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GLOBALLY ORTHOGONAL REGULAR FRACTIONS OF THE s^n FACTORIAL

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ABSTRACT

Dependence of the global orthogonality of a regular fraction of the s^n factorial, as introduced by Raktoe et al. [1980], on the basic matrix of contrasts used is brought out and studied in this paper. Some basic matrices for $s = 3, 4, 5, 8$, and a series of other higher values are presented for which any *subspace-type* regular design is globally orthogonal. For $s = 4$, under suitable basic matrices of contrasts, all regular fractions of the 4^n factorial are shown to be globally orthogonal. A similar result for $s = 2$ is obtained by Raktoe et al. [1980]. Finally subspace-type fractions that are globally orthogonal under any basic matrix of contrasts are also identified.

1. INTRODUCTION

Raktoe et al. [1980] mention various open problems for further research, one of which is to characterize the globally orthogonal fractions of the s^n factorial. Some attempts are made in that direction and the present paper is an outcome of those investigations.

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We consider in this paper only the symmetrical prime powered factorial s^n, i.e., the factorial in which each of the n factors has s levels, where s is a prime or a prime power. The levels of a factor are identified with the s elements of the Galois field $GF(s) = F = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$. The s^n treatment combinations are then represented by the s^n vectors of the n-dimensional vector space $F^n = \{\underline{x} = (x_1\ x_2\ \ldots\ x_n)' : x_i \epsilon F$ for all $i = 1, 2, \ldots, n\}$. A subset D of F^n, defined by the solutions of any system of $r (r < n)$ linear equations $B\underline{x} = \underline{c}$, where (i) B is a $r \times n$ matrix whose rows consist of r independent vectors from F^n and (ii) $\underline{c}$ is a given vector from F^n is called a *regular fraction* or *regular design* of the s^n factorial. Note that D is a subspace or a coset of a subspace of the vector space F^n. For this reason, we shall call D a *subspace-type* regular design when $\underline{c} = \underline{0}$ and when $\underline{c} \neq \underline{0}$ D will be called a *coset-type design*. Since a regular design D has s^{n-r} treatment combinations, the name "s^{n-r} fraction" has been reserved for it in the literature.

Let $\underline{\beta}$ denote the $N \times 1$ parametric vector of N factorial effects $\{A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n} : a_i \epsilon F, i = 1, 2, \ldots, n\}$, where $N = s^n$, A_i stands for the i-th factor, and $A_1^0 A_2^0 \ldots A_n^0$ is the general mean effect μ. We shall adopt for our analysis the geometric definition of factorial effects, which in turn depends on the selected basic matrix M of contrasts, e.g.:

$$M^{(s)} = \begin{array}{c} \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{s-1} \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_0 & \alpha_1 & \cdots & \alpha_{s-1} \end{array} \\ \begin{bmatrix} 1 & m_{\alpha_0,\alpha_1} & \cdots & m_{\alpha_0,\alpha_{s-1}} \\ 1 & m_{\alpha_1,\alpha_1} & \cdots & m_{\alpha_1,\alpha_{s-1}} \\ \vdots & \vdots & & \vdots \\ 1 & m_{\alpha_{s-1},\alpha_1} & \cdots & m_{\alpha_{s-1},\alpha_{s-1}} \end{bmatrix}_{s\times s} \end{array},$$

where (i) any two columns are mutually orthogonal and (ii) the rows and columns of $M^{(s)}$ are indexed by the s elements of $GF(s)$.

Following the geometric definition of factorial effects define $X_{D,M^{(s)}}$ to be the $k \times N$ design matrix of a given design D of k points relative to a given basic matrix $M^{(s)}$. The rows of $X_{D,M^{(s)}}$ correspond to the k treatments of D and the columns correspond to the N factorial effects, such that the entry corresponding to the treatment $\underline{x}' = (x_1\ x_2\ \ldots\ x_n)$ and the factorial effect $A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n}$ is given by $m_{\alpha,\gamma}$, where $\alpha = (\Sigma a_i x_i)/\gamma$ and γ is the first non-zero element of the vector $(a_1\ a_2\ \ldots\ a_n)$. A nice property of $X_{D,M^{(s)}}$, for a regular design D, is that each column in it is either proportional or orthogonal to $\underline{1}$, the vector of 1's of appropriate order. Although a proof of this result is available in Raktoe et al. [1980], an alternative proof is given here to familiarize the reader with the type of reasoning which will be used in the various proofs in this paper.

Suppose $B\underline{x} = \underline{c}$ defines an s^{n-r} fraction D. Then in the column of $X_{D,M^{(s)}}$ determined by $A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n}$ and D, the element $m_{\alpha,\gamma}$ of $M^{(s)}$ appears as many times as there are solutions to the system of equations:

$$\left.\begin{aligned} B\underline{x} &= \underline{c} \\ \sum_{i=1}^{n} a_i x_i &= \alpha\gamma \end{aligned}\right\} . \tag{1.1}$$

If $(a_1\ a_2\ \ldots\ a_n)$ does not lie in the row space of B, then $m_{\alpha,\gamma}$ appears s^{n-r-1} times, for all $\alpha \in F$, and hence this column is orthogonal to $\underline{1}$. On the other hand, if $(a_1\ a_2\ \ldots\ a_n)$ lies in the row space of B, then for the particular α that makes (1.1) a consistent system, $m_{\alpha,\gamma}$ appears s^{n-r} times and hence this column is proportional to $\underline{1}$.

