Globally Orthogonal Regular Fractions of the sn Factorial

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FRACTIONS OF THE $s^n$ FACTORIAL

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Key Words & Phrases: factorial experiments, fractional replication, globally orthogonal fractions.

ABSTRACT

Dependence of the global orthogonality of a regular fraction of the $s^n$ factorial, as introduced by Raktoe et al. [1980], on the basic matrix of contrasts used is brought out and studied in this paper. Some basic matrices for $s = 3, 4, 5, 8$, and a series of other higher values are presented for which any subspace-type regular design is globally orthogonal. For $s = 4$, under suitable basic matrices of contrasts, all regular fractions of the $4^n$ factorial are shown to be globally orthogonal. A similar result for $s = 2$ is obtained by Raktoe et al. [1980]. Finally, subspace-type fractions that are globally orthogonal under any basic matrix of contrasts are also identified.

1. INTRODUCTION

Raktoe et al. [1980] mention various open problems for further research, one of which is to characterize the globally orthogonal fractions of the $s^n$ factorial. Some attempts are made in that direction and the present paper is an outcome of those investigations.

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We consider in this paper only the symmetrical prime powered factorial \( s^n \), i.e., the factorial in which each of the \( n \) factors has \( s \) levels, where \( s \) is a prime or a prime power. The levels of a factor are identified with the \( s \) elements of the Galois field \( GF(s) = F = \{ \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1} \} \). The \( s^n \) treatment combinations are then represented by the \( s^n \) vectors of the \( n \)-dimensional vector space \( F^n = \{ x = (x_1 x_2 \ldots x_n)' : x_i \in F \ \text{for all} \ i = 1, 2, \ldots, n \} \). A subset \( D \) of \( F^n \), defined by the solutions of any system of \( r(r < n) \) linear equations \( Bx = c \), where (i) \( B \) is an \( r \times n \) matrix whose rows consist of \( r \) independent vectors from \( F^n \) and (ii) \( c \) is a given vector from \( F^n \) is called a regular fraction or regular design of the \( s^n \) factorial. Note that \( D \) is a subspace or a coset of a subspace of the vector space \( F^n \). For this reason, we shall call \( D \) a subspace-type regular design when \( \mathbf{c} = \mathbf{0} \) and when \( \mathbf{c} \neq \mathbf{0} \) \( D \) will be called a coset-type design. Since a regular design \( D \) has \( s^{n-r} \) treatment combinations, the name "\( s^n \)-\( r \) fraction" has been reserved for it in the literature.

Let \( \mathbf{\beta} \) denote the \( N \times 1 \) parametric vector of \( N \) factorial effects

\[
A^1 A^2 \ldots A^n : a_i \in F, i = 1, 2, \ldots, n, \quad \text{where} \quad N = s^n, A_1 \text{ stands for} \text{the} i-th \text{factor, and } A_1^0 A_2^0 \ldots A_n^0 \text{ is the general mean effect } u. \]

We shall adopt for our analysis the geometric definition of factorial effects, which in turn depends on the selected basic matrix \( M \) of contrasts, e.g.:

\[
M(s) = \begin{bmatrix}
\alpha_0 & \alpha_1 & \cdots & \alpha_{s-1} \\
1 & m_{\alpha_0, \alpha_1} & \cdots & m_{\alpha_0, \alpha_{s-1}} \\
\alpha_1 & 1 & m_{\alpha_1, \alpha_1} & \cdots & m_{\alpha_1, \alpha_{s-1}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\alpha_{s-1} & 1 & m_{\alpha_{s-1}, \alpha_1} & \cdots & 1 \\
\end{bmatrix}_{s \times s}
\]
where (i) any two columns are mutually orthogonal and (ii) the rows and columns of \( M(s) \) are indexed by the \( a \) elements of \( CF(s) \).

Following the geometric definition of factorial effects define \( X_{D,M(s)} \) to be the \( k \times N \) design matrix of a given design \( D \) of \( k \) points relative to a given basic matrix \( M(s) \). The rows of \( X_{D,M(s)} \) correspond to the \( k \) treatments of \( D \) and the columns correspond to the \( N \) factorial effects, such that the entry corresponding to the treatment \( x'_i = (x_{1i}, x_{2i}, \ldots, x_{ni}) \) and the factorial effect \( A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n} \) is given by \( m_{a,y} \), where \( \alpha = (x_{1i}) / \gamma \) and \( \gamma \) is the first non-zero element of the vector \((a_1, a_2, \ldots, a_n)\). A nice property of \( X_{D,M(s)} \), for a regular design \( D \), is that each column in it is either proportional or orthogonal to \( 1 \), the vector of \( 1 \)'s of appropriate order. Although a proof of this result is available in Rak toe et al. [1980], an alternative proof is given here to familiarize the reader with the type of reasoning which will be used in the various proofs in this paper.

Suppose \( Bx = c \) defines an \( s^{n-r} \) fraction \( D \). Then in the column of \( X_{D,M(s)} \) determined by \( A_1^{a_1} A_2^{a_2} \ldots A_n^{a_n} \) and \( D \), the element \( m_{\alpha,y} \) of \( M(s) \) appears as many times as there are solutions to the system of equations:

\[
\begin{align*}
Bx &= c \\
\sum_{i=1}^{n} a_i x_i &= \gamma \\
\end{align*}
\]

(1.1)

If \((a_1, a_2, \ldots, a_n)\) does not lie in the row space of \( B \), then \( m_{\alpha,y} \) appears \( s^{n-r} \) times, for all \( \alpha \in F \), and hence this column is orthogonal to \( 1 \). On the other hand, if \((a_1, a_2, \ldots, a_n)\) lies in the row space of \( B \), then for the particular \( \alpha \) that makes (1.1) a consistent system, \( m_{\alpha,y} \) appears \( s^{n-r} \) times and hence this column is proportional to \( 1 \).

