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ON INVARIANCE AND RANDOMIZATION UNDER FACTOR PERMUTATION IN FRACTIONAL FACTORIAL DESIGNS[1]

by

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ABSTRACT

This paper establishes the spectrum invariance of the information matrix under an arbitrary subgroup Γ of the group Λ of factor permutations. In addition, it provides the randomized unbiased estimation of a linear parametric function under the composition Ω∘Γ, where Ω is the group of level permutations. These two results are achieved for the most practical partioning of the whole parametric vector using the concepts of Γ-closed and admissibility of a parametric subvector. Applications are given with an explicit illustration using the minimal resolution III design setting for the 2^3 factorial.

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1. Introduction. The problems of spectrum invariance and randomized unbiased estimation were earlier studied by Ehrenfeld and Zacks (1961), Zacks (1963, 1964) and Paik and Federer (1970). These authors dealt mostly with main effect fractional factorial designs from the symmetrical factorial and the operational group used was the group Ω of level permutations. More recently, Srivastava, Raktoe and Pesotan (1976) and Pesotan and Raktoe (1981) have extended their results under Ω to the general mixed factorial for all practical partionings of the full parametric vector. Joiner (1973) initiated the study of the above two problems relative to the group Δ of factor permutations under fairly restrictive conditions on a subvector of parameters and using a particular subgroup of Δ. The present paper generalizes Joiner's results in several directions: (i) a more flexible choice of the parametric subvector is provided for the invariance problem, (ii) any arbitrary subgroup of Δ can be used rather than a fixed subgroup, (iii) the randomization problem is resolved in conjunction with an arbitrary subgroup of Δ and Ω, and (iv) the results are applicable to the most practical partionings of the complete parametric vector, including odd and even-resolution designs.

Specifically, the second section provides the general framework of the mixed factorial, the group Δ of factor permutations along with the notion of a Γ-closed subset P of parameters, where Γ is a subgroup of Δ. In addition the concept of a primitive parametric set is introduced. Three theorems are then established, the first one presenting a characterization of Δ-closedness of a parametric subset P. The second theorem provides a decomposition of the Γ-closed set $\Gamma(P)$ in terms of primitive parametric sets and the third theorem establishes the cardinality of the Δ-closed set $\Delta(P)$. Finally, the concepts of admissibility of a parametric subset, as used by Srivastava, Raktoe and Pesotan (1976), is recalled and its connection with Γ-closedness of P is stated. In Section 3, three connected results are given using an arbitrary subgroup Γ of Δ. The culmination is the establishment of the invariance of the spectrum of the information matrix under any subgroup Γ of Δ relative to a Γ-closed subvector $\underline{P}$. In Section 4, the randomized unbiased estimation of a linear parametric function of a subvector $\underline{P}$ is established. This is achieved using the composition $\Omega \circ \Gamma$ and an admissible Γ-closed subvector in the most general partioning of the whole parametric vector $\underline{P}_0$. In the final section, some applications are given from the viewpoints of randomization, optimality and minimal cataloguing.

2. Preliminary definitions and results. Consider the $s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k}$ factorial with $s_1 < s_2 < \ldots s_k$ ($s_i \geq 2$), where each of the n_i-factors in the i-th grouping has s_i levels from the set $S_i = \{0,1,2,\ldots,s_i-1\}$.

Let $S = S_1^{n_1} \times S_2^{n_2} \times \ldots \times S_k^{n_k}$ be the Cartesian product of the sets $S_i^{n_i}$, where $S_i^{n_i}$ is the Cartesian product of S_i with itself n_i-times. A treatment combination $\underline{x} = (\underline{x}_1\ \underline{x}_2\ \ldots\ \underline{x}_k)$ in the complete replicate S will be written as a k-tuple, where $\underline{x}_j$ in S^{n_j} is a treatment combination of the $s_j^{n_j}$ factorial. With each treatment $\underline{x}$ in S we associate a factorial effect $\underline{A}^{\underline{x}} = \underline{A}_1^{\underline{x}_1}\ \underline{A}_2^{\underline{x}_2} \ldots \underline{A}_k^{\underline{x}_k}$ of the $\prod_{i=1}^{k} s_i^{n_i}$ factorial, where $\underline{A}_j^{\underline{x}_j}$ is a factorial effect associated with the $s_j^{n_j}$ factorial. In this case $\underline{x}$ and $\underline{x}_j$ will be called the exponents of the factorial effects $\underline{A}^{\underline{x}}$, $\underline{A}^{\underline{x}_j}$ respectively. Let P_0 be the set of all factorial effects of the $\prod_{i=1}^{k} s_i^{n_i}$ factorial.

Subsets of P_0 will be referred to as parametric sets.

For each i, let Δ_i be the symmetric group of all permutations of the set $\{1,2,\ldots,n_i\}$ and let $\Delta = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_k$ be the direct product of these groups. Each element of Δ is a k-tuple and will be called a factor permutation and any subgroup of Δ will be called a factor permutation group.

