King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 016

January 1981

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TANGENTIAL APPROXIMATION OF CONTINUOUS FUNCTIONS ON OPEN SETS (*)

by

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Let X be any non-empty subset of the n-dimensional real Euclidean space R^n. The family of all real valued continuous functions on X will be denoted by $C(X)$.

We will consider the following specific problem.

PROBLEM 1: Let G be a non-empty open subset of R^n, $n \geq 1$. Determine "nice" subclasses of $C(G)$, denoting such a subclass by Z, such that for every pair of functions f and u in $C(G)$, $u > 0$, there corresponds a function g in Z satisfying

$$|f(x) - g(x)| < u(x), \quad x \in G.$$

Let $C^\infty(G)$ be the class consisting of all functions in $C(G)$ which have continuous partial derivatives of all orders on G.

THEOREM 1. $Z = C^\infty(G)$ is a solution to Problem 1.

An elementary proof of Theorem 1 is presented in [3] for the case when $G = R^n$; a standard argument utilizing an exhaustion of G by compact sets shows that the proof as given in [3] remains valid for arbitrary open G.

(*) This article will appear in the Journal of the London Mathematical Society in 1981.

A sharper result for the case when $G = R$ was given by T. Carleman [7] (also, see [13] and [14]) in 1927:

<u>CARLEMAN'S THEOREM</u>. For every pair of functions f and u in $C(R)$, $u > 0$, there corresponds an entire (complex analytic) function g on the plane satisfying

$$|f(x) - g(x)| < u(x), \quad x \in R.$$

It is the purpose of this work to obtain an extension of Carleman's result for arbitrary open sets in R^n. Setting $h = \text{Real } g$ in Carleman's theorem, we obtain the following inequality

$$|f(x) - h(x,0)| < u(x), \quad x \in R.$$

We first observe that the function $h(.,0)$ is an entire real analytic function on R (i.e., it has an absolutely convergent Taylor series representation valid on the whole domain R), a concept which extends to R^n. We obviously may consider such functions as a possible solution to Problem 1 when $G = R^n$, and locally real analytic functions when G is an arbitrary open subset of R^n. However, the harmonicity of the function h on the plane RxR suggests consideration of a still nicer class, namely, the class $H(G)$ consisting of all restrictions to G of functions which are harmonic on GxR. The class $H(G)$ is indeed nicer than the class of all (locally) real analytic functions on G since the former is properly contained in the latter; (for the intimate connection between these two classes, see [2] and [4]).

In 1934, H. Whitney [19; page 86, Theorem III] showed, in particular, that the class of all real analytic functions on G is a solution to Problem 1.

In this article, we will prove that the class $H(G)$ provides uniform and tangential approximations to functions in $C(G)$, as required in Problem 1. Restating, we will prove the following theorem.

THEOREM 2. Let G be a non-empty open subset of R^n, $n \geq 1$. For every pair of functions f and u in $C(G)$, $u > 0$, there corresponds a function h harmonic on $G \times R$ (i.e., $h(.,0) \in H(G)$) such that

$$|f(x) - h(x,0)| < u(x), \qquad x \in G.$$

In view of Tietze's extension theorem, we note that Theorem 2 remains true even if G is a closed set, in which case it is clear that the approximating function h can be chosen to be harmonic on the whole space R^{n+1}.

The following corollary is an immediate consequence of the better than uniform (or tangential) character of the approximation given in Theorem 2.

COROLLARY. Let X be a non-empty open or closed subset of R^n, $n \geq 1$, and let μ be a Borel measure on X (i.e., μ is non-negative and finite on compact sets). Let a function f in $C(X)$ and a number $p > 0$ be given. Then, for each $\epsilon > 0$, there corresponds a function h, which is harmonic on $X \times R$ if X is open, and is harmonic on R^{n+1} if X is closed, such that

$$\int_X |f - h(.,0)|^p \, d\mu < \epsilon.$$

To prove the corollary when X is open, it is an easy exercise to construct a positive and continuous function $u \in L^p(X,\mu)$; using an appropriate multiple of this u in Theorem 2, we get the desired conclusion at once. When X is closed, we argue similarly, but also use Tietze's extension theorem.

The special case of Theorem 2 when $G = R^n$ was apparently known to M. Brelot in 1955 as evidenced in a private communication to W. Kaplan [14; page 44].