Note that in the usual terminology of confounded designs this simply means that for a given regular design under the geometric definition of effects either a factorial effect is unconfounded with the mean or completely confounded with the mean. Indeed all the factorial effects which are completely confounded with the mean are determined by the $(r-1)$ — flat of the projective geometry $PG(n-1, s)$ generated by the r independent rows of B.

Let us now consider all the partitions of $\underline{\beta}' = (\underline{\beta}_1' \vdots \underline{\beta}_2')$ where $\underline{\beta}_1$ is a $N_1 \times 1$ and β_2 is a $N_2 \times 1$, column vector, $N_1 + N_2 = N$, such that (i) the mean μ always belongs to $\underline{\beta}_1$ and is its first entry, and (ii) for a given design D, X_1 is of full column rank, where X_1 is given by

$$X_{D,M^{(s)}} = \left[X_{D,M^{(s)},\underline{\beta}_1} \;\middle|\; X_{D,M^{(s)},\underline{\beta}_2} \right] = \left[X_1 \;\middle|\; X_2 \right].$$

Obviously these two conditions imply that $k \geq N_1$, where k is the number of points in D.

Definition 1.1 [Raktoe et al. [1980]]. A design D will be called *globally orthogonal* iff $X_1'X_1$ is diagonal under all the partitions of $\underline{\beta}'$ that satisfy the above two conditions.

Globally orthogonal designs are important from the viewpoint of constructing optimal designs. For example, from Kiefer (1975) it follows that a globally orthogonal design of the 2^n factorial is d-, a-, and e-optimal. Also globally orthogonal designs lead to uncorrelated estimators of the parameters in $\underline{\beta}_1$ satisfying the two conditions stated above.

Theorem 1.1. [Raktoe et al. [1980]]. *A design* D *of* F^n *is globally orthogonal if and only if any two columns of the design matrix* $X_{D,M^{(s)}}$ *are either proportional or orthogonal to each other.*

Raktoe et al. [1980] have also shown that any regular fraction of the 2^n factorial is globally orthogonal under any basic matrix of contrasts. It is shown here that the same is also true for the 4^n factorial under a suitably chosen basic matrix of contrasts. For $s = 3, 5, 8$ and a series of other higher values basic matrices of contrasts are presented under which any subspace-type regular fraction is globally orthogonal. In addition, those regular designs which are globally orthogonal relative to *any basic matrix* of *contrasts* have been characterized.

2. GLOBAL ORTHOGONALITY AND THE BASIC MATRIX OF CONTRASTS

Let D be a regular s^{n-r} fraction of the s^n factorial defined by the matrix equation $B\underline{x} = \underline{c}$ and let $M^{(s)}$, as before, be a given $s \times s$ basic matrix of contrasts. Let $\underline{u}' = (u_1\ u_2\ \dots\ u_n)$ and $\underline{v}' = (v_1\ v_2\ \dots\ v_n)$ be any two non-zero vectors in F^n and let γ_1, γ_2 respectively be their first non zero elements in F. Let $X_D(\underline{u}), X_D(\underline{v})$ respectively denote the columns in the design matrix $X_{D,M^{(s)}}$ determined by D and the effects $A_1^{u_1} A_2^{u_2} \dots A_n^{u_n}$ and $A_1^{v_1} A_2^{v_2} \dots A_n^{v_n}$. Let $A = [B' \vdots\ \underline{u} \vdots\ \underline{v}]'$. By Theorem 1.1, the global orthogonality of a design D is equivalent to the existence of proportionality or orthogonality between any two columns of the design matrix $X_{D,M^{(s)}}$. The following theorem details when $X_D(\underline{u})$ and $X_D(\underline{v})$ are proportional or orthogonal to each other in terms of the rank of A and the dependency of $\underline{v}'$ and $\underline{u}'$ on the row space of B. In the following $\underline{b}_1', \underline{b}_2', \dots, \underline{b}_r'$ denote the r linearly independent rows of B.

Theorem 2.1. *Let* A *be the augmented matrix given above.*

(i) *if the rank of* A *is* r *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are proportional;*

(ii) *if the rank of* A *is* $r + 2$ *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal;*

(iii) *if the rank of* A *is* $r + 1$ *then (a) if* $\underline{v}'$ *or* $\underline{u}'$ *is in the row space of* B, $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal, (b) if* $\{\underline{u}, \underline{v}\}$ *are linearly dependent in* F^n *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal, (c) if, without loss of generality,* $\underline{v}'$ *lies in the row space of* $[B' \vdots\ \underline{u}]'$ *but* $\underline{v}'$ *does not lie in the row space of* B *and* $\{\underline{u}, \underline{v}\}$ *are linearly independent then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are proportional or orthogonal according as the* γ_1*-st column* $(m_{\alpha_0,\gamma_1}, m_{\alpha_1,\gamma_1}, \dots, m_{\alpha_{s-1},\gamma_1})'$ *of* $M^{(s)}$

is proportional or orthogonal to the permuted γ_2-nd *column* $(m_{\theta_0,\gamma_2}, m_{\theta_1,\gamma_2}, \ldots, m_{\theta_{s-1},\gamma_2})'$ *of* $M^{(s)}$, *where*

$$\theta_j = \left(\frac{d_{r+1}\,\gamma_1}{\gamma_2}\right)\alpha_j + \frac{\sum_{i=1}^{r} d_i c_i}{\gamma_2}, \; j=0,\,1,\,2,\ldots,\,s-1 \quad (2.1)$$

and $\underline{d}' = (d_1, d_2, \ldots, d_r) \neq \underline{0}'$ *and* $d_{r+1} \neq 0$ *are defined by*

$$\underline{v}' = \sum_{i=1}^{r} d_i\,\underline{b}_i' + d_{r+1}\,\underline{u}' \,. \qquad (2.2)$$

