King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 013

November 1980

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Abstract.

This paper extends the operational calculus of Meller for the operator $B_\alpha = t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}$ to the case where $\alpha \in (0, \infty)$. The development is à la Mikusinski calculus and uses Meller's convolution process with a fractional derivative operator.

1. INTRODUCTION

Meller [1], [2] constructed an operational calculus for the operator $B_\alpha = t^{-\alpha}\frac{d}{dt}\, t^{\alpha+1}\, \frac{d}{dt}$ with $-1 < \alpha < 1$ by embedding it in a field of convolution quotients. The convolution process was given by the formula:

$$f(t)*g(t) = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\, \frac{d}{dt}\int_0^t (t-\xi)^{-\alpha} d\xi\, \frac{d}{d\xi}\, \xi\, \frac{d}{d\xi}\int_0^\xi d\eta \,\times$$

$$\times \int_0^1 dx\ \eta^\alpha (1-x)^\alpha f(x\eta) g[(1-x)(\xi-\eta)]. \tag{1}$$

This calculus reduces to Ditkin's calculus [3], [4] for $\frac{d}{dt}\, t\, \frac{d}{dt}$ when $\alpha = 0$. Recently, Koh [5], [6] and Conlan [7] extended Meller's calculus to the case $\alpha \in (-1, \infty)$. A modified convolution process was used which yields results analogous to Meller's. In the present work, we give a direct extension of Meller's calculus by treating the operator $\frac{1}{\Gamma(1-\alpha)}\, \frac{d}{dt}\int_0^t (t-\xi)^{-\alpha} d\xi$ in (1) as a fractional derivative.

Specifically, we let n be the least integer greater than $\alpha \geq 0$. For any n-times differentiable function $f(t)$, the αth-order derivative of $f(t)$ is:

$$D^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) \tag{2}$$

where D is $\frac{d}{dt}$ and I^{ν} is the Riemann-Liouville integral of order $\nu > 0$ given in Ross [8] by

$$I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi) d\xi. \tag{3}$$

It is easy to see that I^{ν} satisfies the semigroup property

$$I^{\alpha} I^{\beta} = I^{\alpha+\beta} \tag{4}$$

but D^{α} does not. Thus $D^{\alpha}D$ in (5) below cannot be written $D^{\alpha+1}$.

§2. THE CONVOLUTION QUOTIENTS

Let $\alpha \geq 0$ be a fixed real number. Let C^{∞} denote the linear space of infinitely differentiable functions on $[0, \infty)$. For every pair of functions $\phi(t)$ and $\psi(t)$ in C^{∞}, define their convolution by

$$\phi(t)*\psi(t) = \frac{1}{\Gamma(\alpha+1)} D^{\alpha}DtD\int_0^t\int_0^1 \eta^{\alpha}(1-x)^{\alpha}\phi(x\eta)\psi[(1-x)(t-\eta)]dxd\eta. \qquad (5)$$

From this definition, the following properties are clear: (i) C^{∞} is closed under convolution, (ii) convolution is bilinear on $C^{\infty}\times C^{\infty}$, (iii) convolution is distributive with respect to the usual addition of functions. It also follows immediately that equation (5) specializes to Meller's convolution for $\alpha < 1$ and to Ditkin's for $\alpha = 0$. Not so immediate are these properties.

Proposition 1. Convolution is commutative.

Proof. Let $x = 1 - \frac{v}{t}$ and $\eta = t - t\xi$ in (5) and noting that the Jacobian $\frac{\partial(x,\ \eta)}{\partial(v,\ \xi)} = 1$ for all $t \in (0, \infty)$, we have

$$\phi(t)*\psi(t) = \frac{1}{\Gamma(\alpha+1)}D^{\alpha}DtD\int_0^1\int_0^t (t-t\xi)^{\alpha}(\tfrac{v}{t})^{\alpha}\phi[(1-\tfrac{v}{t})(t-t\xi)]\psi[\tfrac{v}{t}(t\xi)]dvd\xi$$

$$= \frac{1}{\Gamma(\alpha+1)}D^{\alpha}DtD\int_0^t\int_0^1 v^{\alpha}(1-\xi)^{\alpha}\psi(\xi v)\phi[(1-\xi)(t-v)]d\xi dv$$

$$= \psi(t)*\phi(t).$$

q.e.d.

Proposition 2. For every complex number λ, and any $\phi(t) \in C^{\infty}$, $\lambda * \phi(t) = \lambda\phi(t)$.