Note that in the usual terminology of confounded designs this simply means that for a given regular design under the geometric definition of effects either a factorial effect is unconfounded with
the mean or completely confounded with the mean. Indeed all the factorial effects which are completely confounded with the mean are determined by the \((r-1)\) flat of the projective geometry \(\text{PG}(n-1, s)\) generated by the \(r\) independent rows of \(B\).

Let us now consider all the partitions of \(\bar{\beta}' = (\bar{\beta}'; \bar{\beta}')\) where \(\bar{\beta}_1\) is a \(N_1 \times 1\) and \(\bar{\beta}_2\) is a \(N_2 \times 1\) column vector, \(N_1 + N_2 = N\), such that (i) the mean \(\mu\) always belongs to \(\bar{\beta}_1\) and is its first entry, and (ii) for a given design \(D\), \(X_{1}\) is of full column rank, where \(X_{1}\) is given by

\[
X_{D,\mathcal{M}(s)} = \begin{bmatrix}
    X_{D,\mathcal{M}(s)} & \bar{\beta}_1 \\
    X_{D,\mathcal{M}(s)} & \bar{\beta}_2
  \end{bmatrix} = \begin{bmatrix}
    X_{1} \mid X_{2}
  \end{bmatrix}.
\]

Obviously these two conditions imply that \(k \geq N_1\), where \(k\) is the number of points in \(D\).

**Definition 1.1** [Raktoe et al. [1980]]. A design \(D\) will be called **globally orthogonal** iff \(X'X_{1}\) is diagonal under all the partitions of \(\bar{\beta}'\) that satisfy the above two conditions.

Globally orthogonal designs are important from the viewpoint of constructing optimal designs. For example, from Kiefer (1975) it follows that a globally orthogonal design of the \(2^n\) factorial is \(d\)-, \(a\)-, and \(e\)-optimal. Also globally orthogonal designs lead to uncorrelated estimators of the parameters in \(\bar{\beta}_1\) satisfying the two conditions stated above.

**Theorem 1.1.** [Raktoe et al. [1980]]. A design \(D\) of \(F^N\) is globally orthogonal if and only if any two columns of the design matrix \(X_{D,\mathcal{M}(s)}\) are either proportional or orthogonal to each other.

Raktoe et al. [1980] have also shown that any regular fraction of the \(2^n\) factorial is globally orthogonal under any basic matrix of contrasts. It is shown here that the same is also true for the \(4^n\) factorial under a suitably chosen basic matrix of contrasts. For \(s = 3, 5, 8\) and a series of other higher values basic matrices of contrasts are presented under which any subspace-type regular fraction is globally orthogonal. In addition, those regular designs which are
globally orthogonal relative to any basic matrix of contrasts have 
been characterized.

2. GLOBAL ORTHOGONALITY AND THE BASIC MATRIX OF CONTRASTS

Let $D$ be a regular $s^{n-r}$ fraction of the $s^n$ factorial defined 
by the matrix equation $Bx = c$ and let $M(s)$, as before, be a given 
$s \times s$ basic matrix of contrasts. Let $u' = (u_1, u_2, \ldots, u_n)$ and 
v' = $(v_1, v_2, \ldots, v_n)$ be any two non-zero vectors in $F^n$ and let 
$Y_1, Y_2$ respectively be their first non-zero elements in $F$. Let 
$X_D(u), X_D(v)$ respectively denote the columns in the design matrix 
$X_{D, M(s)}$ determined by $D$ and the effects $A_1, A_2, \ldots, A_n$ and 
v' and v, respectively. Let $A = [B': u: v]'$. By Theorem 1.1, the global 
orthogonality of a design $D$ is equivalent to the existence of 
proportionality or orthogonality between any two columns of the 
design matrix $X_{D, M(s)}$. The following theorem details when $X_D(u)$ and 
$X_D(v)$ are proportional or orthogonal to each other in terms of the 
rank of $A$ and the dependency of $v'$ and $u'$ on the row space of $B$. 
In the following $b_1', b_2', \ldots, b_r'$ denote the $r$ linearly 
independent rows of $B$.

Theorem 2.1. Let $A$ be the augmented matrix given above.

(i) if the rank of $A$ is $r$ then $X_D(u)$ and $X_D(v)$ are 
proportional;

(ii) if the rank of $A$ is $r + 2$ then $X_D(u)$ and $X_D(v)$ are 
orthogonal;