Proof. It is fairly easy to see the truth of the first two parts of the theorem. Hence we provide an explicit proof of the third part. Without loss of generality assume $\underline{v}'$ lies in the row space of $[B' \,\vdots\, \underline{u}]'$. Then there exists scalars $d_1, d_2, \ldots, d_r, d_{r+1}$ in F satisfying equation (2.2). Now if $d_{r+1} = 0$ then from (2.2) we get that for any treatment $\underline{x}' = (x_1\ x_2\ \ldots\ x_n)$ in D, the inner product

$\underline{v}' \cdot \underline{x}' = \sum_{i=1}^{r} d_i(\underline{b}_i' \cdot \underline{x}') = \sum_{i=1}^{r} d_i c_i$. Hence $X_D(\underline{v})$ is proportional to $\underline{1}$.

On the other hand, for any $\alpha \in F$, the system of equations $B\underline{x} = \underline{c}$, $\underline{u}' \cdot \underline{x}' = \alpha$ has s^{n-r-1} solutions since the rank of $[B' \,\vdots\, \underline{u}]'$ is equal to $r + 1$. Hence $X_D(\underline{u})$ consists of all the entries of the γ_1-st column of $M^{(s)}$ each repeated s^{n-r-1} times. Hence $X_D(\underline{u})$ is orthogonal to $X_D(\underline{v})$. Suppose next that $\underline{v} = \theta\underline{u}$ for some $\theta \in F$. Then $\gamma_2 = \theta\gamma_1$. Since $\underline{u}$ is a non-null vector we may write $\underline{u}' \cdot \underline{x}' = \theta_0\gamma_1$, then $\underline{v}' \cdot \underline{x}' = \theta\theta_0\gamma_1 = \theta_0\gamma_2$. Hence each treatment in D determines the pair $(m_{\theta_0,\gamma_1}, m_{\theta_0,\gamma_2})$ under the columns $X_D(\underline{u})$ and $X_D(\underline{v})$, each such pair being repeated s^{n-r-1} times. Therefore $X_D(\underline{u})$ and $X_D(\underline{v})$ are orthogonal. Now suppose that $\underline{d} \neq \underline{0}$ and $d_{r+1} \neq 0$. Then from equation (2.2) we have for any $\underline{x}' = (x_1\ x_2\ \ldots\ x_n)$ in D that $\underline{v}' \cdot \underline{x}' = \sum_{i=1}^{r} d_i c_i + d_{r+1}\,(\underline{u}' \cdot \underline{x}')$. Recall that $F = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$. Take any α_j in F. Then since the rank of $[B' \,\vdots\, \underline{u}]'$

is $r+1$, we can select an $\underline{x}'$ in D such that $\underline{u}' \cdot \underline{x}' = \gamma_1\alpha_j$. For this $\underline{x}$, let us write $\underline{v}' \cdot \underline{x}' = \gamma_2\,\theta_j$ for some θ_j in F. Then from the above equation which holds for any $\underline{x}$ we have

$$\theta_j = \left(\frac{d_{r+1}\gamma_1}{\gamma_2}\right)\alpha_j + \left(\frac{\sum_{i=1}^{r} d_i c_i}{\gamma_2}\right).$$

Corresponding to each α_j there are precisely s^{n-r-1} treatments $\underline{x}'$ in D satisfying the equation $\underline{u}' \cdot \underline{x}' = \gamma_1\ \alpha_j$. Hence the columns $X_D(\underline{u})$ and $X_D(\underline{v})$ determine for a given $\underline{x}'$ in D, the pair $(m_{\alpha_j,\gamma_1}, m_{\theta_j,\gamma_2})$, where θ_j is related to α_j as in equation (2.1), each such pair being repeated s^{n-r-1} times. Hence $X_D(\underline{u})$ and $X_D(\underline{v})$ are proportional or orthogonal according as, the γ_1-st column $(m_{\alpha_0,\gamma_1}, m_{\alpha_1,\gamma_1}, \ldots, m_{\alpha_{s-1},\gamma_1})$ of $M^{(s)}$ is proportional or othogonal to the permuted γ_2-nd column $(m_{\theta_0,\gamma_2}, m_{\theta_1,\gamma_2}, \ldots, m_{\theta_{s-1},\gamma_2})'$ of $M^{(s)}$ where θ_j is related to α_j by equation (2.1). This completes the proof.

Write the regular s^{n-r} fraction D as $D = D_0 + \underline{z}' = \{\underline{x}' + \underline{z}' : \underline{x}' \in D_0\}$, where D is defined by the matrix equation $B\underline{x} = \underline{c}$, D_0 is the subspace of the vector space F^n defined by $B\underline{x} = \underline{0}$ and $\underline{z}'$ is in D. For any α in F and $\underline{u}'$ in F^n, let $C_{\underline{u}}(\alpha) = \{\underline{x}' : \underline{x}' \in D_0, \underline{x}' \cdot \underline{u}' = \alpha\}$. Then $C_{\underline{u}}(\alpha_0)$ is a subspace of F^n, where $\alpha_0 = 0$ is the zero element of F. For $\underline{u}, \underline{v}$ nonzero vectors in F^n, let $\delta_1 = \underline{u}' \cdot \underline{z}'$, $\delta_2 = \underline{v}' \cdot \underline{z}'$, $\tau_i = \gamma_1\alpha_i - \delta_1$, $\mu_i = \gamma_2\alpha_i - \delta_2$ $(i=0,1,2,\ldots,s-1)$, where γ_1, γ_2 respectively are the first nonzero elements of $\underline{u}, \underline{v}$ and $F = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$. Observe that τ_i and μ_i respectively range through F as $i = 0,1,2,\ldots,s-1$.

