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ESTIMATION OF THE PARAMETERS OF BURR DISTRIBUTION
BASED ON ORDER STATISTICS

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Abstract

Asymptotically best unbiased estimators of the two Burr parameters α and β based on some selected order statistics from a sample of size n are considered. It is shown that the minimum variance of the estimator of β for the known α is $1.54416\beta^2/n$ with efficiency of about 64 % as compared to the maximum likelihood estimator of β. The minimum variance of the estimator of α for known β is attained when 76th sample order statistics is used. When α and β are unknown, variances are essentially minimized using three order statistics (.24,.54,.77) for α and (.27,.61,.81) for β. An example is given to illustrate the method.

1. Introduction

Consider a random sample of size n from the Burr distribution first given by Burr (1942) having the probability density function

$$f(x) = \alpha\beta x^{\alpha-1}/(1 + x^{\alpha})^{\beta+1} \ , \quad x > 0, \quad , \beta \geq 1 \qquad (1)$$

and the distribution function

$$F(x) = 1 - (1 + x^{\alpha})^{-\beta} \ , \quad x > 0 \qquad (2)$$

Properties of the distribution (1) have been investigated by Austin (1971), Burr (1968), Burr and Cislak (1968) and Hatke (1949). Recently Austin (1971) has applied the distribution to the control charts for the maximum and minimum value in sampling from a normal distribution. In modeling tensile loads on a machine element, Weibull and log normal distributions have been tried (see Bury, 1975). In this paper we have used some order statistics to estimate the Burr parameters for use in the modeling of tensile loads on a machine element.

2. Estimation

Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be the sample ordered statistics and let $u_1, u_2, \cdots u_n$ be the corresponding population quantities. Let $x_{n_1} < x_{n_2} \cdots < x_{n_k}$, $k = 1, 2, \cdots$ be the selected k sample quantities where $k < n$. A set of k real numbers such that $0 = \lambda_0 < \lambda_1 < \cdot \cdot \cdot \cdot < \lambda_{k+1} = 1$ is called a spacing. We define λ_i by $\lambda_i = F(u_i)$, $i = 1, 2, \cdot \cdot \cdot k$. Then $u_i = F^{-1}(\lambda_i)$ and therefore from (2)

$$u_i = [(1 - \lambda_i)^{-1/\beta} - 1]^{1/\alpha} \tag{3}$$

and from (1)

$$f[F^{-1}(\lambda_i)] = \alpha\beta(1 - \lambda_i)^{1+1/\beta} u_i^{\alpha-1}. \tag{4}$$

Taking the logarithm of (3) and (4), we get

$$\ln u_i = 1/\alpha \ln[(1 - \lambda_i)^{-1/\beta} - 1] \tag{5}$$

and

$$\ln f[F^{-1}(\lambda_i)] = \ln(\alpha\beta) + (1 + 1/\beta)\ln(1 - \lambda_i) + (\alpha - 1)\ln u_i \tag{6}$$

Using some selected order statistics, we shall obtain best unbiased estimators of α and β by minimizing the asymptotic variances of the estimators.

<u>Case 1</u>. (β is known): Given a spacing $\{\lambda_i\}$, the estimator of α when $\beta = \beta_0$, using (3) is

$$\tilde{\alpha}_k = (k)^{-1}\sum_{i=1}^{k}\{\ln[(1-\lambda_i)^{-1/\beta_0}-1]y_i\} = (k)^{-1}\sum_{i=1}^{k} a_i y_i \qquad (7)$$

where $y_i = (\ln x_{n_i})^{-1}$, $a_i = \ln[(1-\lambda_i)^{-1/\beta_0}-1]$ and x_{n_i} is a selected sample ordered statistic corresponding to u_{n_i}. The asymptotic variance of $\hat{\alpha}_k$ is given by

$$\mathrm{var}(\tilde{\alpha}_k) = k^{-2}\Sigma a_i^2\ \mathrm{var}\,y_i + \sum_{i\neq j}\sum a_i a_j\ \mathrm{cov}(y_i,y_j) \qquad (8)$$

where $\quad \mathrm{var}(y_i) = y_i^4 x_{n_i}^{-2}\ \mathrm{var}(x_{n_i})$

and $\quad \mathrm{cov}(y_i,y_j) = y_i^2 y_j^2 (x_{n_i} x_{n_j})^{-1}\ \mathrm{cov}(x_{n_i},x_{n_j})$.

It is well known [see Mosteller (1946)] that given a spacing, the joint distribution of the k sample quantities is asymptotically normal. Suppose $k = 1$. The ith sample quantity is asymptotically normal with mean u_i and variance $\lambda_i(1-\lambda_i)/[nf^2(x_{n_i})]$. Minimizing $\mathrm{var}(\tilde{\alpha}_1)$ with respect to λ_i, we get

$$(1-2\lambda_i)[1-(1-\lambda_i)^{1/\beta_0}]\ln[(1-\lambda_i)^{-1/\beta_0}-1] + \lambda_i/\beta_0 = 0 \quad (9)$$

Table 1 gives the solution of λ_i of equation (9) that minimizes var($\tilde{\alpha}_1$) for some given values of β.