(iii) if the rank of $A$ is $r + 1$ then (a) if $v'$ or $u'$ is in the 
row space of $B$, $X_D(u)$ and $X_D(v)$ are orthogonal, (b) if 
$(u, v)$ are linearly dependent in $F^n$ then $X_D(u)$ and $X_D(v)$ 
are orthogonal, (c) if, without loss of generality, $v'$ lies 
in the row space of $[B': u']$, but $v'$ does not lie in the 
row space of $B$ and $(u, v)$ are linearly independent then 
$X_D(u)$ and $X_D(v)$ are proportional or orthogonal according as 
the $Y_1$-st column $(m_0, Y_1, m_1, Y_1, \ldots, m_{s-r}, Y_1)'$ of $M(s)$
is proportional or orthogonal to the permuted \( \gamma_2 \)-nd column 
\[
(m_0, \gamma_2, m_1, \gamma_2, \ldots, m_{s-1}, \gamma_2)^t
\]
of \( M(\bar{s}) \), where
\[
\theta_j = \left( \frac{d_{r+1}}{\gamma_2} \right) \alpha_j + \sum_{i=1}^{r} \frac{d_i c_{i1}}{\gamma_2}, \quad j = 0, 1, 2, \ldots, s-1 \quad (2.1)
\]
and \( d' = (d_1, d_2, \ldots, d_r) \neq 0' \) and \( d_{r+1} = 0 \) are defined by
\[
v' = \sum_{i=1}^{r} d_i b_i + d_{r+1} u'.
\quad (2.2)
\]

Proof. It is fairly easy to see the truth of the first two parts of the theorem. Hence we provide an explicit proof of the third part. Without loss of generality assume \( v' \) lies in the row space of \( [B' : u]' \). Then there exists scalars \( d_1, d_2, \ldots, d_r, d_{r+1} \) in \( F \) satisfying equation (2.2). Now if \( d_{r+1} = 0 \) then from (2.2) we get that for any treatment \( x' = (x_1 x_2 \ldots x_n) \) in \( D \), the inner product
\[
v' \cdot x' = \sum_{i=1}^{r} d_i (b_i' \cdot x') = \sum_{i=1}^{r} d_i c_{i1}.
\]
Hence \( X_D(v) \) is proportional to \( 1 \).

On the other hand, for any \( \alpha \in F \), the system of equations \( Bx = \bar{c} \), \( u' \cdot x' = \alpha \) has \( \bar{s} \)-r-1 solutions since the rank of \( [B' : u]' \) is equal to \( r+1 \). Hence \( X_D(u) \) consists of all the entries of the \( \gamma_1 \)-st column of \( M(\bar{s}) \) each repeated \( \bar{s} \)-r-1 times. Hence \( X_D(u) \) is orthogonal to \( X_D(v) \). Suppose next that \( v = \theta u \) for some \( \theta \in F \). Then \( \gamma_2 = \theta \gamma_1 \). Since \( u \) is a non-null vector we may write \( u' \cdot x' = \theta \gamma_1 \), then \( v' \cdot x' = \theta \theta \gamma_1 = \theta \gamma_2 \). Hence each treatment in \( D \) determines the pair \( (m_0, \gamma_1, m_1, \gamma_2) \) under the columns \( X_D(u) \) and \( X_D(v) \), each such pair being repeated \( \bar{s} \)-r-1 times. Therefore \( X_D(u) \) and \( X_D(v) \) are orthogonal. Now suppose that \( d \neq 0 \) and \( d_{r+1} \neq 0 \). Then from equation (2.2) we have for any \( x' = (x_1 x_2 \ldots x_n) \) in \( D \)
\[
v' \cdot x' = \sum_{i=1}^{r} d_i c_{i1} + d_{r+1} (u' \cdot x').
\]
Recall that \( F = \{ \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1} \} \). Take any \( \alpha_j \) in \( F \). Then since the rank of \( [B' : u]' \),
is \( r + 1 \), we can select an \( x' \) in \( D \) such that \( u' \cdot x' = \gamma_1 \alpha_j \). For this \( x' \), let us write \( u' \cdot x' = \gamma_1 \theta_j \) for some \( \theta_j \) in \( F \). Then from the above equation which holds for any \( x \) we have

\[
\theta_j = \frac{d + 1}{\gamma_2} \gamma_1 \alpha_j + \frac{1}{1} \frac{d + 1}{\gamma_2} \gamma_1 \gamma_2 
\]

Corresponding to each \( \alpha_j \), there are precisely \( s^{n-r-1} \) treatments \( x' \) in \( D \) satisfying the equation \( u' \cdot x' = \gamma_1 \alpha_j \). Hence the columns \( X_D(u) \) and \( X_D(v) \) determine for a given \( x' \) in \( D \), the pair \( (m_{\alpha_j}, m_{\theta_j}, \gamma_2) \), where \( \theta_j \) is related to \( \alpha_j \) as in equation (2.1), each such pair being repeated \( s^{n-r-1} \) times. Hence \( X_D(u) \) and \( X_D(v) \) are proportional or orthogonal according as, the \( \gamma_1 \)-st column \( (m_{\alpha_j}, m_{\theta_j}, \gamma_2) \), \( s-1 \), \( \gamma_2 \)-rd column \( (m_{\theta_j}, m_{\gamma_2}, \gamma_2, \ldots, m_{\theta_j}, \gamma_2) \) of \( M(s) \), where \( \theta_j \) is related to \( \alpha_j \) by equation (2.1). This completes the proof.