The following may now be easily verified:

(a) if $C_{\underline{u}}(\alpha_0) \neq D_0$, then the cardinality of $C_{\underline{u}}(\alpha_0)$ is s^{n-r-1} and $D = \bigcup_{\alpha\in F} [C_{\underline{u}}(\alpha) + \underline{z}']$, a disjoint union, where $C_{\underline{u}}(\alpha) + \underline{z}' = \{\underline{t}' + \underline{z}' : t' \in C_{\underline{u}}(\alpha)\}$ for each α in F, and $C_{\underline{u}}(\alpha)$ is a coset of $C_{\underline{u}}(\alpha_0)$. Thus in particular

$$D = \bigcup_{i=0}^{s-1} [C_{\underline{u}}(\tau_i) + \underline{z}'] = \bigcup_{i=0}^{s-1} [C_{\underline{v}}(\mu_i) + \underline{z}'] , \tag{2.3}$$

where each of these set unions are pairwise disjoint.

(b) For $\underline{u}$, $\underline{v}$ in F^n, if $C_{\underline{u}}(\alpha_0) = C_{\underline{v}}(\alpha_0) \neq D_0$ then

$$\underline{v}' = \delta\, \underline{u}' + \underline{e}', \tag{2.4}$$

where $\delta \neq 0$ in F and $\underline{e}'$ is a vector in the row space of B.

(c) $C_{\underline{u}}(\alpha_0) = D_0$ iff rank $[B' \vdots \underline{u}]'$ is r, and if $\underline{u}$ lies in the row space of B then $X_D(\underline{u})$ is proportional to $\underline{1}$; otherwise it is orthogonal to $\underline{1}$. Also if $C_{\underline{u}}(\alpha_0) = C_{\underline{v}}(\alpha_0) \neq D_0$ then the rank of A is $r + 1$, and if $C_{\underline{u}}(\alpha_0) \neq C_{\underline{v}}(\alpha_0)$, $C_{\underline{u}}(\alpha_0) \neq D_0$, $C_{\underline{v}}(\alpha_0) \neq D_0$ then the rank of A is $r + 2$.

The theorem below gives conditions for $X_D(\underline{u})$, $X_D(\underline{v})$ to be orthogonal or proportional to each other in terms of the subspaces $C_{\underline{u}}(\alpha_0)$, $C_{\underline{v}}(\alpha_0)$, instead of the rank of A as done in the previous theorem, along the lines of Theorem 4.2 of Raktoe <u>et</u> <u>al</u>. [1980].

Theorem 2.2. (i) *If* $C_{\underline{u}}(\alpha_0) = C_{\underline{v}}(\alpha_0) = D_0$, *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are proportional, indeed each column is proportional to* $\underline{1}$, (ii) *if* $C_{\underline{u}}(\alpha_0) = D_0$, *and* $C_{\underline{v}}(\alpha_0) \neq D_0$ *or* $C_{\underline{u}}(\alpha_0) \neq D_0$ *and* $C_{\underline{v}}(\alpha_0) = D_0$, *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal to each other. Indeed in the first case* $X_D(\underline{u})$ *is proportional to* $\underline{1}$ *and* $X_D(\underline{v})$ *consists of all the entries of the* γ_2*-nd column of* $M^{(s)}$ *each being repeated* s^{n-r-1} *times,* (iii) *if* $C_{\underline{u}}(\alpha_0) \neq D_0$ *and* $C_{\underline{v}}(\alpha_0) \neq D_0$ *then the following holds: (1) if* $C_{\underline{u}}(\alpha_0) \neq C_{\underline{v}}(\alpha_0)$ *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal, (2) if* $C_{\underline{u}}(\alpha_0) = C_{\underline{v}}(\alpha_0)$ *and* $\underline{v} = \delta\underline{u}$ *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are orthogonal, (3) if* $C_{\underline{u}}(\alpha_0) = C_{\underline{v}}(\alpha_0)$ *and* $\underline{v} \neq \delta\underline{u}$ *then* $X_D(\underline{u})$ *and* $X_D(\underline{v})$ *are proportional or orthogonal to each other according as the* γ_1*-st column* $(m_{\alpha_0,\gamma_1}, m_{\alpha_1,\gamma_1}, \ldots, m_{\alpha_{s-1},\gamma_1})'$ *of* $M^{(s)}$ *is proportional or orthogonal to the permuted* γ_2*-nd column* $(m_{\theta_0,\gamma_2}, m_{\theta_1,\gamma_2}, \ldots, m_{\theta_{s-1},\gamma_2})'$ *of* $M^{(s)}$, *where* θ_j *is related to* α_j *by*

$$\theta_j = \left(\frac{\delta\gamma_1}{\gamma_2}\right)\alpha_j + \frac{(\underline{e}' \cdot \underline{z}')}{\gamma_2}, \quad j = 0, 1, 2, \ldots, s-1 . \tag{2.5}$$

Proof. Follows immediately using Theorem 2.1 and part (c) above.