Proof.
$$\lambda * \phi(t) = \frac{1}{\Gamma(\alpha+1)} D^{\alpha} D t D \int_0^t \eta^{\alpha} \int_0^1 (1-x)^{\alpha} \lambda\phi(x\eta)\,dx\,d\eta$$

$$= \frac{\lambda}{\Gamma(\lambda+1)} D^{\alpha} D t^{\alpha+1} \int_0^1 (1-x)^{\alpha} \phi(xt)\,dx$$

$$= \frac{\lambda}{\Gamma(\lambda+1)} D^{\alpha} D \int_0^t (t-u)^{\alpha} \phi(u)\,du$$

$$= \frac{\lambda}{\Gamma(\lambda+1)} D^{n} I^{n-\alpha} \int_0^t \alpha(t-u)^{\alpha-1} \phi(u)\,du$$

$$= \lambda D^{n} I^{n-\alpha} I^{\alpha} \phi(t) = \lambda\phi(t).$$

The last step follows from (4) and (2). q.e.d.

In view of proposition 2, there is no distinction between constants and constant functions in our calculus.

Proposition 3. Convolution is associative.

Proof. A direct calculation shows that for nonnegative integers q and r,

$$t^q * t^r = \frac{q!r!\Gamma(q + \alpha + 1)\Gamma(r + \alpha + 1)}{(q + r)!\Gamma(\alpha + 1)\Gamma(q + r + \alpha + 1)}\, t^{q+r}. \tag{6}$$

Hence on using (6) again,

$$t^p*(t^q*t^r) = \frac{p!q!r!\ \Gamma(p+\alpha+1)!\Gamma(q+\alpha+1)\Gamma(r+\alpha+1)}{(p+q+r)!\Gamma(\alpha+1)\Gamma(p+q+r+\alpha+1)\Gamma(\alpha+1)}\, t^{p+q+r}$$

$$= (t^p * t^q) * t^r\ . \tag{7}$$

Due to the bilinearity of our convolution, equation (7) still holds for polynomials. Our proposition follows from Weierstrass's Approximation Theorem and the fact [9] that the space of C^∞ functions with compact support is dense in C^∞. q.e.d.

<u>Proposition 4</u>. C^∞ has no zero divisors, i.e. if $\phi(t)$ and $\psi(t)$ belong to C^∞ and $\phi(t) * \psi(t) = 0$, then either $\phi(t) = 0$ or $\psi(t) = 0$.

<u>Proof</u>. $\phi(t) * \psi(t) = 0$ implies that

$$\int_0^t (t-\xi)^{n-\alpha-1}\ \frac{d}{d\xi}\,\xi\,\frac{d}{d\xi}\int_0^\xi\int_0^1 \eta^\alpha(1 - x)^\alpha\phi(x\eta)\psi[(1 - x)(\xi - \eta)]\,dx\,d\eta$$

$$= C_1 \frac{t^{n-1}}{(n-1)!} + C_2 \frac{t^{n-2}}{(n-2)!} + \cdots + C_n . \tag{8}$$

As $t \to 0$, $C_n = 0$. Now, by an argument leading to equation (7), we see that if $C_i \neq 0$ for some i, then $\phi(t)$ and $\psi(t)$ have to be polynomials. But if they are polynomials, the left side of equation (8) will be of degree at least n. Hence the right side of (8) has to be zero. A similar argument, together with Titchmarsh's Theorem [10], yields

$$\int_0^t \int_0^1 \eta^\alpha (1-x)^\alpha \phi(x\eta) \psi[(1-x)(t-\eta)] dx d\eta = 0. \tag{9}$$

To complete the proof, let $x = \frac{y}{\tau}$, $\eta = z\tau$ and $t = v\tau$ in (9). We then have

$$\int_0^v \int_0^\tau z^\alpha (\tau - y)^\alpha \phi(yz) \psi[(\tau - y)(v - z)] dy dz = 0$$

By a theorem of Mikusinski and Ryll-Nardzewski [11], it follows that $z^\alpha \phi(yz) = 0$ or $y^\alpha \psi(yz) = 0$. Thus, $\phi(t) = 0$ or $\psi(t) = 0$. q.e.d.

