similar to $-\Gamma_*'$, then Γ has all eigenvalues in the right half plane. But, by Proposition 3.1, there is only one such feedback matrix, which is Γ^*: the one corresponding to P^*. By the assumption $\Phi(i\omega) > 0$, $P^* - P_* > 0$ i.e. $P^* \in \mathcal{P}_o^+$. The proof of the other part is analogous. □

Indeed, Corollary 3.7 may be reformulated as: $\mathcal{P}_o^+ = \{P^*\}$ and $\mathcal{P}_o^- = \{P_*\}$.

Next, we shall discuss the relationships between the solutions of (ARE) : $\Lambda(P) = 0$ and (QMI) : $\Lambda(P) \leq 0$.

For any $\varepsilon > 0$ and any matrix M, define the ball $\mathcal{U}(M,\varepsilon) = \{L : L = M + N, \|N\| < \varepsilon\}$, where $\|\cdot\|$ is the usual matrix norm associated to the vector Euclidean norm.

Definition 3.8. An $n \times n$ symmetrix matrix P belongs to the *boundary* of $\mathcal{P}$ (denoted by $\partial\mathcal{P}$) if, for all $\varepsilon > 0$, there exist two matrices P_1 and P_2 belonging to $\mathcal{U}(P,\varepsilon)$ such that $P_1 \in \mathcal{P}$ and $P_2 \notin \mathcal{P}$. (Since $\mathcal{P}$ is closed, $\partial\mathcal{P} \subset \mathcal{P}$.)

The following theorem, which provides a complete characterization of the boundary points of $\mathcal{P}$, is due to Germain [13].

Theorem 3.9. *Let* $P_o \in \mathcal{P}$ *and set* $-Q_o = FP_o + P_oF'$ *and* $S_o = G - P_oH'$. *Then* $P_o \in \partial\mathcal{P}$ *if and only if the matrix* $M_o = \begin{bmatrix} Q_o & S_o \\ S_o' & R \end{bmatrix}$ *is singular.*

As a corollary, we obtain the following result linking $\mathcal{P}_o$ with $\mathcal{P}$. A sharper result will be given later in this section.

Corollary 3.10. $\mathcal{P}_o \subset \partial\mathcal{P}$.

Proof. Let $P_o \in \mathcal{P}_o$. Then $\Lambda(P_o) = 0$, consequently, the matrix M_o (see Theorem 3.9) is singular. □

In fact, if $m < n$ (which is usually the case in application), we have a stronger result, namely

Corollary 3.1 . *Let* $m < n$ *and let* P_1 *and* P_2 *be arbitrary elements of* $\mathcal{P}_o$. *Then the segment* $[P_1, P_2]$ *is contained in* $\partial\mathcal{P}$. *(In particular,* $[P_*, P^*] \subset \partial\mathcal{P}$.)

Proof. Let $\alpha \in [0,1]$ and define $P(\alpha) = \alpha P_1 + (1-\alpha)P_2$. Then $\Lambda(P(\alpha))$ may be written

$$\Lambda(P(\alpha)) = \alpha\Lambda(P_1) + (1-\alpha)\Lambda(P_2) - \alpha(1-\alpha)(P_1 - P_2)H'R^{-1}H(P_1 - P_2). \quad (3.5)$$

Since $P_1, P_2 \in \mathcal{P}_o$, $\Lambda(P_1) = \Lambda(P_2) = 0$. Let $Q(\alpha) = -FP(\alpha) - P(\alpha)F'$ and $S(\alpha) = G - P(\alpha)H'$. Then it is easy to see that

$-\Lambda(P(\alpha) = Q(\alpha) - S(\alpha)R^{-1}S(\alpha)' = \alpha(1-\alpha)(P_1 - P_2)\, H'R^{-1}H(P_1 - P_2).$

If $m < n$, $H'R^{-1}H$ is not full rank and hence the matric $M(\alpha) = \begin{bmatrix} Q(\alpha) & S(\alpha) \\ S(\alpha)' & R \end{bmatrix}$ is singular. Then $P(\alpha) \in \partial P \quad \forall \alpha \in [0,1]$. ||

The final task of this section is to prove that solutions of the (ARE) are extreme points of the set of solutions of the (QMI). This is a much stronger result than Corollary 3.10. (The extreme points of a set are contained in its boundary.) To this end, we shall need the following lemma.