One extension of Carleman's theorem which yields Theorem 2 with $n = 1$ is due to W. Kaplan [14; page 44, Theorem 2] in 1955, where it is shown, in particular, that if I is an open interval, then every function in $C(I)$ can be approximated tangentially on I by complex analytic functions on $I\times R$. Since every open subset of R is the countable union of disjoint open intervals, we deduce, as a special case of Kaplan's result, a very nice solution to Problem 1 when $n = 1$, as follows:

KAPLAN'S THEOREM: Let G be a non-empty open subset of R. For every pair of functions f and u in $C(G)$, $u > 0$, there corresponds a complex analytic function g on $G\times R$ satisfying

$$|f(x) - g(x)| < u(x), \qquad x \in G.$$

Various other extensions of Carleman's theorem are available in the literature. The planar case, where the approximating functions are required to be complex analytic, has been thoroughly studied by several authors. We simply refer to N.U. Arakelyan's [1] summary of his doctoral dissertation which provides an excellent survey of the progress of contributions related to uniform and tangential approximations, on relatively closed subsets of arbitrary plane regions, by holomorphic functions; for similar approximations by meromorphic functions, one can consult, for instance, A.H. Nersesian [15] and A. Roth [16], while the recent article by P.M. Gauthier [11] deals with their generalizations to open Riemann surfaces. We shall not dwell on these topics because they are out of the scope of Problem 1.

In the proof of Theorem 2 we will need a lemma concerning uniform approximation of an appropriate type on compact sets; recalling that a box in R^n is the cartesian product of n closed intervals in R, we state the lemma.

<u>LEMMA</u>. Let $\{B_j : 1 \leq j \leq J\}$ be a finite collection of boxes B_J in R^n, $n \geq 1$, and let K_0 be a compact subset of R^n. For a positive number s, define the sets B and K as follows:

$$B = \bigcup_{j=1}^{J} B_j x[-s,s] \qquad \text{and} \qquad K = B \cup (K_0 x\{0\}).$$

If the function f in $C(K)$ is harmonic on the interior K^o (= B^o) of K, then f can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^{n+1}.

Postponing the discussion and the proof of the lemma, we will use it to prove Theorem 2.

Our general approach to the proof of Theorem 2 is motivated by the scheme first employed by T. Carleman [7], which has since been used by different mathematicians in various approximation problems on non-compact sets (e.g., [14] and [17]); in fact, A. Sinclair [17] presents a formalization of the method.

PROOF OF THEOREM 2. Exhaust the open set $G = \bigcup_{0<i} G_i$ by an increasing sequence of bounded non-empty open subsets G_i of R^n such that the closure $\overline{G}_i$ of each G_i is contained in the next one G_{i+1}, i.e., $\overline{G}_i \subset G_{i+1}$. Each $\overline{G}_i$, being compact, can be covered by the interiors of finitely many boxes contained in G_{i+1}; for each i, let B_i be the union of one set of such boxes; thus, we have $G_i \subset B_i \subset G_{i+1}$ for all $i \geq 1$.

For each $i \geq 1$, define

$$K_0 = \text{the empty set}, \qquad K_i = (B_i \times [-i,i]) \cup (B_{i+1} \times \{0\}),$$

$$h_0 = 0 \text{ on } G \times R, \quad \epsilon_0 = 0 \quad \text{and} \quad \epsilon_i = 2^{-i-2} \operatorname*{infimum}_{B_{i+1}} u.$$

By induction, we will define a sequence of functions h_i in $C(G \times R)$ satisfying for all $i \geq 1$ the following four conditions, where $K_i - K_{i-1}$ denotes the complement of K_{i-1} relative to K_i, and ∂B_{i+1} is the boundary of B_{i+1}:

(1) h_i is harmonic on the interior K_i^0 of K_i.

(2) $|h_i(w) - h_{i-1}(w)| < \epsilon_{i-1}$, $w \in K_{i-1}$.