TABLE 1

β	λ_i	a_i
1.0	.823958	1.54340
1.5	.869111	1.05748
2.0	.904915	0.80773
2.5	.931742	0.65567
3.0	.951335	0.55332
3.5	.965447	0.47972
4.0	.975530	0.42421
5.0	.987780	0.34583
6.0	.993912	0.29282
7.0	.996969	0.25439
8.0	.998491	0.22514
9.0	.999248	0.20192
10.0	.999626	0.18355

In case of large β, the largest observation may be used to estimate α. For $k = 2, 3, \cdots$ λ_5 can be determined by minimizing (8) numerically.

Case 2. (α is known): Suppose $\alpha = \alpha_0$. For a given spacing $\{\lambda_i\}$ satisfying (2), we find that

$$\tilde{\beta}_k = k^{-1} \sum_{i=1}^{k} b_i z_i \tag{10}$$

where $b_i = \ln(1 - \lambda_i)$ and $z_i = -\{\ln(1 + x_{n_i}^{\alpha_0})\}^{-1}$.

Its asymptotic variance is given by

$$\mathrm{var}(\tilde{\beta}_k) = k^{-2}[\sum_i^k b_i^2 \, \mathrm{var}(z_i) + \sum_{i \neq j}\sum b_i b_j \, \mathrm{cov}(z_i, z_j)]$$

where $\mathrm{var}(z_i) = z_i^4 f_i^2 [\beta(1 - \lambda_i)]^{-2} \, \mathrm{var}(x_{n_i})$, $f_i = f(x_{n_i})$

and $\mathrm{cov}(z_i, z_j) = z_i^2 z_j^2 f_i f_j \, \beta^{-2} [(1 - \lambda_i)(1 - \lambda_j)]^{-1} \, \mathrm{cov}(x_{n_i}, x_{n_j})$.

Let $k = i$, $\tilde{\beta}_i = b_i z_i$ and the asymptotic variance is given by

$$\mathrm{var}(\tilde{\beta}_i) = [\frac{\ln(1 - \lambda_i)}{\ln^2(1 + x_i^{\alpha_0})} \; \frac{\alpha_0 x^{\alpha_0 - 1}}{1 + x_i^{\alpha_0}}]^2 \mathrm{var}(x_i) \tag{11}$$

Using $\mathrm{var}(x_i) = \lambda_i(1 - \lambda_i)/nf_i^2$,

$f_i = \alpha_0\beta x_i^{\alpha_0-1}(1 - \lambda_i)/[1 + x_i^{\alpha_0}]$ and $\ln(1 + x^{\alpha_0}) = -(1/\beta)\ln(1 - \lambda_i)$.

We have

$$\mathrm{var}(\tilde{\beta}_i) = \beta^2\lambda/[n(1-\lambda_i)\ln^2(1-\lambda_i)] \tag{12}$$

We choose λ_i by minimizing $\mathrm{var}(\hat{\beta}_i)$ with respect to λ_i and derive an equation in λ_i only

$$2\lambda_i + \ln(1 - \lambda_i) = 0$$

giving a value of $\lambda_i = 0.79681$.

The minimum variance of $\tilde{\beta}$ for any α_0 is 1.54416 β^2/n. The maximum likelihood estimate of β for $\alpha = \alpha_0$ from a sample of size n is

$$\hat{\beta} = n/\sum_{i=1}^{n} \ln(1 + x_i^{\alpha})$$

and its asymptotic variance is β^2/n. The efficiency of $\tilde{\beta}$ as compared to $\hat{\beta}$ is given by

$$E = (1 - \lambda_i) \ln (1 - \lambda_i)/\lambda_i$$

and is about 65 % for $\lambda_i = 0.79681$.

For $k = 2, 3, \ldots$ λ_i can be determined such that (11) is minimized. Some special cases have been investigated separately.

Case 3. (α and β are unknown): Suppose α and β are unknown. We write $y_{n_i} = \ln x_{n_i} = (1/\alpha)[(1 - \lambda_i)^{1/\beta} - 1]$ and

$\ln f_{n_i} = \ln\tilde{\alpha} + \ln\tilde{\beta} + (\alpha - 1)y_{n_i} + (1 + 1/\tilde{\beta})\ln(1 - \lambda_i)$, $i = 1, 2, 3$.