Remark. A proof of Theorem 2.2 can be given independent of rank arguments using (2.3) and (2.4). For instance consider part 3 of (iii) and suppose $\underline{v}' = \delta\underline{u}' + \underline{e}'$, with $\underline{e}' \neq \underline{0}$. Then for any $\underline{x}' \in D$, from (2.3), $\underline{x}' \in C_{\underline{u}}(\tau_j) + \underline{z}'$ for some j. Hence $\underline{x}' \cdot \underline{u}' = \alpha_j$, $\underline{x}' \cdot \underline{v}' = (\delta\gamma_1)\alpha_j + \underline{z}' \cdot \underline{e}'$. Thus each $\underline{x}'$ in D determines the pair $(m_{\alpha_j,\gamma_1}, m_{\theta_j,\gamma_2})$ where θ_j is related to α_j as in (2.5). Moreover, each block $C_{\underline{u}}(\tau_j) + \underline{z}'$ in the decomposition of D given in (2.3), produces the same pair. In all s distinct pairs are produced one for each j with θ_j related to α_j as in (2.5), concluding the proof of part 3 of (iii).

Note that for a subspace-type design D, and any pair of vectors $\underline{u}$, $\underline{v}$ satisfying equation (2.4), if $d_{r+1}\gamma_1 = \gamma_2$ then equation (2.5) reduces to $\theta_j = \alpha_j$ for each j and hence $X_D(\underline{u})$ and $X_D(\underline{v})$ are proportional or orthogonal for any columnwise orthogonal basic matrix $M^{(s)}$ with first column $\underline{1}$.

Further observe that when the vectors $\underline{u}$ and $\underline{v}$ satisfy equation (2.4), which corresponds to case (iii), part 3 of Theorem 2.2 then the proportionality or orthogonality respectively of the columns $X_D(\underline{u})$, $X_D(\underline{v})$ depends on the corresponding condition being satisfied by the γ_1-st and γ_2-nd columns of $M^{(s)}$. This leads to the following:

Corollary 2.1. *A sufficient condition that a regular fraction be globally orthogonal is that the basic matrix* $M^{(s)}$ *possesses the following property:*

(*) the γ_1*-st column* $(m_{\alpha_0,\gamma_1}, \ldots, m_{\alpha_{s-1},\gamma_1})'$ *and the permuted* γ_2*-nd column* $(m_{\theta_0,\gamma_2}, \ldots, m_{\theta_{s-1},\gamma_2})'$ *of* $M^{(s)}$ *must be proportional or orthogonal where* $\theta_j = \gamma\alpha_j + \omega$, $j = 0, 1, 2, \ldots, s-1$, *for all* $\gamma, \gamma_1, \gamma_2$ *in* $F-\{0\}$ *and for all* ω *in* F *such that* $(\gamma,\omega) \neq (1,0)$.

Observe that the Hadamard matrices H_2, H_4 of orders two and four respectively with first columns $\underline{1}$ satisfy the condition imposed on the basic matrix mentioned in Corollary 2.1. Thus these matrices can be used as basic matrices for the 2^n and the 4^n factorials respectively and we then have the following:

Corollary 2.2. (i) [Raktoe et al. [1980]]. *Any regular fraction* D *in the* 2^n *factorial under the basic matrix* H_2 *is globally orthogonal,* (ii) *any regular fraction* D *in the* 4^n *factorial under the basic matrix* H_4 *is globally orthogonal.*

Note that, in fact in the above corollary H_2 and H_4 can be replaced by matrices $\tilde{H}_2$, $\tilde{H}_4$ respectively obtained by post-multiplying H_2, H_4 by diagonal matrices with each diagonal entry nonzero.

A question which now arises is: for which s^n factorials $(s > 2)$ do there exist globally orthogonal designs which are independent of the basic orthogonal matrix used? For co-set type designs this question remains unresolved. However, for subspace-type regular designs we provide the following characterizations.

Let D be a s^{n-r} subspace-type regular fraction defined by $B\underline{x} = 0$. Let the j-th column of B be its first non-null column. Then, clearly $1 \le j \le n - r + 1$. Now we have the following:

Theorem 2.3. *When* $j < n - r + 1$, D *is globally orthogonal if and only if the basic matrix of contrasts* $M^{(s)}$ *satisfies condition* (*) *of Corollary (2.1) with* $\theta_k = \gamma\,\alpha_k$, $\omega = 0$, γ_1, γ_2 *in* $F-\{0\}$ *and* γ *in* $F-\{0,1\}$.

Proof. The sufficiency follows from Corollary 2.1. To show the necessity part, assume there exists γ_1, γ_2 not zero in F and γ in $F-\{0,1\}$ such that the γ_1-st column $(m_{\alpha_0,\gamma_1}, \ldots, m_{\alpha_{s-1},\gamma_1})'$ of $M^{(s)}$ is neither proportional nor orthogonal to the permuted γ_2-nd column $(m_{\theta_0,\gamma_2}, \ldots, m_{\theta_{s-1},\gamma_2})'$ where $\theta_k = \gamma\,\alpha_k$, $k = 0, 1, 2, \ldots, s-1$. Since $j < n - r + 1$, we can always select a vector $\underline{u}$ in F^n such that its first $j-1$ entries are zero and its first nonzero entry is γ_1. Define d_{r+1} by $d_{r+1} = \frac{\gamma\,\gamma_2}{\gamma}$. Determine $\underline{d}' = (d_1, d_2, \ldots, d_r)$ as a solution to the equation $\underline{d}' \cdot \underline{b}_j' = \gamma_2$ where $\underline{b}_j$ is the j-th column of B. Such a solution always exists as $\underline{b}_j \ne \underline{0}$. Define $\underline{v}'$ by $\underline{v}' = \sum_{i=1}^{r} d_i\, \underline{a}_i' + d_{r+1}\, \underline{u}'$, where $\underline{a}_i'$ is the i-th row of B. Then clearly for these $\underline{u}$, $\underline{v}$, the columns $X_D(\underline{u})$ and $X_D(\underline{v})$ are neither proportional nor orthogonal in view of Theorem 2.1 or Theorem 2.2. This completes the proof.