Lemma 3.12 *Let P be an arbitrary element of* P_o . *Suppose there exist two elements* P_1 *and* P_2 *belonging to* P *such that* $P = \alpha P_1 + (1-\alpha)P_2$ *for some* $\alpha \in (0,1)$. *Then,* $P_1 \in P_o$, $P_2 \in P_o$ *and* $\Delta PH'R^{-1}H\Delta P = 0$, *where* $\Delta P = P_1 - P_2$.

Proof. Let P, P_1 and P_2 be as in the lemma. Then, by (3.5) $\Lambda(P) = \alpha\Lambda(P_1) + (1-\alpha)\Lambda(P_2) - \alpha(1-\alpha)\Delta PH'R^{-1}H\Delta P$. Since $\alpha \in (0,1)$, the last term is ≤ 0. On the other hand, since $\Lambda(P) = 0$ (for $P \in P_o$), $\Lambda(P_1) \le 0$ and $\Lambda(P_2) \le 0$ (for both belong to P), the last term must be ≥ 0. As $\alpha > 0$, $\Delta PH'R^{-1}H\Delta P = 0$. Consequently, $\Lambda(P_1) = \Lambda(P_2) = 0$. which implies P_1 and P_2 belong to P_o. □

The above lemma says in essence that elements of P_o cannot be written as the convex combination of elements other than those of P_o.

The next theorem is the main result of this section.

Theorem 3.13. *Let $P \in P_o$. Then P is an extreme point of P.*

Proof. Let $P \in P_o$ and assume there exist P_1 and $P_2 \in P$ such that $P = \alpha P_1 + (1-\alpha)P_2$ for $\alpha \in (0,1)$. We shall show that $P = P_1 = P_2$. By Lemma 3.12, $P_1 \in P_o$, $P_2 \in P_o$ and $\Delta P H' R^{-1} H \Delta P = 0$. The last of these facts implies $\Delta P H' R^{-\frac{1}{2}} = 0$, i.e. $P_1 H' R^{-\frac{1}{2}} = P_2 H' R^{-\frac{1}{2}}$. This in turn implies that $(G - P_1 H')R^{-1}(G - P_1 H')' = (G - P_2 H')R^{-1}(G - P_2 H')' := E$. However, P_1 and P_2 are in P_o implies $FP_1 + P_1 F' = -E = FP_2 + P_2 F'$. Hence $F\Delta P + \Delta P F' = 0$. But F is a stability matrix i.e. $\text{Re}\lambda(F) < 0$. Then F and $-F'$ have no eigenvalue in common, which implies $\Delta P = 0$ (see e.g. [19]). Hence $P_1 = P_2 = P$. □

If the eigenvalues of F were distinct, the above result may alternatively be proved by the following.

Proposition 3.14. *Let P_1 and P_2 be two elements of P_o such that $\Delta P H' R^{-1} H \Delta P = 0$, where $\Delta P = P_1 - P_2$. Then, $\Gamma_1 = \Gamma_2$ and*

$$\Gamma_2 \Delta P + \Delta P \Gamma_2' = 0 , \tag{3.6}$$

where Γ_1 and Γ_2 are the feedback matrices corresponding to P_1 and P_2 respectively.

Proof. Recall that $\Gamma_1 = F - GR^{-1}H + P_1 H' R^{-1} H$. As was indicated in the proof of Theorem 3.13, $\Delta P H' R^{-1} H \Delta P = 0$ implies $P_1 H' R^{-\frac{1}{2}} = P_2 H' R^{-\frac{1}{2}}$. Then $\Gamma_1 = \Gamma_2$. The rest of the result then follows by (3.3). □

Therefore, if the eigenvalues of F are distinct, so are those of Γ_2 (Corollary 3.5). Then, by Corollary 3.6, Γ_2 has no opposite eigenvalues, and consequently ΔP in (3.6) will be zero [19]. Hence, in view of Lemma 3.12 any $P \in P_0$ is an extreme point of P.

Of course, if $m = n$, Theorem 3.13 would follow trivially from Lemma 3.12 since then $H'R^{-1}H$ is full rank and hence the condition $\Delta PH'R^{-1}H\Delta P = 0$ implies $\Delta P = 0$.

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