Write the regular \( \alpha^{n-r} \) fraction \( D \) as \( D = D_0 + z' \)

\( \{ x' + z' : x' \in D_0 \} \), where \( D \) is defined by the matrix equation

\( \sum_{x = 0}^n \) is the subspace of the vector space \( F^n \) defined by

\( Bx = 0 \) and \( z' \) is in \( D \). For any \( \alpha \) in \( F \) and \( u' \) in \( F^n \), let

\( C_u(\alpha) = \{ x' : x' \in D_0 \} \cdot \gamma_2 \cdot u' = \alpha \}. \) Then \( C_u(\alpha) \) is a subspace of \( F^n \), where \( \lambda = 0 \) is the zero element of \( F \). For \( u, v \) nonzero vectors in \( F^n \), let \( \delta_1 = u' \cdot z', \delta_2 = v' \cdot z', \tau_1 = \gamma_1 \alpha_1 - \delta_1 \),

\( \mu_1 = \gamma_2 \alpha_1 - \delta_2 \) \( (i = 1, 2, \ldots, s-1) \), where \( \gamma_1, \gamma_2 \) respectively are the first nonzero elements of \( u \) and \( v \) and \( F = \{ \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1} \} \). Observe that \( \tau_1 \) and \( \mu_1 \) respectively range through \( F \) as \( i = 0, 1, 2, \ldots, s-1 \).

The following may now be easily verified:

(a) if \( C_u(\alpha) = D_0 \), then the cardinality of \( C_u(\alpha) \) is \( s^{n-r-1} \)

and \( D = u [C_u(\alpha) + z'] \), a disjoint union, where

\( C_u(\alpha) + z' = \{ x' + z' : x' \in C_u(\alpha) \} \) for each \( \alpha \) in \( F \), and

\( C_u(\alpha) \) is a coset of \( C_u(\alpha_0) \). Thus in particular
\[ D = \bigcup_{i=0}^{s-1} [C_u(\tau_i) + z'] = \bigcup_{i=0}^{s-1} [C_v(\mu_i) + z'], \quad (2.3) \]

where each of these set unions are pairwise disjoint.

(b) For \(\mu, \nu\) in \(F^n\), if \(C_u(\alpha_0) = C_v(\alpha_0) = D_0\) then
\[ \nu' = \delta \mu' + e', \quad (2.4) \]
where \(\delta = 0\) in \(F\) and \(e'\) is a vector in the row space of \(B\).

(c) \(C_u(\alpha_0) = D_0\) iff rank \([B':u']\) is \(r\), and if \(u\) lies in the row space of \(B\) then \(X_D(u)\) is proportional to \(1\); otherwise it is orthogonal to \(1\). Also if \(C_u(\alpha_0) = C_v(\alpha_0) = D_0\) then the rank of \(A\) is \(r + 1\), and if \(C_u(\alpha_0) = C_v(\alpha_0)\), \(C_u(\alpha_0) = D_0\), \(C_v(\alpha_0) = D_0\) then the rank of \(A\) is \(r + 2\).

The theorem below gives conditions for \(X_D(\mu)\) and \(X_D(\nu)\) to be orthogonal or proportional to each other in terms of the subspaces \(C_u(\alpha_0)\) and \(C_v(\alpha_0)\), instead of the rank of \(A\) as done in the previous theorem, along the lines of Theorem 4.2 of Raktos et al. (1980).

**Theorem 2.2.** (i) If \(C_u(\alpha_0) = C_v(\alpha_0) = D_0\), then \(X_D(\mu)\) and \(X_D(\nu)\) are proportional, indeed each column is proportional to \(1\).

(ii) If \(C_u(\alpha_0) = D_0\) and \(C_v(\alpha_0) = D_0\) or \(C_u(\alpha_0) = D_0\) and \(C_v(\alpha_0) = D_0\), then \(X_D(\mu)\) and \(X_D(\nu)\) are orthogonal to each other. Indeed in the first case \(X_D(\mu)\) is proportional to \(1\) and \(X_D(\nu)\) consists of all the entries of the \(\gamma_2\)th column of \(M(u)\) each being repeated \(\alpha^{n-r-1}\) times, (iii) if \(C_u(\alpha_0) = D_0\) and \(C_v(\alpha_0) = D_0\) then the following holds: (1) if \(C_u(\alpha_0) = C_v(\alpha_0)\) then \(X_D(\mu)\) and \(X_D(\nu)\) are orthogonal, (2) if \(C_u(\alpha_0) = C_v(\alpha_0)\) and \(\nu = \delta \mu\) then \(X_D(\mu)\) and \(X_D(\nu)\) are orthogonal, (3) if \(C_u(\alpha_0) = C_v(\alpha_0)\) and \(\nu = \delta \mu\) then \(X_D(\mu)\) and \(X_D(\nu)\) are proportional or orthogonal to each other according as the \(\gamma_1\)th column \((m_0, \gamma_1', m_1, \gamma_1', \ldots, m_{s-1}, \gamma_1')\) of \(M(s)\) is proportional or orthogonal to the permuted \(\gamma_2\)th column \((m_0, \gamma_2', m_1, \gamma_2', \ldots, m_{s-1}, \gamma_2')\) of \(M(s)\), where \(\gamma_j\) is related to \(\alpha_j\) by

\[ \theta_j = \left( \begin{array}{c} 5\gamma_1' \\ \gamma_2' \end{array} \right) \alpha_j + \frac{(e' \cdot z')}{\gamma_2}, \quad j = 0, 1, 2, \ldots, s-1. \quad (2.5) \]
Proof. Follows immediately using Theorem 2.1 and part (c) above.