If $\tau \epsilon \Delta_i$ is any permutation and $\underline{y} = (y_1\ y_2\ \dots\ y_{n_i})$ is a treatment combination of the $s_i^{n_i}$ factorial then define

$$(2.1)\qquad \tau(\underline{y}) = (y_{\tau(1)}\ y_{\tau(2)}\ \cdots\ y_{\tau(n_i)}) ,$$

the treatment combination obtained by permuting the entries in $\underline{y}$ according to τ.

Let $\delta = (\delta_1, \delta_2, \dots, \delta_k)$ be in Δ, $\underline{x} = (\underline{x}_1\ \underline{x}_2\ \dots\ \underline{x}_k)$ in S, Γ a factor permutation group and P any parametric set. Define

$$(2.2)\qquad \delta(\underline{y}) = \left(\delta_1(\underline{x}_1)\delta_2(\underline{x}_2)\ \dots\ \delta_k(\underline{x}_k)\right),$$

and

$$(2.3)\qquad \Gamma(P) = \{\underline{A}^{\delta(\underline{x})} : \underline{A}^{\underline{x}} \epsilon P,\ \delta \epsilon \Gamma\}.$$

If $P = \{\underline{A}^{\underline{x}}\}$ is a singleton set the notation $\Gamma(\underline{x})$ will be used to denote $\Gamma(\{\underline{A}^{\underline{x}}\})$.

The parametric set P will be called δ-<u>closed</u> if and only if $\delta(P) = P$, where $\delta(P) = \{\underline{A}^{\delta(\underline{x})} : A^{\underline{x}} \epsilon P\}$, and P will be called Γ-<u>closed</u> if and only if $\Gamma(P) = P$. Observe that P is Γ-closed if and only if P is δ-closed for each δ in Γ and that the set $\Gamma(P)$ is always Γ-closed.

The two extremes, namely the entire parametric set P and the singleton set consisting of the mean $A^{\underline{0}}$ are Γ-closed for any factor permutation group Γ. Parametric sets which are Δ-closed include the interesting and most often studied sets, such as, the set of all main effects, the set of all main effects and the mean, and more generally the set of all t-factor interactions.

Let ϵ denote the identity permutation, $\tau_1 = (12)$ the two cycle interchanging 1 and 2, and for $1 \le i \le k$, let τ_{2i} be the n_i-cycle $\tau_{i2} = (12\ldots n_i)$ in Δ. Let ϵ_{i1} and ϵ_{i2} be the k-tuples in Δ which have in their i-th entry the permutations τ_1 and τ_{i2} respectively, and ϵ in each of the remaining positions. Then since τ_1 and τ_{i2} generates Δ_{n_i}, the following may easily be verified.

<u>THEOREM</u> 2.1. A parametric set P is Δ-closed if and only if P is δ-closed for each factor permutation δ in the set $\{\epsilon_{i1} : 1 \le i \le k\} \cup \{\epsilon_{i2} : 1 \le i \le k\}$.

A parametric set P will be called a <u>primitive parametric set generated by Γ and $\underline{x}$</u> if and only if $\Gamma(\underline{x}) = P$, for some factor permutation group Γ and some treatment $\underline{x}$ in S, which is an exponent of a factorial effect in P. It will be shown below that primitive parametric sets can be used as basic building blocks to generate any Γ-closed set. Since for a given $\underline{x}$ and Γ, in principle $\Gamma(\underline{x})$ is easy to generate, this result provides a systematic way of generating $\Gamma(P)$.

For $\underline{A}^{\underline{x}}$, $\underline{A}^{\underline{y}}$ in P_0, and any factor permutation group Γ, define $\underline{A}^{\underline{x}}$ to be Γ-equivalent to $A^{\underline{y}}$, in symbols $\underline{A}^{\underline{x}} \sim_\Gamma \underline{A}^{\underline{y}}$ if and only if $\delta(\underline{x}) = \underline{y}$ for some δ in Γ. Clearly Γ-equivalence is an equivalence relation on any parametric set P. If $P/\sim_\Gamma$ denotes the quotient set of P modulo the relation $\sim_\Gamma$ then

(2.5) $$P = Q_1 \cup Q_2 \cup \dots \cup Q_r ,$$

a disjoint union, where $r = |P/\sim_\Gamma|$ is the cardinality of the quotient set and each of the subsets Q_j of P is an equivalence class of Γ-equivalent effects. Select $\underline{A}^{\underline{x}(j)}$ in Q_j for each j and let $R = \{\underline{A}^{\underline{x}(j)} : 1 \le j \le r\}$ be a set of distinct representatives one from each equivalence class. The following is now the main result of this section:

THEOREM 2.2. *For any parametric set P and any factor permutation group Γ*

(2.6) $$\Gamma(P) = \Gamma(\underline{x}_{(1)}) \cup \Gamma(\underline{x}_{(2)}) \cup \dots \cup \Gamma(\underline{x}_{(r)}),$$

a disjoint set union of primitive parametric sets generated by Γ and the set of exponents of the effects

<u>in R, where R is a distinct set of representatives one from each Γ-equivalent equivalence class.</u>