(3) $\quad |h_i(x,0) - f(x)| < \varepsilon_i \;, \quad (x,0) \in K_i - K_{i-1}.$

(4) $\quad h_i(x,0) = f(x), \quad x \in \partial B_{i+1}.$

For $i = 1$: by the classical Weierstrass approximation theorem, there exists a polynomial p on R^n such that

$$|p(x) - f(x)| < \varepsilon_1 \;, \quad x \in B_1 .$$

By the continuity of the function $p-f$, the last inequality holds on some neighborhood, say N, of B_1 and we can choose $N \subset G_2 \subset B_2$. By Urysohn's lemma, there exists a function g in $C(G)$ such that $g = 1$ on B_1, $g = 0$ on $G-N$ and $0 \leq g \leq 1$ on G. For all (x,z) in $G\times R$, define the function h_1 as follows:

$$h_1(x,z) = (1-g(x))f(x) + g(x) \sum_{0 \leq j} \frac{(-1)^j}{(2j)!} z^{2j} \Delta^j p(x),$$

where $\Delta^j = \Delta(\Delta^{j-1})$ is the j-th iterate of the n-dimensional Laplace operator Δ with Δ^0 as the identity operator. Observing that the above series is a harmonic polynomial extension of p to R^{n+1}, it is easily verified that the function h_1 satisfies conditions (1) through (4), as required.

Now, assume that for some $i \geq 1$, the functions h_j, $1 \leq j \leq i$, satisfy conditions (1) through (4). Since the function $h_i \in C(K_i)$ is harmonic on K_i^o, the lemma asserts that there exists a harmonic polynomial q on R^{n+1} such that

$$|q(w) - h_i(w)| < 2^{-1} \varepsilon_{i+1} \;, \quad w \in K_i .$$

Thus,

$$|q(x,0) - h_i(x,0)| < 2^{-1}\varepsilon_{i+1}, \quad x \in B_{i+1}.$$

As before, by the continuity of the function $q(.,0) - h_i(.,0)$, the last inequality holds on some neighborhood, say M, of B_{i+1}, and since $h_i(x,0) = f(x)$ for all $x \in \partial B_{i+1}$, we can choose $M \subset G_{i+2} \subset B_{i+2}$ such that

$$|h_i(x,0) - f(x)| < 2^{-1}\varepsilon_{i+1}, \quad x \in M-B_{i+1}.$$

By Urysohn's lemma, there exists a function g in $C(G)$ such that $g = 1$ on B_{i+1}, $g = 0$ on $G-M$ and $0 \leq g \leq 1$ on G. It follows that the function h_{i+1}, defined by

$$h_{i+1}(x,z) = g(x)q(x,z) + (1 - g(x))f(x), \quad (x,z) \in GxR,$$

satisfies conditions (1) through (4).

The induction is completed.

It is clear from the definition of the sets K_i that $GxR = \bigcup_{0<i} K_i$, and since the series $\sum_{0<i} \varepsilon_i$ converges, condition (2) implies that the sequence of functions h_i converges locally uniformly on GxR to a function h which is necessarily harmonic on GxR (via [12; page 32, Lemma 2.17]).

Next, for fixed x in G, let us estimate $|f(x) - h(x,0)|$. Since $(x,0)$ is in $K_m - K_{m-1}$ for some $m > 0$, we can choose $i > m$ so large that

$$|h_i(x,0) - h(x,0)| < \varepsilon_m.$$

Using conditions (2) and (3), we have

$$|f(x)-h(x,0)| = \left|[f(x)-h_m(x,0)] + [h_i(x,0)-h(x,0)] - \sum_{j=m}^{i-1} [h_{j+1}(x,0)-h_j(x,0)]\right|$$

$$< \quad \varepsilon_m \quad + \quad \varepsilon_m \quad + \sum_{j=m}^{i-1} \varepsilon_j$$

$$< \; 4\varepsilon_m \; < \; u(x).$$

Since x in G was arbitrary, we have shown that

$$|f(x) - h(x,0)| < u(x), \quad x \in G,$$

which completes the proof of Theorem 2.

Returning to the lemma, we observe that it is a special case of the following general problem.

PROBLEM 2. Let K be a compact subset of R^n, $n > 1$, and consider a function f in $C(K)$. What additional conditions on K and f will ensure that f can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^n ?