In the next result we consider the case when the first nonnull column j of the defining matrix B of the subspace-type s^{n-r} fraction D, satisfies $j = n - r + 1$.

Theorem 2.4. *Let* D *be a subspace-type* s^{n-r} *fraction with* $s \neq 2$. *Then* D *is globally orthogonal under any columnwise orthogonal matrix* $M^{(s)}$ *with a leading column of ones if and only if* $j = n - r + 1$, *where the* j*-th column of the defining matrix* B *of* D *is its first nonnull column.*

Proof. The necessity follows from Theorem 2.3. To show the sufficiency assume $j = n - r + 1$. Then without loss of generality, we have $B = [0 \vdots I_r]$, where 0 is the $r \times n - r$ zero matrix and I_r is the identify matrix of order r. Then clearly any vector $\underline{u}'$ not in the row space of B must have at least one nonzero element among its first $n - r$ coordinates. Take such a vector $\underline{u}'$ with its first nonzero entry γ_1 and define a vector $\underline{v}$ by $\underline{v}' = \sum_{i=1}^{r} d_i \underline{b}_i' + d_{r+1} \underline{u}'$,

where $(d_1, d_2, \ldots, d_r) \neq \underline{0}'$ and $d_{r+1} \neq 0$ are chosen arbitrarily from F. Then $d_{r+1} \gamma_1$ is the first nonzero entry of $\underline{v}'$ and for such $\underline{u}'$, $\underline{v}'$ it follows that $\theta_k = \alpha_k$, $k = 0, 1, 2, \ldots, s-1$. Hence for such vectors $\underline{u}$, $\underline{v}$, namely, those satisfying equation (2.4), the columns $X_D(\underline{u})$ and $X_D(\underline{v})$ are either proportional or orthogonal to each other. Thus by Theorem 1.1, D is globally orthogonal. This completes the proof.

Note that when D is a subspace-type s^{n-r} fraction then when $j < n - r + 1$, the property of global orthogonality of D is equivalent to the basic matrix $M^{(s)}$ satisfying the condition given in Theorem 2.3 and thus in this case the global orthogonality of D is in this sense $M^{(s)}$-dependent. If $j = n - r + 1$, then according to Theorem 2.4, the global orthogonality of D is $M^{(s)}$-independent and moreover in this case observe that $D = F^{n-r} \times \{0\}^r$ where F^{n-r} and $\{0\}^r$ are the respective Cartesian products of F with itself and $\{0\}$ with itself to $n - r$ and r factors respectively.

3. CONSTRUCTION OF BASIC ORTHOGONAL MATRICES

In the previous section we saw that the basic matrix $M^{(s)}$ plays a significant role in the construction of globally orthogonal designs. Hence in this section we study such matrices in some detail.

Throughout this section, the following notation will be used: (i) x will be a primitive element of GF(s), (ii) the rows of $M^{(s)}$ will be labelled in the following way: the first row is labelled by $\alpha_0 = 0$, the second row by $\alpha_1 = 1$ and for each $i(1 \le i \le s - 1)$, the $(i + 1)$st row is labelled by $\alpha_i = x^{i-1}$. If $F = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}) = (0\ 1\ x\ x^2\ \ldots\ x^{s-1})$ is the cyclic permutation sending $\alpha_0 \to \alpha_1$, $\alpha_1 \to \alpha_2$, $\ldots$, $\alpha_{s-2} \to \alpha_{s-1}$, $\alpha_{s-1} \to \alpha_0$ then for each $\phi \neq 0$ in F we define $F(\phi) = (\phi\alpha_0, \phi\alpha_1, \ldots, \phi\alpha_{s-1})$ to mean the cyclic permutation sending $\phi\alpha_0 \to \phi\alpha_1$, $\ldots$, $\phi\alpha_{s-2} \to \phi\alpha_{s-1}$, $\phi\alpha_{s-1} \to \phi\alpha_0$ and we call $F(\phi)$ the *cyclic multiplicative permutation* (cmp) determined by ϕ.

If $M^{(s)} = (m_{ij})_{s\times s}$ is a basic matrix, relative to which every subspace-type design is globally orthogonal, then according to Theorem 2.3 of the previous section, the columns of $M^{(s)}$ must satisfy the following conditions:

(i) $m_{i1} = 1$, for all $i = \alpha_0, \alpha_1, \ldots, \alpha_{s-1}$, and all other column vectors are nonnull,

(ii) the inner product $\underline{m}_j' \cdot \underline{m}_k' = 0$, for all $j \neq k$, where $\underline{m}_j$, $\underline{m}_k$ are the j-th and k-th columns of $M^{(s)}$,

(iii) for all j, k in $\{2, 3, \ldots, s\}$ and all $\phi \neq 0$ in F, the j-th column $(m_{\alpha_0 j}, m_{\alpha_1 j}, \ldots, m_{\alpha_{s-1} j})'$ and the permuted k-th column $(m_{\theta_0,k}, m_{\theta_1,k}, \ldots, m_{\theta_{s-1},k})'$ are either proportional or orthogonal to each other where $F(\phi) = (\theta_0, \theta_1, \ldots, \theta_{s-1})$ is the cmp determined by ϕ.