PROOF. From the definition of Γ-equivalence it is easy to verify that for each j, $\Gamma(Q_j) = \Delta(\underline{x}_{(j)})$. Thus the equation (2.6) follows from (2.5). Finally, suppose $\underline{A}^{\underline{y}}$ lies in the intersection of $\Gamma(\underline{x}_{(j)})$ and $\Gamma(\underline{x}_{(\ell)})$ with $j \neq \ell$. Then there exists δ_1, δ_2 in Γ such that $\underline{y} = \delta_1\ \underline{x}_{(j)}$ and $\underline{y} = \delta_2(\underline{x}_{(\ell)})$. Hence $\delta_2^{-1}\ \delta_1\ (\underline{x}_{(j)}) = \underline{x}_{(\ell)}$. Since Γ is a group, $\delta_2^{-1}\ \delta_1 \in \Gamma$ and hence $\underline{x}_{(j)}$ is Γ-equivalent to $\underline{x}_{(\ell)}$ contradicting the choice of these exponents. Hence the equation (2.6) gives a disjoint union, completing the proof.

In general the cardinality of Γ(P) depends on the cardinalities of Γ and P and appears difficult to predict. However, for the case when Γ = Δ, the cardinality of Δ(P) may be calculated in terms of the cardinality of the quotient set $P/\sim_\Delta$. Consider the decomposition of Δ(P) as a disjoint union of primitive parametric sets as stated in Theorem 2.2, say

$$(2.7) \qquad \Delta(P) = \Delta(\underline{y}_{(1)}) \cup \Delta(\underline{y}_{(2)}) \cup \ \dots \ \cup \Delta(\underline{y}_{(\ell)}),$$

where $\ell = |P/\sim_\Delta|$. For each j, $1 \le j \le \ell$, suppose $\underline{y}_{(j)} = (\underline{y}_{(j)_1}\ \underline{y}_{(j)_2}\ \cdots\ \underline{y}_{(j)k})$. In $\underline{y}_{(j)i}$ let $t_{(j)i_1}, t_{(j)i_2}, \dots, t_{(j)im_{ij}}$ be the repetition counts of the distinct entries in $\underline{y}_{(j)i}$. Then, if $\Delta_i(\underline{y}_{(j)i})$ denotes the set of factorial effects of the $s_i^{n_i}$ factorial with exponents from the set $\{\delta(\underline{y}_{(j)i}) : \delta \in \Delta_i\}$ then

$$(2.8) \qquad |\Delta_i(\underline{y}_{(j)i})| = \frac{n_i!}{\prod_{h=1}^{m_{ij}} (t_{(j)ih}!)}$$

Now, since $\Delta = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_k$, it follows that

$$(2.9) \qquad |\Delta(\underline{y}_{(j)})| = \prod_{i=1}^{k} |\Delta_i(\underline{y}_{(j)i})| \ .$$

From equations (2.7), (2.8) and (2.9) one immediately obtains the following theorem.

THEOREM 2.3. For any parametric set P, the cardinality of $\Delta(P)$ is

$$|\Delta(P)| = \sum_{j=1}^{\ell} \left[\frac{\prod_{i=1}^{k} (n_i!)}{\prod_{i=1}^{k} \prod_{h=1}^{m_{ij}} (t_{(j)ih}!)} \right] ,$$

where ℓ is the cardinality of the quotient set $P/\sim_\Delta$.

The concepts of an admissible set of factorial effects and a basic collection of factorial effects were introduced in Srivastava et al. (1976). These definitions are reproduced here for the sake of completeness. A set of factorial effects P in the $t_1 \times t_2 \times \ldots \times t_m$ factorial is called admissible iff whenever $A_1^{i_1} A_2^{i_2} \ldots A_m^{i_m}$ belongs to P and $i_j \neq 0$ $(1 \leq j \leq m)$ then $A_1^{i_1} A_2^{i_2} \ldots A_{j-1}^{i_{j-1}} A_j^{\ell} A_{j+1}^{i_{j+1}} \ldots A_m^{i_m}$ belongs to P for all $\ell \neq 0$ in the set of levels $T_j = \{0,1,2,\ldots,t_j-1\}$.

Define formal row vectors of factorial effects of the i-th factor as follows

$$v_i^0 = (A_i^0),$$

$$v_i^1 = (A_i^1, A_i^2, \ldots, A_i^{t_i-1}).$$

For $e_j \in \{0,1\}$ and $1 \leq j \leq m$, let

$$\underline{\alpha}' = \alpha'(e_1, e_2, \ldots, e_m) = v_1^{e_1} \otimes v_2^{e_2} \otimes \ldots \otimes v_m^{e_m},$$

be the formal Kronecker product of these formal row vectors. A set of factorial effects α is called a basic collection iff there exists a sequence $(e_j)_{j=1}^m$ with $e_j \in \{0,1\}$, such that α is the set of effects formed from the entries of the row vector $\underline{\alpha}'(e_1, e_2, \ldots e_m)$. It is noted in Srivastava et al. (1976) that a set of factorial effects is admissible iff it is a disjoint union of basic collections of effects. Further, the set complement of an admissible set is admissible.