Additional conditions on K and f are indeed necessary. Since the uniform limit of harmonic functions is harmonic, the function f should be harmonic on the interior of K for Problem 2 to have a solution. The connectedness of the complement of K is another essential condition for some sets K, but it is not necessary for others, as can be seen in the following two examples. Let $K = \{w \in R^n : \frac{1}{2} \le |w| \le 1\}$ and f be the fundamental harmonic function on R^n with pole $w = 0$ (i.e., $f(w) = -\ln|w|$ if $n = 2$, and $f(w) = |w|^{2-n}$ if $n > 2$); if there were a harmonic polynomial h on R^n which is within $\epsilon = \frac{1}{4}\ln 2$ of f on K, then it is easy to see that h violates the maximum principle for harmonic functions ([12; page 19, Corollary 1.13]). On the other hand, if $K = \{w \in R^n : |w| = 1\}$, then every function f in $C(K)$ can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^n, because f admits a harmonic extension to the closed ball $D = \{w \in R^n : |w| \leq 1\}$ ([12; page 25, Theorem 2.8]) and the set D satisfies the conditions of Propositions 1 and 4 below.

For planar compact K, an answer to Problem 2 was given in 1929 by J.L. Walsh [18; page 503] : every function in $C(K)$ can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^2 if and only if K is the boundary of an unbounded open connected set, namely, the unbounded component of R^2-K. In particular, if R^2-K is connected, then the boundary of K satisfies the condition in Walsh's result just cited, and utilizing the maximum principle, we get : if the complement of the compact $K \subset R^2$ is connected, then every function f in $C(K)$ which is harmonic on the interior of K can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^2. An elegent direct proof for this last assertion was presented in 1964 by L. Carleson [8; page 170, Lemma 3]. The corresponding result for $K \subset R^3$ remains true providing f is further restricted to be harmonic on a neighborhood of K; this was established again by Walsh [18; page 541], and as Walsh himself also points out, his method of proof holds, with only minor modifications, in all dimensions $n > 2$. We thus have the next proposition which will be used in the proof of our lemma.

<u>PROPOSITION 1</u>. If K is a compact subset of R^n, $n > 1$, having a connected complement, then every function which is harmonic on a neighborhood of K can be approximated uniformly on K, as closely as desired, by harmonic polynomials on R^n.

Problem 2 in the general case $(n > 1)$ has been thoroughly examined and substantial contributions were obtained by M. Brelot [5;6] and J. Deny [9]. These authors introduce the concept of "stability" of the points on the boundary ∂K of K and they prove that the "stability" of all the points of ∂K is a necessary and sufficient condition for every function in $C(\partial K)$ to be approximable uniformly on ∂K, as closely as desired, by functions which are harmonic on neighborhoods of K ([6; page 62, Theorem 1] and [9; page 110, Theorem 3]). This major result of Brelot and Deny and the maximum principle for harmonic functions yield the next proposition.

<u>PROPOSITION 2</u>. If all the points of the boundary of the compact $K \subset R^n$, $n > 1$, are "stable", then every function in $C(K)$ which is harmonic on the interior of K can be approximated uniformly on K, as closely as desired, by functions which are harmonic on neighborhoods of K.

The concept of "stability" admits different, but equivalent, definitions (e.g., [5; page 131] and [6; page 60]) and various characterizations (e.g., [5; pages 132-137], [6; pages 60-61], [9; page 108] and [10; page 113]). Following the characterization by Brelot [6; page 60], a point x on the boundary of the compact $K \subset R^n$, $n > 1$, is stable if (and only if) the complement $G = R^n - K$ of K is <u>not thin</u> ("non effile") at x, where G is <u>thin</u> at x means, according to Brelot [6; page 57, footnote (9)], that there is a subharmonic function u on a neighborhood of x such that

$$(*) \qquad \limsup_{\substack{z \to x \\ z \in G - \{x\}}} u(z) < u(x);$$

since $-u$ is superharmonic if and only if u is subharmonic, and recalling the obvious properties of sup and inf, we see that the above condition for thinness at x is equivalent to saying that there exists a superharmonic function v on a neighborhood of x such that $u = -v$ satisfies the inequality expressed in (*), which makes the set G thin at x according to L.L. Helms [12; page 209, Theorem 10.3]. We borrow the following result from Helms [12; page 211, Corollary 10.5].

PROPOSITION 3. If a Borel set $E \subset R^n$, $n > 1$, is thin at a limit point x of E, then

$$\lim_{r \to 0} \frac{\sigma(D \cap E)}{\sigma(D)} = 0,$$

where $D = D(x,r) = \{w \in R^n : |w - x| = r\}$ and $\sigma(X)$ denotes the usual surface area for any Borel subset X of D.

Finally, we will prove Proposition 4 below which provides a simple geometric criterion for stability; though quite restrictive, it is sufficient to show that the boundary of the compact set K in our lemma consists of stable points.