Observe that if $M^{(s)}$ is a matrix satisfying (i), (ii) and (iii) then so does $M^{(s)}V$ also where V is a diagnol matrix, $V = \text{diag}\ \{d_1, d_2, \ldots, d_s\}$ with $d_1 = 1$ and all other $d_i \neq 0$.

Let $\underline{x}' = (x_1, x_2, \ldots, x_m)'$ be any nonzero real contrast vector and let $\tau_k(\underline{x}') = (x_{\tau_k(1)}, x_{\tau_k(2)}, \ldots, x_{\tau_k(m)})'$ where τ_k is a permutation on $\{1, 2, \ldots, m\}$ defined by $\tau_k(i) = i + k \pmod{m}$, for $k = 1, 2, \ldots, m-1$.

The following lemma will be useful to prove certain results of this section:

Lemma 3.1. *When* s *is a prime greater than* 2, *then the contrast vector* $\underline{x}'$ *is neither proportional nor orthogonal to each member in the set* $\{\tau_k(\underline{x}') : k = 1, 2, \ldots, s-1\}$, *where* $\underline{x}' = (x_1, x_2, \ldots, x_s)$.

Proof. Suppose that for some k, $\tau_k(\underline{x}') = c\,\underline{x}'$ for some real nonzero constant c. Then it follows that $\tau_k^s(\underline{x}') = c^s(\underline{x}')$. Since s is a prime, τ_k^s is the identity permutation and hence $c^s = 1$. The only real root of this equation is $c = 1$ and hence $\tau_k(\underline{x}') = \underline{x}'$ implying equality of all the components of $\underline{x}$. However, this is impossible and $\underline{x}'$ and $\tau_k(\underline{x}')$ can never be proportional for any prime $s > 2$, wherever $\underline{x}$ is a non zero contrast vector.
Hence for each $k = 1, 2, \ldots, s-1$ we must have that $\underline{x}' \cdot \tau_k(\underline{x}') = 0$. Then by substituting $k = 1, 2, \ldots, s-1$ in this inner product equation one obtains $s-1$ homogeneous equations which upon summing results in $(-x_1^2) + (-x_2^2) + \ldots + (-x_s^2) = 0$, since $\underline{x}'$ is a contrast vector. This implies $\underline{x} = \underline{0}$, However, $\underline{x} \neq \underline{0}$ and the lemma follows.

Of course when s is not a prime, there exist many contrast vectors $\underline{x}$ such that $\underline{x}'$ and $\tau_k(\underline{x}')$ are either proportional or orthogonal to each other for each $k = 1, 2, \ldots, s-1$. We shall now describe a method of construction of $M^{(s)}$ satisfying (i), (ii), (iii) utilizing such contrasts vectors in the next theorem. The proof of this theorem is immediate from direct computation.

Theorem 3.1. *Let* A *be a* $(s-1) \times (s-2)$ *matrix whose columns are contrast vectors of order* $s-1$ *such that if* $\underline{a}$ *and* $\underline{b}$ *are any two columns of* A *(not necessarily distinct) then* $\underline{a}$ *is either proportional or orthogonal to* $\tau_k(\underline{b})$ *for each* $k = 1, 2, \ldots, s-1$.

Then the matrix

$$M^{(s)} = \left[\begin{array}{c|c|c} 1 & -(s-1) & \underline{0}' \\ \hline & & \\ \underline{1} & \underline{1} & A \\ & & \end{array}\right]_{s\times s}$$

satisfies the conditions (i), (ii) *and* (iii) *above.*

The following example illustrates Theorem 3.1 for $s = 5$.

$$M^{(5)} = \begin{bmatrix} 1 & -4 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

If this $M^{(5)}$ is used as a basic orthogonal matrix for the 5^n factorial, then any subspace-type design of the 5^n factorial will be globally orthogonal.

The following theorem provides necessary conditions for the existence of a basic orthogonal matrix $M^{(s)}$ satisfying conditions (i), (ii), (iii) in case $s-1$ is a prime number.

Theorem 3.2. *For any* s $(s \neq 3)$ *a prime or prime power such that* $s-1$ *is a prime, if* $M^{(s)}$ *exists satisfying conditions* (i), (ii), (iii) *above then*

$$M^{(s)} = \left[\begin{array}{c|c|c|c|c|c} 1 & x_1 & x_1 & x_1 & \cdots & x_1 \\ \hline \underline{1} & \underline{y}_1 & \delta_1(\underline{y}_1) & \delta_2(\underline{y}_1) & \cdots & \delta_{s-2}(\underline{y}_1) \end{array}\right] V,$$

where $V = \text{diag.}\ \{1, c_1, c_2, \ldots, c_{s-1}\}$, c_i *nonzero real,*

$$\underline{y}_1' = (y_{11}\ y_{21}\ \ldots\ y_{(s-1)1}),\quad \delta_k(\underline{y}_1') = (y_{\delta_k(1)1}\ y_{\delta_k(2)1}\ \cdots\ y_{\delta_k(s-1)1}),$$

and δ_k *are permutations on* $\{1, 2, \ldots, s-1\}$ *defined by* $\delta_k(i) = i + k(\mathrm{mod}(s-1))$ *for* $k = 1, 2, \ldots, s-2$.