Let δ be a factor permutation and let $\underline{\alpha}'(\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_k)$ be a basic collection of factorial effects for the $\prod_{i=1}^{k} s_i^{n_i}$ factorial, where for each j ($1 \le j \le k$) $\underline{m}_j = (m_{j1}, \ldots, m_{jn_j})$ is a (0-1) vector. Then, clearly $\delta(\underline{\alpha}'(\underline{m}_1, \ldots, \underline{m}_k)) = \underline{\alpha}'(\delta_1(\underline{m}_1), \delta_2(\underline{m}_2), \ldots, \delta_k(\underline{m}_k))$ where $\delta_j(\underline{m}_j) = (m_{j\delta_j(1)}\ m_{j\delta_j(2)} \cdots m_{j\delta_j(n_j)})$. It follows immediately from this and the above remarks that if Γ is any factor permutation group and P <u>is an admissible set of factorial effects then $\Gamma(P)$ is an admissible set of factorial effects which is Γ-closed</u>. Moreover, since the parametric set P_0 is Γ-closed it follows from Theorem 2.2 that <u>if</u> P <u>is a</u> Γ-<u>closed set then so is the complement</u> $P^c = P_0 - P$. In particular if P is both Γ-closed and admissible then so is P^c. These remarks will be used in the ensuing development.

3. <u>The factor interchange invariance theorem</u>. Joiner (1973) established the characteristic roots invariance of the information matrix of a fraction from the $\prod_{i=1}^{k} s_i^{n_i}$

factorial with respect to a Γ-closed admissible vector of effects under factor permutations from a factor permutation group Γ, where $\Gamma = (\delta_1) \times (\delta_2) \times \ldots \times (\delta_k)$, $\delta_j \in \Delta_j$ and (δ_j) is the cyclic subgroup of Δ_j generated by δ_j. The purpose of this section is to generalize this result where the information matrices of fractions are taken relative to Γ-closed sets for any factor permutation group Γ.

With each treatment $\underline{x} = (\underline{x}_1\ \underline{x}_2\ \ldots\ \underline{x}_k)$ of the $\prod_{i=1}^{k} s_i^{n_i}$ factorial associate an observation denoted by $y(\underline{x}) = y(\underline{x}_1\ \underline{x}_2\ \ldots\ \underline{x}_k)$ and let $\underline{Y}_0$ be the set of all observations. Let $X = X_1^{n_1} \otimes X_2^{n_2} \otimes \ldots \otimes X_k^{n_k}$ be the Kronecker product of real orthogonal matrices $X_j^{n_j}$, where $X_j^{n_j} = X_j \otimes \ldots \otimes X_j$ is the Kronecker product of X_j with itself n_j times, and each X_j is a real orthogonal matrix of order s_j with each first column entry equal to $1/(s_j)^{1/2}$. Then X is a real orthogonal matrix of order $s = \prod_{i=1}^{k} s_i^{n_i}$ with each first column entry equal to $1/(s)^{1/2}$. Associate with the observation vector $\underline{Y}_0$ and the column vector $\underline{P}_0$ obtained from the entire parametric set P_0 the well-known linear model

$$E(\underline{Y}\ ) = X\,\underline{P}_0\ , \quad \mathrm{Cov}\ (\underline{Y}_0) = \sigma^2 I_s.$$

Let $\underline{P}$ be a $K \times 1$ parametric vector and let $P_0' = (\underline{P}', \underline{Q}')$.

Then the implied linear model for a $N \times 1$ observation vector $\underline{Y}$ under the assumption that $\underline{Q} = \underline{0}$ is

$$E(\underline{Y}) = X(\underline{Y},\underline{P})\ \underline{P}, \quad \mathrm{Cov}\ (\underline{Y}) = \sigma^2 I_N,$$

where $X(\underline{Y},\underline{P})$ is the $N \times K$ submatrix of X read off from X relative to $\underline{Y}$ and $\underline{P}$.

If $\underline{Y}$ is any observation vector and $\delta = (\delta_1, \delta_2, \ldots, \delta_k)$ in Δ is any factor permutation then $\delta(\underline{Y})$ will denote the observation vector obtained from $\underline{Y}$ wherein each component $y(\underline{x}) = y(\underline{x}_1\ \underline{x}_2\ \ldots\ \underline{x}_k)$ in $\underline{Y}$ is replaced by $y(\delta(\underline{x})) = y(\delta_1(\underline{x}_1)\delta_2(\underline{x}_2)\ldots\delta_k(x_k))$.

Let $\underline{x}$ be a treatment in S and let Γ be any factor permutation group. Consider the primitive parametric set $\Gamma(\underline{x})$ generated by Γ and $\underline{x}$. The column vector $\underline{\Gamma}(\underline{x})$ whose entries are precisely the elements of $\Gamma(\underline{x})$ will be called a primitive vector.

The main result of this section is the following:

THEOREM 3.1. For each factor permutation δ in a factor permutation group Γ, a primitive vector $\Gamma(\underline{x})$ and any observation vector $\underline{Y}$, there exists an orthogonal matrix $V(\Gamma(\underline{x}),\delta)$ such that

$$X(\underline{Y},\Gamma(\underline{x}))V(\Gamma(\underline{x}),\delta) = X(\delta(\underline{Y}),\Gamma(\underline{x})).$$

Further, when $\Gamma = \Delta$, and if $t = |\Delta(\underline{x})|$,

$$\sum_{\delta \in \Delta} V(\Delta(\underline{x}),\delta) = (t-1)!\, J$$

where J is a matrix of plus ones of order t.