The above properties establish C^∞ as an integral domain under the operations of addition and convolution as multiplication. By virtue of proposition 2, the multiplicative identity for C^∞ is the number 1. We may now extend C^∞ into the field F of convolution quotients consisting of equivalence classes of ordered pairs (ϕ, ψ) of elements in C^∞ with $\psi \neq 0$. The equivalence relation is given by

$$(\phi_1, \psi_1) \sim (\phi_2, \psi_2) \quad \text{if} \quad \phi_1 * \psi_2 = \phi_2 * \psi_1.$$

As usual, convolution quotients are called operators [10] and are denoted by $\frac{\phi}{\psi}$. Operators of the form $\frac{\phi(t)}{1}$ constitute a subring of F isomorphic to C^∞ through the canonical maps $\frac{\phi(t)}{1} \longleftrightarrow \phi(t)$.

3. AN OPERATIONAL CALCULUS

We now show that the operator B_α belongs to F. First, note that a right inverse to B_α is given by

$$\Lambda\phi = \int_0^t \xi^{-\alpha-1}\int_0^\xi \eta^\alpha \phi(\eta)d\eta d\xi,$$

i.e., $B_\alpha \Lambda \phi = \phi$, for $\phi \in C^\infty$. If we restrict the domain of B_α to

$\{\phi \in C^\infty | \phi(0) = 0\}$, then Λ is also a left inverse, i.e. $\Lambda B_\alpha \phi = \phi$.

Proposition 5. For any $\phi(t) \in C^\infty$, $\frac{t}{\alpha + 1} * \phi(t) = \Lambda\phi(t)$.

Proof. We shall assume that α is not an integer. Otherwise, the proof is more straightforward, obviating the use of fractional integrals.

$$\frac{t}{\alpha+1} * \phi(t) = \frac{1}{\Gamma(\alpha+1)} D^\alpha D t D \int_0^t \int_0^1 \eta^\alpha (1-x)^\alpha \phi(x\eta) \frac{(1-x)(t-\eta)}{\alpha+1}\, dx\, d\eta$$

$$= \frac{1}{\Gamma(\alpha+2)} D^\alpha D t D \int_0^t \frac{(t-\eta)}{\eta^2} \int_0^\eta (\eta-\xi)^{\alpha+1} \phi(\xi)\, d\xi\, d\eta$$

$$= \frac{1}{\Gamma(\alpha+2)} D^\alpha \{\int_0^t \frac{1}{\eta^2} \int_0^\eta (\eta-\xi)^{\alpha+1} \phi(\xi)\, d\xi\, d\eta + \frac{1}{t} \int_0^t (t-\xi)^{\alpha+1} \phi(\xi)\, d\xi\}$$

$$= \frac{1}{\Gamma(\alpha+1)} D^\alpha \int_0^t \frac{1}{\eta} \int_0^\eta (\eta-\xi)^\alpha \phi(\xi)\, d\xi\, d\eta. \qquad (10)$$

Let $\psi(\xi) = I^n \phi(\xi)$ where n = least integer greater than α. Then (10) becomes

$$\frac{t}{\alpha+1} * \phi(t) = D^\alpha \int_0^t \frac{1}{\eta} \int_0^\eta \frac{(\eta - \xi)^{\alpha-n}}{\Gamma(\alpha - n + 1)} \psi(\xi)\, d\xi\, d\eta$$

$$= D^n \int_0^t \frac{(t-u)^{n-\alpha-1}}{(n-\alpha)} \int_0^u \frac{1}{\eta} \int_0^\eta \frac{(\eta-\xi)^{\alpha-n}}{\Gamma(\alpha-n+1)} \psi(\xi)\, d\xi\, d\eta\, du$$

$$= \frac{D^n}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_0^t \psi(\xi) \int_\xi^t \eta^{-1}(\eta-\xi)^{\alpha-n}(t-\eta)^{n-\alpha}\, d\eta\, d\xi .$$

The inner integral reduces, via the Beta function, to

$\{(\frac{\xi}{t})^{\alpha-n} - 1\}\Gamma(\alpha - n + 1)\Gamma(n - \alpha)$. Thus

$$\frac{t}{\alpha + 1} * \phi(t) = \frac{D^n}{n - \alpha} \int_0^t \psi(\xi)\{(\frac{\xi}{t})^{\alpha-n} - 1\}\, d\xi$$

$$= \frac{D^n}{n - \alpha} \int_0^1 t\psi(wt)(w^{\alpha-n} - 1)\, dw$$

$$= \int_0^1 (\frac{w^{\alpha-n} - 1}{n - \alpha})(w^n t\phi(wt) + nw^{n-1} I\phi(wt))\, dw$$

$$= \int_0^t \frac{1}{n - \alpha} [(\frac{\xi}{t})^\alpha - (\frac{\xi}{t})^n]\phi(\xi)\, d\xi$$