Proof. Let $M^{(s)} = [\underline{1} \vdots A]$ be an $s \times s$ matrix that satisfies conditions (i), (ii) and (iii). Let $\underline{a} = (a_1\ a_2\ \ldots\ a_s)'$ and $\underline{b} = (b_1\ b_2\ \ldots\ b_s)'$ be any two distinct columns of A. Since $(s-1)$ is a prime, by Lemma 3.1, no element in the first row of M can be zero. By the same lemma, the contrast vector $\underline{a}$ must be of the form $\gamma(-(s-1)\ 1\ 1\ 1\ \ldots\ 1)$, $\gamma \neq 0$ if it is proportional to any of its own cmp. But then $\underline{a}$ orthogonal to $\underline{b}$ implies $b_1 = 0$. Therefore, each column of A must be orthogonal to all of its own cmp's. Now since $\underline{a}$ is orthogonal to $\underline{b}$ and if $\underline{a}$ is orthogonal to all of its own $(s-1)$ cmp's then $a_1 b_1 = 0$. Hence $\underline{a}$ must be proportional to at least one cmp. of $\underline{b}$. And if $\underline{a}$ is proportional to one cmp. of $\underline{b}$ then automatically it is orthogonal to the remaining cmp's of $\underline{b}$, since $\underline{b}$ is orthogonal to all its own cmp. Hence $\underline{a}$ must be proportional to exactly one cmp of $\underline{b}$ and orthogonal to all other cmp's of $\underline{b}$. This establishes the theorem.

Hence whenever $s-1$ is a prime greater than 2, the existence of an $s \times s$ matrix $M^{(s)}$ satisfying the conditions (i), (ii) and (iii) depends on the existence of a matrix

$$M^{*}(s) \quad \begin{bmatrix} x_1 & x_1 & \cdots & x_1 \\ x_2 & x_s & \cdots & x_3 \\ x_3 & x_2 & \cdots & x_4 \\ \vdots & \vdots & & \vdots \\ x_s & x_{s-1} & \cdots & x_2 \end{bmatrix}_{s\times(s-1)}$$

of $(s-1)$ mutually orthogonal contrast vectors of dimension s such that the 2nd, 3rd, ..., $(s-1)$th columns are obtained by applying all possible cmp's to the first column. One may ask whether matrices like $M^{*}(s)$ exist or not. The answer is in the affirmative. For terminology and other aspects of block designs refer to Raghavarao [1971]. In the following result, we describe a method of construction of a series of such $M^{*}(s)$ matrices:

Theorem 3.3. *Let there exist a symmetric* BIB *design, constructible through an "initial block" by the method of differences, with parameters* v, k, λ *which satisfy* (i) $v = 2k + 1$ *and* (ii) $v - 4(k - \lambda) + 1 = 0$. *Let* N *be the incidence matrix of such a design so that the columns of* N *are obtained by cyclically permuting its first column. Define* N^* *to be the matrix obtained from* N *by replacing its* 0's *by* -1's . *Then*

$$M^* = \left[\begin{array}{c|c} 1 & \underline{1}' \\ \hline \underline{1} & N^* \end{array}\right]$$

has the desired properties.

Proof. Since under any two columns of N, the pairs (0,0), (0,1), (1,0) and (1,1) appear respectively $v - 2k + \lambda$, $(k-\lambda)$, $(k-\lambda)$ and λ times, the inner product of any two columns of N^* is equal to $v - 4(k-\lambda) = -1$. Also each column of N^* has k plus ones and $(k+1)$ minus ones. Hence the theorem follows.

The result below is an immediate consequence of Theorem 3.3. It provides a series of $\overset{*}{M}(s)$-matrices for certain values of s.

Corollary 3.1. *A matrix* $\overset{*}{M}(s)$ *of order* s *where* $s-1 = 4t + 3$, *a prime number always exists and can be constructed from the incidence matrix of a symmetric* BIBD $(v = 4t + 3, k = 2t + 1, \lambda = t)$ *obtained through the development of the initial block* $(x^0, x^2, x^4, \ldots, x^{4t})$ *or* $(x^1, x^3, x^5, \ldots, x^{4t+1})$, *where* x *is a primitive element of* GF$(4t + 3)$.

To illustrate the Corollary, let $s = 8$. Now $x = 3$ is a primitive element of GF(7), and thus {1, 2, 4} is an "initial block" of the SBIBD $(v = 7, k = 3, \lambda = 1)$. From the incidence matrix of this SBIBD and the above results we have

$$M^*(8) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Using the above matrix $M^*(8)$, we may conclude that all subspace-type designs of the 8^n factorial are globally orthogonal.

Note that if one allows $\lambda = 0$, then an $M^*(4)$ also exists, which is given by $M^*(4) = H_2 \otimes H_2$ with $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Finally observe that the matrices $M^*(s)$ described in Corollary 3.1 are normalized Hadamard matrices. We do not know, however, whether every normalized Hadamard matrix can be arranged as a M^*-matrix.

4. DISCUSSION

In this paper we have focused attention to global orthogonality of subspace-type of designs in relation to existence and construction of basic orthogonal matrices. It is desirable to carry out a similar study for coset-type of designs. In both cases further research will shed light on the intimate relationship between group-theoretic generation of the aliasing structure and global orthogonality of designs. All these problems are currently under study.

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