PROOF. Select any factor permutation τ in Γ. Since Γ is a group $\tau\delta$ is in Γ, and since $\Gamma(\underline{x})$ is Γ-closed the factorial effects with exponents $\tau(\underline{x})$ and $\tau(\delta(\underline{x}))$ both belong to $\Gamma(\underline{x})$. Using the product definition of factorial effects it may be easily verified that the columns determined by the factorial effect $A^{\tau(\underline{x})}$ and $\underline{Y}$, and the factorial effect $\underline{A}^{\tau(\delta(\underline{x}))}$ and $\delta(\underline{Y})$ in the matrices $X(\underline{Y},\Gamma(\underline{x}))$ and $X(\delta(\underline{Y}),\Gamma(\underline{x}))$ respectively are the same. Thus δ determines a permutation $\phi_\delta : \Gamma(\underline{x}) \to \Gamma(\underline{x})$ given by $\phi_\delta(\underline{A}^{\tau(\underline{x})}) = \underline{A}^{\tau(\delta(\underline{x}))}$ such that each column of $X(\underline{Y},\Gamma(\underline{x}))$ is assigned the identical column in $X(\delta(\underline{Y}), \Gamma(\underline{x}))$ under ϕ_δ. Hence there exists a column permutation matrix $V(\Gamma(\underline{x}),\delta)$ of the identity matrix of order $|\Gamma(\underline{x})|$ determined by ϕ_δ satisfying the first part of the theorem and the second part follows from the definition of $V(\Delta(\underline{x}),\delta)$.

Let P be any parametric set and Γ any factor permutation group. Then, by Theorem 2.2 the Γ-closed set $\Gamma(P)$ is a disjoint union of primitive parametric sets as given in equation (2.6). The column vector

$\Gamma(\underline{P})$ given by

$$(3.1) \qquad (\Gamma(\underline{P}))' = [\Gamma(\underline{x}_{(1)})', \Gamma(\underline{x}_{(2)})', \ldots, \Gamma(\underline{x}_{(r)})']$$

and obtained from the Γ-closed set Γ(P) will be called a Γ-closed vector. If $\underline{Y}$ is any observation vector and $\Gamma(\underline{P})$ is a Γ-closed parametric vector, then define the design matrix corresponding to $\underline{Y}$ and $\Gamma(\underline{P})$ by

$$(3.2) \qquad X_{\Gamma(\underline{P})}(\underline{Y}) = [X(\underline{Y},\Gamma(\underline{x}_{(1)}))|X(\underline{Y}, \Gamma(\underline{x}_{(2)}))|\ldots|X(Y,\Gamma(\underline{x}_{(r)}))].$$

COROLLARY 3.1. *For each factor permutation δ in Γ, a Γ-closed vector Γ(P) and any observation vector Y there exists an orthogonal matrix* $V(\Gamma(\underline{P}),\delta)$ *such that*

$$X_{\Gamma(\underline{P})}(\underline{Y})V(\Gamma(\underline{P}),\delta) = X_{\Gamma(\underline{P})}(\delta(\underline{Y}))$$

Further, if $\Gamma = \Delta$ *and* $t_j = |\Gamma(\underline{x}_{(j)}|$, *then* $\sum_{\delta\in\Delta} V(\Delta(P),\delta)$ *is a block diagonal matrix consisting of r blocks whose (j-j)-th block is* $(t_j-1)!\,J$, *where J is a matrix of plus ones of order* t_j.

PROOF. By Theorem 3.1, for each δ in Γ and each j $(1 \le j \le r)$, there exists an orthogonal matrix $V(\Gamma(\underline{x}_{(j)}),\delta)$ such that $X(\delta(\underline{Y})), \Gamma(\underline{x}_{(j)}) = X(\underline{Y},\Gamma(\underline{x}_{(j)}))$ $V(\Gamma(\underline{x}_{(j)}), \delta)$. Define $V(\Gamma(\underline{P}),\delta)$ to be a diagonal block matrix with $V(\Gamma(\underline{x}_{(j)}),\delta)$ being the (j-j)-th block. The assertions in the corollary can now be easily verified.

An immediate corollary to Corollary 3.1 is the following:

COROLLARY 3.2. Let $\underline{Y}$ be any observation vector and $\Gamma(\underline{P})$ any Γ-closed vector. Then any pair of matrices in the set of information matrices $\{X'_{\Gamma(\underline{P})}(\delta(\underline{Y}))X_{\Gamma(\underline{P})}(\delta(\underline{Y})): \delta\epsilon\Gamma\}$ have the same characteristic roots.