$$+ \int_0^t \frac{1}{n - \alpha} [(\frac{\xi}{t})^\alpha - (\frac{\xi}{t})^n](\frac{n}{\xi}) \int_0^\xi \phi(u)\, du\, d\xi$$

$$= \int_0^t \frac{\phi(\xi)}{-\alpha} [(\frac{\xi}{t})^\alpha - 1]\, d\xi$$

$$= \int_0^t \xi^\alpha \phi(\xi) \int_0^t \eta^{-\alpha-1} d\eta d\xi$$

$$= \int_0^t \eta^{-\alpha-1} \int_0^\eta \xi^\alpha \phi(\xi) d\xi d\eta = \Lambda\phi(t). \qquad \text{q.e.d.}$$

This result implies that operators of the form $\frac{\phi(t)}{t}$ with $\phi(0) = 0$ may be identified with locally integrable functions $f(t)$ such that $\Lambda f(t) < \infty$ for every $t > 0$. Indeed,

$$\frac{\phi(t)}{t} = f(t) \in L_{loc}[0,\infty) \quad \text{iff} \quad \phi(t) = t*f(t) = (\alpha+1)\Lambda f < \infty, \quad \forall t > 0,$$

The next result follows from proposition 5 and equation (6) by induction.

Proposition 6. Let k be a positive integer. Then, for any $\phi(t) \in C^\infty$,

$$\frac{\Gamma(\alpha+1)t^k}{\Gamma(\alpha+k+1)k!} * \phi(t) = \Lambda^k\phi(t) \quad \text{where} \quad \Lambda^k\phi(t) = \underbrace{\Lambda(\Lambda(\cdots(\Lambda\phi)))}_{\text{k-times}}.$$

Let V be the operator $\frac{\alpha+1}{t}$ and V^k the k-times application of V.

Proposition 7. For any 2k times differentiable function $\phi(t)$,

$$V^k\phi(t) = B_\alpha^k\phi(t) + \sum_{j=1}^{k} B_\alpha^{k-j}\phi(t)|_{t\to0^+}\, V^j \tag{11}$$

<u>Proof</u>. $\phi(t) = \Lambda B_\alpha\phi(t) + \phi(0) = \frac{t}{\alpha+1} * B_\alpha\phi(t) + \phi(0)$.

Thus $V\phi(t) = B_\alpha\phi(t) + \phi(0)V$ and (11) is proved for $k = 1$. Suppose now that (11) is true for $k = m - 1$. Then for any 2m times differentiable function $\phi(t)$,

$$V^m\phi(t) = V\Big(B_\alpha^{m-1}\phi(t) + \sum_{j=1}^{m-1} B_\alpha^{m-1-j}\phi(t)|_{t\to0^+}\, V^j\Big)$$

$$= B_\alpha B_\alpha^{m-1}\,\phi(t) + B_\alpha^{m-1}\,\phi(t)|_{t\to0^+}\, V + \sum_{j=1}^{m-1} B_\alpha^{m-1-j}\phi(t)|_{t\to0^+}\, V^{j+1}$$

$$= B_\alpha^m\,\phi(t) + \sum_{j=1}^{m} B_\alpha^{m-j}\phi(t)|_{t\to0^+} V^j.$$

The proposition follows by induction.

A number of operational formulas such as those in Theorems 5 and 6 of [5] may be generated by using equation (11). The proofs are similar mutatis mutandis. A generalization of Theorem 5 of [5] is obtained by parametric differentiation.

Proposition 8.

$$\frac{V}{(V-a)^{m+1}} = \frac{\Gamma(\alpha+1)}{m!} t^m (at)^{-\frac{\alpha+m}{2}} I_{\alpha+m}(2\sqrt{at})$$

$$\frac{V}{(V+a)^{m+1}} = \frac{\Gamma(\alpha+1)}{m!} t^m (at)^{-\frac{\alpha+m}{2}} J_{\alpha+m}(2\sqrt{at})$$

where $I_\nu(x)$ and $J_\nu(x)$ are Bessel functions of order ν.

Remarks. 1. All the results of Meller are extendible to the case $\alpha \in (0, \infty)$ via the method given in this paper.

2. The operational calculus may be applied to certain time-varying systems and to Kratzel's problem as done in [6].

3. In [12], a convolution for the operator $\Lambda = t^{-n-1} \frac{d}{dt} t^{n+1} \frac{d}{dt}$, n a natural number is given which is associative, commutative, and distributive with respect to addition. However, the ring under convolution as multiplication contains zero divisors.

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