Remark. A proof of Theorem 2.2 can be given independent of rank arguments using (2.3) and (2.4). For instance consider part 3 of (iii) and suppose \( \mathbf{v}' = \delta \mathbf{u}' + \mathbf{e}' \), with \( \mathbf{e}' \neq 0 \). Then for any \( \mathbf{x}' \in \mathcal{D} \), from (2.3), \( \mathbf{x}' \in C(\tau_j) + \mathbf{z}' \) for some \( j \). Hence \( \mathbf{x}' \cdot \mathbf{u}' = \alpha_j, \mathbf{x}' \cdot \mathbf{v}' = (\delta \gamma) \alpha_j + \mathbf{z}' \cdot \mathbf{e}' \). Thus each \( \mathbf{x}' \) in \( \mathcal{D} \) determines the pair \( (\alpha_j, \gamma_1, \gamma_1, \gamma_2) \) where \( \gamma_j \) is related to \( \alpha_j \) as in (2.5). Moreover, each block \( C(\tau_j) + \mathbf{z}' \) in the decomposition of \( \mathcal{D} \) given in (2.3), produces the same pair. In all \( s \) distinct pairs are produced one for each \( j \) with \( \gamma_j \) related to \( \alpha_j \) as in (2.5), concluding the proof of part 3 of (iii).

Note that for a subspace-type design \( \mathcal{D} \), and any pair of vectors \( \mathbf{u}, \mathbf{v} \) satisfying equation (2.4), if \( d_{t+1} \gamma_1 = \gamma_2 \) then equation (2.5) reduces to \( \gamma_j = \alpha_j \) for each \( j \) and hence \( \mathbf{x}_d(\mathbf{u}) \) and \( \mathbf{x}_d(\mathbf{v}) \) are proportional or orthogonal for any columnwise orthogonal basic matrix \( \mathbf{H}(s) \) with first column \( \mathbf{1} \).

Further observe that when the vectors \( \mathbf{u} \) and \( \mathbf{v} \) satisfy equation (2.4), which corresponds to case (iii), part 3 of Theorem 2.2 then the proportionality or orthogonality respectively of the columns \( \mathbf{x}_d(\mathbf{u}), \mathbf{x}_d(\mathbf{v}) \) depends on the corresponding condition being satisfied by the \( \gamma_2 \)-st and \( \gamma_2 \)-nd columns of \( \mathbf{H}(s) \). This leads to the following:

Corollary 2.1. A sufficient condition that a regular fraction be globally orthogonal is that the basic matrix \( \mathbf{H}(s) \) possesses the following property:

\(^{(4)}\) the \( \gamma_1 \)-st column \( (m_{\alpha_1}, \ldots, m_{\alpha_{s-1}}, \gamma_2) \) and the permuted \( \gamma_2 \)-nd column \( (m_{\alpha_1}, \ldots, m_{\alpha_{s-1}}, \gamma_2) \) of \( \mathbf{H}(s) \) must be proportional or orthogonal where \( \gamma_j = \gamma_0 \alpha_j + \omega, \quad j = 0, 1, 2, \ldots, s-1, \) for all \( \gamma, \gamma_1, \gamma_2 \) in \( F-\{0\} \) and for all \( \omega \) in \( F \) such that \( (\gamma, \omega) \neq (1, 0) \).

Observe that the Hadamard matrices \( H_2, H_4 \) of orders two and four respectively with first columns \( \mathbf{1} \) satisfy the condition imposed on the basic matrix mentioned in Corollary 2.1. Thus these matrices can be
used as basic matrices for the $2^n$ and the $4^n$ factorials respectively and we then have the following:

Corollary 2.2. (i) [Raktoe et al. [1980]]. Any regular fraction $D$ in the $2^n$ factorial under the basic matrix $H_2$ is globally orthogonal; (ii) any regular fraction $D$ in the $4^n$ factorial under the basic matrix $H_4$ is globally orthogonal.

Note that, in fact in the above corollary $H_2$ and $H_4$ can be replaced by matrices $\tilde{H}_2, \tilde{H}_4$ respectively obtained by post-multiplying $H_2, H_4$ by diagonal matrices with each diagonal entry nonzero.

A question which now arises is: for which $s^n$ factorials ($s > 2$) do there exist globally orthogonal designs which are independent of the basic orthogonal matrix used? For co-set type designs this question remains unresolved. However, for subspace-type regular designs we provide the following characterizations.

Let $D$ be a $s^{n-r}$ subspace-type regular fraction defined by $Bx = 0$. Let the $j$-th column of $B$ be its first non-null column. Then, clearly $1 \leq j \leq n - r + 1$. Now we have the following:

Theorem 2.3. When $j < n - r + 1$, $D$ is globally orthogonal if and only if the basic matrix of contrasts $H(s)$ satisfies condition (*) of Corollary (2.1) with $\theta_k = \gamma \alpha_k$, $\omega = 0$, $\gamma_1, \gamma_2$ in $F\{0\}$ and $\gamma$ in $F\{0,1\}$.

Proof. The sufficiency follows from Corollary 2.1. To show the necessity part, assume there exists $\gamma_1, \gamma_2$ not zero in $F$ and $\gamma$ in $F\{0,1\}$ such that the $\gamma_1$-st column $(m_{\theta_0, \gamma_1}, \ldots, m_{\theta_s, \gamma_1})^t$ of $H(s)$ is neither proportional nor orthogonal to the permuted $\gamma_2$-nd column $(m_{\theta_0, \gamma_2}, \ldots, m_{\theta_s, \gamma_2})^t$ where $\theta_k = \gamma \alpha_k$, $k = 0, 1, 2, \ldots, s-1$. Since $j < n - r + 1$, we can always select a vector $u$ in $F^n$ such that its first $j-1$ entries are zero and its first nonzero entry is $\gamma_1$. Define $d_{r+1}$ by $d_{r+1} = \frac{\gamma_2}{\gamma_1}$. Determine $d' = (d_1', d_2', \ldots, d'_r)$ as a solution to the equation $d'_i \cdot b_j = \gamma_2$ where $b_j$ is the $j$-th column of $B$. Such a solution always exists as $b_j \neq 0$. Define $v'$ by $v' = \sum_{i=1}^{r} d_i s'_i + d_{r+1} u_i$ where $s'_i$ is the $i$-th row of $B$. Then clearly
for these \( u, v \), the columns \( X_D(u) \) and \( X_D(v) \) are neither proportional nor orthogonal in view of Theorem 2.1 or Theorem 2.2. This completes the proof.