4. The randomization theorem under the groups of level and factor permutations. In this section the complete parametric vector $\underline{P}_0$ will be partitioned as

$$\underline{P}'_0 = (\underline{P}'_1 \mid \underline{P}'_2 \mid \underline{P}'_3)$$

where $\underline{P}_i$ will be Γ-closed admissible vectors defined below, and $\underline{P}_1$ is a vector to be estimated, $\underline{P}_2$ is a vector not of interest and not assumed to be known and $\underline{P}_3$ is a vector assumed to be known (which without loss of generality can be taken to be zero) such that $1 \le |P_1| \le |P_0|$, $0 \le |P_2| \le |P_0| - 1$, and $0 \le |P_3| \le |P_0| - |P_1| - |P_2| \le |P_2| - 1$. Such a partitioning leads explicitly to the following cases:

(i) $|P_1| = |P_0|$, $|P_2| = |P_3| = 0$, (ii) $|P_2| = 0$, $|P_3| \neq 0$, (iii) $|P_2| \neq 0$, $|P_3| \neq 0$, and (iv) $|P_2| \neq 0$, $|P_3| = 0$. Since the cases (i), (ii) and (iv) can clearly be subsumed under case (iii), this section will concern itself with the above type of partitioning for case (iii) only.

In Srivastava et al. (1976) the unbiased estimation of a linear function of an admissible vector is established in the general factorial setting when a randomized observation vector is selected from among the whole class of observation vectors generated by the group of level permutations. This result generalized the randomization results of Ehrenfeld and Zacks (1961) and Zacks (1963, 1964) for regular fractions to arbitrary fractions in the general factorial setting.

The above leads one to ask the question: is it possible to obtain the unbiased estimation of Γ-closed vector in the $\prod_{i=1}^{k} s_i^{n_i}$ factorial setting when a randomized observation vector is selected from among the whole class of observation vectors generated by a factor permutation group Γ? A simple example answers this question in the negative. For instance, let $\underline{Y}$ be the observation vector associated with the design D = {(01), (12)} of the 3^2 factorial and let $\Gamma = \Delta = \{i, (12)\}$ the symmetric group on two symbols.

Consider the following partitioning of the entire parametric set: $\underline{P}_0' = [\underline{P}_1' \mid \underline{P}_2' \mid \underline{P}_3']$, where $P_1 = \{A_1^1 A_2^0, A_1^0 A_2^1\}$, $P_2 = \{A_1^1 A_2^1, A_1^2 A_2^2\}$ and $P_3 = P_0 - (P_1 \cup P_2)$. Now each of the vectors $\underline{P}_i$ are Δ-closed. Select a η in Δ. Since $|\Delta| = 2$, the probability of this selection is ½. Let $\underline{P}_{1,\eta}^0$ be a solution of $\underline{P}_{1,\eta}$ in the equation $\eta(\underline{Y}) = X(\eta(\underline{Y}),\underline{P})\ \underline{P}$, where $\underline{P}' = [\underline{P}_1' \mid P_2']$. Let $\underline{\mu}$ be a column vector such that $[\underline{\mu}' \mid \underline{0}']$ lies in row space of the design matrix $[X(\delta(\underline{Y}),\underline{P}_1) \mid X(\delta(\underline{Y}),P_2]$ for each δ in Δ. Then if G is the probability distribution of the underlying random variable $\delta(\underline{Y})$ and the expected value of $\underline{\mu}'\ \underline{P}_{1,\eta}^0$ is defined by $E_\Delta(\underline{\mu}'P_{1,\eta}^0) = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} E_G(\underline{\mu}'\underline{P}_{1,\eta}^0 \mid \delta(\underline{Y}))$ then one can easily calculate that $E_\Delta(\underline{\mu}'\underline{P}_{1,\eta}^0) \neq \underline{\mu}'\underline{P}_{1,\eta}$.

In view of the above example one is naturally lead to investigate the following situation: is it possible to obtain the unbiased estimation of a <u>Γ-closed admissible vector</u> in the $\prod_{i=1}^{k} s_i^{n_i}$ factorial setting when a randomized observation vector is selected from among the whole class of observation vectors generated by the composition $\Omega \circ \Gamma$ and an observation vector $\underline{Y}$, that is, when the selection is made from the class $\Omega \circ \Gamma(\underline{Y}) = \{\omega(\delta(\underline{Y})) : \omega \text{ in } \Omega, \delta \text{ in } \Gamma\}$? This is, in fact, the case and this is established in this case. A beginning in this direction has been made by Joiner (1973), where the group Γ considered is a direct product of cyclic groups each of order two.