PROPOSITION 4. If K is a compact subset of R^n, $n > 1$, having the property that at each point x on the boundary of K there exists a (closed rectangular) box, say $B_x \subset R^n$, whose interior is contained in the complement of K and B_x has x as one of its corner points, then every function in $C(K)$ which is harmonic on the interior of K can be approximated uniformly on K, as closely as desired, by functions which are harmonic on neighborhoods of K.

To prove Proposition 4, consider $x \in \partial K$. From the geometry of R^n, it is clear that there are at most 2^n boxes in any collection of boxes in R^n having x as a common corner point and having pairwise disjoint interiors. Thus, for sufficiently small $r > 0$ and $D = D(x,r) = \{w \in R^n : |w - x| = r\}$, the surface area of $D \cap B_x$ is equal to $2^{-n}\,\sigma(D)$, and since by assumption the interior of the box B_x is contained in the complement $G = R^n - K$ of K, it follows that

$$\frac{\sigma(D \cap G)}{\sigma(D)} \geqslant \frac{\sigma(D \cap B_x)}{\sigma(D)} = 2^{-n},$$

which implies, by Proposition 3, that $R^n - K$ is not thin at x, that is x is stable, and the conclusion of Proposition 4 follows from Proposition 2.

Recalling the definition of the compact set K in our lemma, we merely observe that the complement of K is connected and K satisfies the conditions of Proposition 4. Therefore, the lemma follows from Propositions 1 and 4.

REFERENCES

[1] N.U. Arakelyan : Some problems in approximation theory and in entire-function theory, Math. Notes, 9 (1971), 267-271. (Russian original appeared in Mat. Zametki, 9 (1971), 467-475.)

[2] D.H. Armitage : On harmonic polynomials, Proc. London Math. Soc. (3), 34 (1977), 173-192.

[3] A. Boghossian : Better-than-uniform approximation of continuous functions by C^{∞} functions, Amer. Math. Monthly, 81 (1974), 999-1001.

[4] A. Boghossian : Harmonic extension of real analytic functions, J. London Math. Soc. (2), 20 (1979), 273-276.

[5] M. Brelot : Critères de régularité et de stabilité, Acad. Roy. Belg. Bull. Cl. Sci., 25 (1939), 125-137.

[6] M. Brelot : Sur l'approximation et la convergence dans la théorie des fonctions harmoniques ou holomorphes, Bull. Soc. Math. France, 73 (1945), 55-70.

[7] T. Carleman : Sur un théorème de Weierstrass, Ark. Mat., 20 (1927), 1-5.

[8] L. Carleson : Mergelyan's theorem on uniform polynomial approximation, Math. Scand., 15 (1964), 167-175.

[9] J. Deny : Systèmes totaux de fonctions harmoniques, Ann. Inst. Fourier (Grenoble), 1 (1949-50), 103-113.

[10] E. Effros & J. Kazdan : Applications of Choquet simplexes to elliptic and parabolic boundary value problems, J. Differential Equations, 8 (1970), 95-134.

[11] P.M. Gauthier : Meromorphic uniform approximation on closed subsets of open Riemann surfaces, Approximation Theory and Functional Analysis (Editor: J.B. Prolla), Notas de Matemática, North-Holland (1979), 139-158.

[12] L.L. Helms : Introduction to Potential Theory, John Wiley & Sons, 1969; reprinted by Robert E. Krieger Publishing Co., Huntington, N.Y., 1975.

[13] L. Hoischen : A note on the approximation of continuous functions by integral functions, J. London Math. Soc., 42 (1967), 351-354.

[14] W. Kaplan : Approximation by entire functions, Michigan Math. J., 3 (1955-56), 43-52.

[15] A.H. Nersesian : On the uniform and tangential approximation by meromorphic functions (Russian), Izv. Akad. Nauk Arm. SSR, 7 (1972), 405-412.

[16] A. Roth : Uniform and tangential approximations by meromorphic functions on closed sets, Canad. J. Math., 28 (1976), 104-111.

[17] A. Sinclair : A general solution for a class of approximation problems, Pacific J. Math., 8 (1958), 857-866.

[18] J.L. Walsh : The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bull. Amer. Math. Soc., 35 (1929), 499-544.

[19] H. Whitney : Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc., 36 (1934), 63-89.

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