In the next result we consider the case when the first nonnull column \( j \) of the defining matrix \( B \) of the subspace-type \( s^{n-r} \) fraction \( D \), satisfies \( j = n - r + 1 \).

Theorem 2.4. Let \( D \) be a subspace-type \( s^{n-r} \) fraction with \( s = 2 \). Then \( D \) is globally orthogonal under any columnwise orthogonal matrix \( M(s) \) with a leading column of ones if and only if \( j = n - r + 1 \), where the \( j \)-th column of the defining matrix \( B \) of \( D \) is its first nonnull column.

Proof. The necessity follows from Theorem 2.3. To show the sufficiency assume \( j = n - r + 1 \). Then without loss of generality, we have \( B = [0 : I_r] \), where \( 0 \) is the \( r \times n - r \) zero matrix and \( I_r \) is the identify matrix of order \( r \). Then clearly any vector \( u' \) not in the row space of \( B \) must have at least one nonzero element among its first \( n - r \) coordinates. Take such a vector \( u' \) with its first nonzero entry \( r_1 \) and define a vector \( v \) by

\[
v = \frac{r}{r_2} d_1 b_1 + d_{r+1} u',
\]

where \( (d_1, d_2, \ldots, d_r) \neq 0 \) and \( d_{r+1} \neq 0 \) are chosen arbitrarily from \( F \). Then \( d_{r+1} r_1 \) is the first nonzero entry of \( v \)' and for such \( u', v \) it follows that \( \theta_k = \alpha_k \), \( k = 0, 1, 2, \ldots, s-1 \). Hence for such vectors \( u, v \), namely, those satisfying equation (2.4), the columns \( X_D(u) \) and \( X_D(v) \) are either proportional or orthogonal to each other. Thus by Theorem 1.1, \( D \) is globally orthogonal. This completes the proof.

Note that when \( D \) is a subspace-type \( s^{n-r} \) fraction then when \( j < n - r + 1 \), the property of global orthogonality of \( D \) is equivalent to the basic matrix \( M(s) \) satisfying the condition given in Theorem 2.3 and thus in this case the global orthogonality of \( D \) is in this sense \( M(s) \)-dependent. If \( j = n - r + 1 \), then according to Theorem 2.4, the global orthogonality of \( D \) is \( M(s) \)-independent and moreover in this case observe that \( D = F^{n-r} \times (0)^r \) where \( F^{n-r} \) and \( (0)^r \) are the respective Cartesian products of \( F \) with itself and \( (0) \) with itself to \( n - r \) and \( r \) factors respectively.
3. CONSTRUCTION OF BASIC ORTHOGONAL MATRICES

In the previous section we saw that the basic matrix \( M(s) \) plays a significant role in the construction of globally orthogonal designs. Hence in this section we study such matrices in some detail.

Throughout this section, the following notation will be used:
(i) \( x \) will be a primitive element of \( GF(s) \), (ii) the rows of \( M(s) \) will be labelled in the following way: the first row is labelled by \( \alpha_0 = 0 \), the second row by \( \alpha_1 = 1 \) and for each \( i(1 \leq i \leq s - 1) \), the \( (i + 1) \)st row is labelled by \( \alpha_i = x^{i-1} \). If \( F = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}) = (0, 1, x, \ldots, x^{s-1}) \) is the cyclic permutation sending \( \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_{s-2} + \alpha_{s-1} + \alpha_0 \) then for each \( \phi \neq 0 \) in \( F \) we define \( F(\phi) = (\phi \alpha_0, \phi \alpha_1, \ldots, \phi \alpha_{s-1}) \) to mean the cyclic permutation sending \( \phi \alpha_0 + \phi \alpha_1 + \ldots + \phi \alpha_{s-2} + \phi \alpha_{s-1} + \phi \alpha_0 \) and we call \( F(\phi) \) the cyclic multiplicative permutation (cmp) determined by \( \phi \).

If \( M(s) = (m_{ij})_{s \times s} \) is a basic matrix, relative to which every subspace-type design is globally orthogonal, then according to Theorem 2.3 of the previous section, the columns of \( M(s) \) must satisfy the following conditions:

(i) \( m_{11} = 1 \), for all \( i = \alpha_0, \alpha_1, \ldots, \alpha_{s-1} \), and all other column vectors are nonnull,

(ii) the inner product \( m_j^t \cdot m_k = 0 \), for all \( j \neq k \), where \( m_j, m_k \) are the \( j \)-th and \( k \)-th columns of \( M(s) \),

(iii) for all \( j, k \) in \( \{2, 3, \ldots, s\} \) and all \( \phi \neq 0 \) in \( F \), the \( j \)-th column \( (m_{0,j}, m_{\alpha_1,j}, \ldots, m_{\alpha_{s-1},j})^t \) and the permuted \( k \)-th column \( (m_{0,k}, m_{\alpha_1,k}, \ldots, m_{\alpha_{s-1},k})^t \) are either proportional or orthogonal to each other where \( F(\phi) = (\theta_0, \theta_1, \ldots, \theta_{s-1}) \) is the cmp determined by \( \phi \).