Consider the $\prod_{i=1}^{k} s_i^{n_i}$ factorial and let Ω_i be the symmetric group of all permutations on the set S_i of s_i levels and let Ω be the direct product of the groups $\Omega_1^{n_1} \times \Omega_2^{n_2} \times \ldots \times \Omega_k^{n_k}$, where $\Omega_j^{n_j} = \Omega_j \times \Omega_j \times \ldots \times \Omega_j$ is the direct product of Ω_j with itself n_j-times. Then $\Omega = \{\omega : \omega = (\underline{\omega}_1, \underline{\omega}_2, \ldots, \underline{\omega}_k)$, where $\underline{\omega}_i = (\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{in_i})$ is a n_i-tuple of permutations each from Ω_i. An element ω in Ω is called a <u>level permutation</u> and Ω is called the <u>group of level permutations</u>. If $\underline{x}_i$ is a treatment in $S_i^{n_i}$ and $\underline{\omega}_i \in \Omega_i^{n_i}$, define $\underline{\omega}_i(\underline{x}_i) = (\omega_{i_1}(x_{i_1}))\omega_{i_2}(x_{i_2})\ldots\omega_{in_i}(x_{in_i}))$ and by extension if $\underline{x}$ is a treatment in S, define $\omega(\underline{x}) = (\underline{\omega}_1(\underline{x}_1)\ \underline{\omega}_2(\underline{x}_2) \ldots \underline{\omega}_k(\underline{x}_k))$. Moreover, if $\underline{Y}$ is an observation vector in $\prod_{i=1}^{k} s_i^{n_i}$ factorial, define $\omega(\underline{Y})$ to be the observation vector obtained from $\underline{Y}$ by replacing each observation $y(\underline{x})$ in $\underline{Y}$ by $y(\omega(\underline{x}))$.

Let $\underline{Y}$ be an observation vector and Γ a factor permutation group. Consider the class of observation vectors $\Gamma\circ\Omega(\underline{Y}) = \{\delta\omega(\underline{Y}) : \delta\in\Gamma, \omega\in\Omega\}$ generated by the action of Γ and Ω on $\underline{Y}$. It might be noted that the class of observation vectors $\Omega\circ\Gamma(\underline{Y}) = \{\omega\delta(\underline{Y}) : \delta\in\Gamma, \omega\in\Omega\}$

coincides with the class $\Gamma \circ \Omega(\underline{Y})$, since clearly for each δ in Γ and ω in Ω, there exists a ω_1 in Ω such that $\delta\omega = \omega_1\delta$.

Let P be an admissible set of factorial effects and let Γ be any factor permutation group. Then as noted earlier $\Gamma(P)$ is an admissible set which is Γ-closed and $(\Gamma(P))^c$ is also an Γ-<u>closed admissible</u> set. The vector $\Gamma(\underline{P})$ whose entries are from the Γ-closed admissible set $\Gamma(P)$ will be called a Γ-<u>closed admissible vector</u>. Throughout this section a partitioning of the entire parametric vector $\underline{P}_0$ into three parts of the following type will always be considered:

$$(4.1) \qquad \underline{P}_0' = [(\Gamma(\underline{P}_1))' | (\Gamma\underline{P}_2))' | \underline{P}_3'],$$

where $\Gamma(\underline{P}_1)$, $\Gamma(\underline{P}_2)$ are Γ-closed admissible vectors, $\Gamma(P_2)$ is a subset of $(\Gamma(P_1))^c$ and $P_3 = P_0 - (\Gamma(P_1) \cup \Gamma(P_2))$. It follows from the above remarks that $P_3 = \Gamma(P_3)$ is also a Γ-closed admissible set, and, P_0 is a disjoint union of the sets $\Gamma(P_1)$, $\Gamma(P_2)$ and P_3. A decomposition of $\underline{P}_0$ in the form (4.1) will be called a Γ-<u>closed admissible triple</u>.

Consider the partition given in (4.1) and let

(4.2) $$M_{ij} = M_{ij}(\underline{Y}) = X'(\underline{Y},\Gamma(\underline{P}_i))X(\underline{Y},\Gamma(\underline{P}_j)),$$

for i = 1,2,3 and j=1,2,3 and let M_{ij}^- be a generalized inverse of M_{ij}. In particular note that M_{ij} is the information matrix of the observation vector $\underline{Y}$ relative to $\underline{\Gamma}(\underline{P}_i)$. If $\underline{\mu}'(\Gamma(\underline{P}_1))$ is linear function of $\Gamma(\underline{P}_1)$ then it is known that $\Gamma(\underline{P}_1)$ is estimable if and only if the vector $[\underline{\mu}'|\underline{0}']$ lies in the row space of the design matrix $[X(\underline{Y},\Gamma(\underline{P}_1))|X(\underline{Y},\Gamma(\underline{P}_2))]$, and the best linear unbiased estimate of $\underline{\mu}'(\Gamma(\underline{P}_1))$ is given by

(4.3) $$\underline{\mu}'(\hat{\Gamma}(\underline{P}_1)) = \underline{\mu}'[M_{11}-M_{12}M_{22}^-M_{21}]^-[X'(Y,\Gamma(\underline{P}_1)) -M_{12}M_{22}^-X'(\underline{Y},\Gamma(\underline{P}_2))]\underline{Y}.$$

For each ω in Ω and $\delta\epsilon\Gamma$, define

(4.4) $$A(\delta\omega(\underline{Y})) = M_{11}(\delta\omega(\underline{Y}))-M_{12}(\delta\omega(\underline{Y}))M_{22}^-(\delta\omega(\underline{Y}))M_{21}(\delta\omega(\underline{Y})).$$