Using $\ln f_{n_i}$, $i = 1, 2, 3$, we have

$$\ell_1 = (\tilde{\alpha} - 1)y_1 + (1 + 1/\tilde{\beta})\alpha_1$$

$$\ell_2 = (\tilde{\alpha} - 1)y_2 + (1 + 1/\tilde{\beta})\alpha_2$$

where $\ell_1 = \ln(f_{n_1}/f_{n_2})$, $\ell_2 = \ln(fn_2/fn_3)$

$$\alpha_1 = \ln[(1 - \lambda_1)/(1 - \lambda_2)], \quad \alpha_2 = \ln[(1 - \lambda_2)/(1 - \lambda_3)]$$

$$y_1 = y_{n_1} - y_{n_2}, \quad y_2 = y_{n_2} - y_{n_3} \text{ and } \lambda_i = F(x_{n_i})$$

Solving for $\tilde{\alpha}$ and $\tilde{\beta}$, we get the estimator of (α, β) as

$$\tilde{\alpha} = 1 + (\ell_1\alpha_2 - \ell_2\alpha_1)/(y_1\alpha_2 - y_2\alpha_1) \tag{13}$$

and

$$\tilde{\beta} = (\alpha_1 y_2 - \alpha_2 y_1)/[(\ell_1 - \alpha_1)y_2 - (\ell_2 - \alpha_2)y_1] \tag{14}$$

The asymptotic variances of $\tilde{\alpha}$ and $\tilde{\beta}$ are

$$\mathrm{var}(\tilde{\alpha}) = (\alpha - 1)^2 \sum_{i,j=1}^{4}\sum^{4} a_i a_j C(n_i, n_j) \tag{15}$$

and

$$\mathrm{var}(\tilde{\beta}) = \beta^2 \sum_{i,j=1}^{4}\sum^{4} b_i b_j C(n_i, n_j) \tag{16}$$

where $C(n_i, n_j) = \begin{cases} \mathrm{Var}(n_i), & j = 1 \\ \mathrm{Cov}(n_i, n_j), & j \neq i \end{cases}$

$n_1 = \ell_1$, $n_2 = \ell_2$, $n_3 = y_1$, $n_4 = y_2$, $a_1 = \alpha_2/(\ell_1\alpha_2 - \ell_2\alpha_1)$

$$a_2 = -\alpha_1/(\ell_1\alpha_2 - \ell_2\alpha_1)$$

$$a_3 = -\alpha_2/(y_1\alpha_2 - y_2\alpha_1)$$

$$a_4 = \alpha_1/(y_1\alpha_2 - y_2\alpha_1)$$

$$b_1 = - y_2/[(\ell_1 - \alpha_1)y_2 - (\ell_2 - \alpha_2)y_1]$$

$$b_2 = y_1/[(\ell_1 - \alpha_2)y_2 - (\ell_2 - \alpha_2)y_1]$$

$$b_3 = (\ell_2 - \alpha_2)/[(\ell_1 - \alpha_1)y_2 - (\ell_2 - \alpha_2)y_1] - a_3$$

$$b_4 = -a_4 - (\ell_1 - \alpha_1)/[(\ell - \alpha_1)y_2 - (\ell_2 - \alpha_2)y_1]$$

Minimum variance can be obtained only numerically using variance values of unknown quantities involved in the expressions (15) and (16). For various values of $\{\lambda_i\}$ the variance functions at (15) and (16) were computed on the IBM 370 computer. For k = 3, we find that the variance functions are essentially minimized by choosing (0.24, 0.54, 0.77) for $var(\tilde{\alpha})$ and by choosing (0.27, 0.61, 0.81) for $var(\tilde{\beta})$.

3. Example

To illustrate the method, suppose that the machine element is subjected to 100 random tensile loads during its mission x and that the load has been represented by the Burr model with $\alpha = 6$ and $\beta = 4$. If β is known to have a value of 4, then α is estimated from (7) when $k = i$ and $\lambda_i = 0.97553$ (see table 1). The values of x_{n_i} and y_i are

$x_{.98} = 1.075$, $y_i = 0.93023$ and $a_i = 0.42421$. $\tilde{\alpha} = 5.87$ and $var(\hat{\alpha}) = 13.07$. If α is known to have a value of 6, then β is estimated from (10) when $k = 1$, $\lambda_i = 0.79681$. The values of $x_{n_i} = 0.853$, $z_i = -3.0689$ and $b_i = -1.5936$. Then $\tilde{\beta} = 4.89$ and $var(\tilde{\beta}) = 0.3696$. In this case, the maximum likelihood estimation of β is $\hat{\beta} = 4.23$ and $var(\hat{\beta}) = 0.1786$. If $\beta = 4$, then efficiency of $\tilde{\beta}$ is about 65%. If we use $\tilde{\beta}$ and $\hat{\beta}$ in the variance formulae respectively the efficiency is about 48.3%. If α and β are unknown then using (13) and (14) as estimating equations we have for α, the values of x_{n_1}, x_{n_2} and x_{n_3} are $x_{n_1} = .585$, $x_{n_2} = .744$ and $x_{n_3} = .840$. Then $\tilde{\tilde{\alpha}} = 5.53$ and $var(\tilde{\tilde{\beta}}) = 20.31$ and for β, the values of x_{n_1}, x_{n_2} and x_{n_3} are $x_{n_1} = .612$, $x_{n_2} = .772$ and $x_{n_3} = .878$. Then $\tilde{\tilde{\beta}} = 3.42$ and $var(\tilde{\tilde{\beta}}) = .2511$.

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