Observe that if \( M(s) \) is a matrix satisfying (i), (ii) and (iii) then so does \( M(s)V \) also where \( V \) is a diagonal matrix, \( V = \text{diag}(d_1, d_2, \ldots, d_s) \) with \( d_1 = 1 \) and all other \( d_i \neq 0 \).
Let \( \mathbf{x}' = (x_1', x_2', \ldots, x_m') \) be any nonzero real contrast vector and let \( \tau_k(\mathbf{x}') = (x_{\tau_k(1)}, x_{\tau_k(2)}, \ldots, x_{\tau_k(m)})' \) where \( \tau_k \) is a permutation on \( \{1, 2, \ldots, m\} \) defined by \( \tau_k(i) = 1 + k \) (mod \( m \)), for \( k = 1, 2, \ldots, m-1 \).

The following lemma will be useful to prove certain results of this section:

**Lemma 3.1.** When \( s \) is a prime greater than 2, then the contrast vector \( \mathbf{x}' \) is neither proportional nor orthogonal to each member in the set \( \{\tau_k(\mathbf{x}') : k = 1, 2, \ldots, s-1\} \), where \( \mathbf{x}' = (x_1', x_2', \ldots, x_s') \).

**Proof.** Suppose that for some \( k \), \( \tau_k(\mathbf{x}') = c \mathbf{x}' \) for some real nonzero constant \( c \). Then it follows that \( \tau_k(\mathbf{x}') = c^s \mathbf{x}' \). Since \( s \) is a prime, \( \tau_k(\mathbf{x}') = c^s \mathbf{x}' \) is the identity permutation and hence \( c^s = 1 \). The only real root of this equation is \( c = 1 \) and hence \( \tau_k(\mathbf{x}') = \mathbf{x}' \) implying equality of all the components of \( \mathbf{x} \). However, this is impossible and \( \mathbf{x}' \) and \( \tau_k(\mathbf{x}') \) can never be proportional for any prime \( s > 2 \), wherever \( \mathbf{x} \) is a nonzero contrast vector.

Hence for each \( k = 1, 2, \ldots, s-1 \) we must have that \( \mathbf{x}' \cdot \tau_k(\mathbf{x}') = 0 \). Then by substituting \( k = 1, 2, \ldots, s-1 \) in this inner product equation one obtains \( s-1 \) homogeneous equations which upon summing results in 
\[
(-x_1^2) + (-x_2^2) + \ldots + (-x_s^2) = 0,
\]
since \( \mathbf{x}' \) is a contrast vector. This implies \( \mathbf{x} = 0 \). However, \( \mathbf{x} \neq 0 \) and the lemma follows.

Of course when \( s \) is not a prime, there exist many contrast vectors \( \mathbf{x} \) such that \( \mathbf{x}' \) and \( \tau_k(\mathbf{x}') \) are either proportional or orthogonal to each other for each \( k = 1, 2, \ldots, s-1 \). We shall now describe a method of construction of \( \mathbf{M}'(s) \) satisfying (I), (II), (III) utilizing such contrasts vectors in the next theorem. The proof of this theorem is immediate from direct computation.

**Theorem 3.1.** Let \( A \) be a \((s-1) \times (s-2)\) matrix whose columns are contrast vectors of order \( s-1 \) such that if \( \mathbf{a} \) and \( \mathbf{b} \) are any two columns of \( A \) (not necessarily distinct) then \( \mathbf{a} \) is either proportional or orthogonal to \( \tau_k(\mathbf{b}) \) for each \( k = 1, 2, \ldots, s-1 \).
Then the matrix

\[
M^{(s)} = \begin{bmatrix}
1 & -(s-1) & 0' \\
1 & 1 & A \\
\vdots & \vdots & \vdots \\
1 & 1 & A
\end{bmatrix}_{s \times s}
\]

satisfies the conditions (i), (ii) and (iii) above.

The following example illustrates Theorem 3.1 for \( s = 5 \).

\[
M^{(5)} = \begin{bmatrix}
1 & -4 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1
\end{bmatrix}
\]

If this \( M^{(5)} \) is used as a basic orthogonal matrix for the \( 5^n \) factorial, then any subspace-type design of the \( 5^n \) factorial will be globally orthogonal.

The following theorem provides necessary conditions for the existence of a basic orthogonal matrix \( M^{(s)} \) satisfying conditions (i), (ii), (iii) in case \( s-1 \) is a prime number.

Theorem 3.2. For any \( s (s=3) \) a prime or prime power such that \( s-1 \) is a prime, if \( M^{(s)} \) exists satisfying conditions (i), (ii), (iii) above then

\[
M^{(s)} = \begin{bmatrix}
1 & x_1 & x_1 & x_1 & \cdots & x_1 \\
1 & y_1 & \delta(y_1) & \delta(y_1) & \cdots & \delta_{s-2}(y_1)
\end{bmatrix} v,
\]

where \( v = \text{diag.} \{1, c_1, c_2, \ldots, c_{s-1}\}, \) \( c_1 \) nonzero real,

\( y_1 = (y_{11}, y_{12}, \ldots, y_{(s-1)1}) \), \( \delta_k(y_1) = (y_{\delta_k(1)}, y_{\delta_k(2)}, \ldots, y_{\delta_k(s-1)}) \),

and \( \delta_k \) are permutations on \( \{1, 2, \ldots, s-1\} \) defined by

\( \delta_k(1) = 1 + k(\text{mod}(s-1)) \) for \( k = 1, 2, \ldots, s-2. \)
Proof. Let $M(s) = [\mathbf{1} : A]$ be an $s \times s$ matrix that satisfies
conditions (i), (ii) and (iii). Let $a = (a_1 \ a_2 \ \ldots \ a_s)'$ and
$b = (b_1 \ b_2 \ \ldots \ b_s)'$ be any two distinct columns of $A$. Since
$(s-1)$ is a prime, by Lemma 3.1, no element in the first row of $M$
can be zero. By the same lemma, the contrast vector $a$ must be of
the form $\gamma(-(s-1) \ 1 \ 1 \ \ldots \ 1)$, $\gamma \neq 0$ if it is proportional to any
of its own cmp. But then $a$ orthogonal to $b$ implies $b_1 = 0$.
Therefore, each column of $A$ must be orthogonal to all of its own
cmp's. Now since $a$ is orthogonal to $b$ and if $a$ is orthogonal to
all of its own $(s-1)$ cmp's then $a \cdot b = 0$. Hence $a$ must be
proportional to at least one cmp of $b$. And if $a$ is proportional
to one cmp of $b$ then automatically it is orthogonal to the
remaining cmp's of $b$, since $b$ is orthogonal to all its own cmp.
Hence $a$ must be proportional to exactly one cmp of $b$ and
orthogonal to all other cmp's of $b$. This establishes the theorem.

Hence whenever $s-1$ is a prime greater than 2, the existence
of an $s \times s$ matrix $M(s)$ satisfying the conditions (i), (ii) and
(iii) depends on the existence of a matrix

$$
M(s) = \begin{bmatrix}
    x_1 & x_1 & \ldots & x_1 \\
    x_2 & x_s & \ldots & x_3 \\
    x_3 & x_2 & \ldots & x_4 \\
    \vdots & \vdots & \ddots & \vdots \\
    x_s & x_{s-1} & \ldots & x_2 \\
\end{bmatrix}_{s \times (s-1)}
$$

of $(s-1)$ mutually orthogonal contrast vectors of dimension $s$
such that the 2nd, 3rd, ..., $(s-1)$th columns are obtained by
applying all possible cmp's to the first column. One may ask whether
matrices like $M(s)$ exist or not. The answer is in the affirmative.
For terminology and other aspects of block designs refer to
Raghavarao [1971]. In the following result, we describe a method of
construction of a series of such $M(s)$ matrices:
Theorem 3.3. Let there exist a symmetric BIB design, constructible through an "initial block" by the method of differences, with parameters $v, k, \lambda$ which satisfy

(i) $v = 2k + 1$ and

(ii) $v - 4(k - \lambda) + 1 = 0$.

Let $N$ be the incidence matrix of such a design so that the columns of $N$ are obtained by cyclically permuting its first column. Define $N^*$ to be the matrix obtained from $N$ by replacing its 0's by -1's. Then

$$M^* = \begin{bmatrix}
1 & 1^t \\
\vdots & \vdots \\
1 & N^*
\end{bmatrix}$$

has the desired properties.

Proof. Since under any two columns of $N$, the pairs $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ appear respectively $v - 2k + \lambda$, $(k - \lambda)$, $(k - \lambda)$ and $\lambda$ times, the inner product of any two columns of $N^*$ is equal to $v - 4(k - \lambda) = -1$. Also each column of $N^*$ has $k$ plus ones and $(k + 1)$ minus ones. Hence the theorem follows.

The result below is an immediate consequence of Theorem 3.3. It provides a series of $M(s)$-matrices for certain values of $s$.

Corollary 3.1. A matrix $M(s)$ of order $s$ where $s - 1 = 4t + 3$, a prime number always exists and can be constructed from the incidence matrix of a symmetric BIBD ($v = 4t + 3$, $k = 2t + 1$, $\lambda = t$) obtained through the development of the initial block ($x^0$, $x^1$, $x^2$, ..., $x^t$) or ($x^1$, $x^3$, $x^5$, ..., $x^{t+1}$), where $x$ is a primitive element of GF(4t + 3).

To illustrate the Corollary, let $s = 8$. Now $x = 3$ is a primitive element of GF(7), and thus $(1, 2, 4)$ is an "initial block" of the SBIBD ($v = 7$, $k = 3$, $\lambda = 1$). From the incidence matrix of this SBIBD and the above results we have
\[
M^*(8) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\end{bmatrix}
\]

Using the above matrix \( M^*(8) \), we may conclude that all subspace-type designs of the \( s^n \) factorial are globally orthogonal.

Note that if one allows \( \lambda = 0 \), then an \( M^*(4) \) also exists, which is given by \( M^*(4) = H \otimes H \) with \( H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \). Finally observe that the matrices \( M^*(s) \) described in Corollary 3.1 are normalized Hadamard matrices. We do not know, however, whether every normalized Hadamard matrix can be arranged as a \( M^*-matrix. \)

4. DISCUSSION

In this paper we have focused attention to global orthogonality of subspace-type of designs in relation to existence and construction of basic orthogonal matrices. It is desirable to carry out a similar study for coset-type of designs. In both cases further research will shed light on the intimate relationship between group-theoretic generation of the aliasing structure and global orthogonality of designs. All these problems are currently under study.

5. ACKNOWLEDGEMENT

This research was supported by NSERC grants No. A8776 and No. A07204.
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