It is then known that the estimability of $\underline{\mu}'(\Gamma(\underline{P}_1))$ is equivalent th the condition that $\underline{\mu}'$ is in the row space of the matrix $A(\underline{Y})$. If $(\Gamma(\underline{P}_1))^0$ is a solution of the equation

$$E_G(\underline{Y}) = X(\underline{Y},\Gamma(\underline{P}_1))\underline{P}_1 + X(\underline{Y},\Gamma(\underline{P}_2))\underline{P}_2,$$

under the assumption $\underline{P}_3 = \underline{0}$ and a given distribution G, then the expected value of $\underline{\mu}'(\Gamma(\underline{P}_1)^0$ using a nonrandomized design is given by (writing A for $A(\underline{Y})$).

$$(4.5) \qquad E_G(\underline{\mu}'(\Gamma(\underline{P}_1))^0|\underline{Y}) = \underline{\mu}'[A^-A\Gamma(\underline{P}_1) + A^-[M_{12}-M_{12}M_{22}^-M_{22}]\Gamma(\underline{P}_2) + A^-[M_{13}-M_{12}M_{22}^-M_{23}]\Gamma(\underline{P}_3)].$$

The following lemmas will be useful in establishing the randomization theorem.

LEMMA 4.1.

$$\frac{1}{\prod_{i=1}^{k}(s_i!)^{n_i}}\ \frac{1}{|\Gamma|}\sum_{\substack{\delta\in\Gamma\\ \omega\in\Omega}} \underline{\mu}'A^-(\delta\omega(\underline{Y}))A(\delta\omega(\underline{Y}))\Gamma(\underline{P}_1) = \underline{\mu}'\Gamma(\underline{P}_1),$$

<u>where for each</u> ω <u>in</u> Ω <u>and each</u> δ <u>in</u> Γ, <u>the vector</u> $[\underline{\mu}'|\underline{0}']$ <u>lies in the row space of the design matrix</u> $[X(\delta\omega(\underline{Y}),\ \Gamma(\underline{P}_1))|X(\delta\omega(\underline{Y}),\ \Gamma(\underline{P}_2))]$.

<u>PROOF</u>. From the facts noted above, it follows that for each ω in Ω and each δ in Γ, the vector $\underline{\mu}'$ lies in the row space of $A(\delta\omega(\underline{Y}))$. Hence $\underline{\mu}' = \underline{\lambda}_{\delta,\omega}'\ A(\delta\omega(\underline{Y}))$ for each δ in Γ and ω in Ω.

Substituting this for μ' in the expression given in the lemma, since $AA^{-}A = A$, the lemma follows.

LEMMA 4.2. <u>For any partitioning of P_0 into a Γ-closed admissible triple</u>

$$\sum_{\substack{\omega\in\Omega \\ \delta\in\Gamma}} A^{-}(\delta\omega(\underline{Y}))[M_{13}(\delta\omega(\underline{Y})) - M_{12}(\delta\omega(\underline{Y}))M_{22}^{-}(\delta\omega(\underline{Y}))M_{23}(\delta\omega(\underline{Y}))] = 0,$$

<u>where 0 is the zero matrix</u>.

PROOF. By Corollary 3.1 there exist orthogonal matrices $V(\Gamma(\underline{P}_i),\delta)$ such that $X(\delta\omega(\underline{Y}),\ \Gamma(\underline{P}_i)) = X(\omega(\underline{Y}),\ \Gamma(\underline{P}_i))V(\Gamma(\underline{P}_i),\delta)$. Further by Corollary 3.1 of Srivastava <u>et al</u>. (1976), there exist orthogonal matrices $U(\Gamma(\underline{P}_i),\omega)$ such that $X(\omega(\underline{Y}),\ \Gamma(\underline{P}_i)) = X(\underline{Y},\Gamma(\underline{P}_i)U(\Gamma(\underline{P}_i),\omega)$. Substituting these equations in the expression given in the lemma reduces that expression to

$$\sum_{\substack{\omega\in\Omega \\ \delta\in\Gamma}} V'(\Gamma(\underline{P}_1),\delta)U'(\Gamma(\underline{P}_1),\omega)\ L\ U'(\Gamma(\underline{P}_3),\omega)\ V\ (\Gamma(\underline{P}_3),\delta)$$

$$= \sum_{\delta\in\Gamma} V'(\Gamma(\underline{P}_1),\delta)\left[\sum_{\omega\in\Omega} U'(\Gamma(\underline{P}_1)\ L\ U\ (\Gamma(\underline{P}_3),\omega)\right] V(\Gamma(\underline{P}_3),\delta),$$

where $L = [M_{11}-M_{12}M_{22}^{-}M_{21}]^{-}[M_{13}-M_{12}M_{22}^{-}M_{23}]$ is a fixed matrix independent of ω in Ω and δ in Γ.

In exactly the same manner as shown in Lemma 4.2 of Srivastava *et al*. (1976) it may be shown that the sum $\Sigma(U'LU)$ in the above expression reduces to zero and this completes the proof.

Select permutations η in Ω, τ in Γ with respective probabilities $\dfrac{1}{\prod_{i=1}^{k} (s_i!)^{n_i}}$ and $\dfrac{1}{|\Gamma|}$. Then the following is the main result of this section.

THEOREM 4.1. *Let* $\underline{P}^0_{1,\eta,\tau}$ *be a solution to the equation* $\tau(\eta(\underline{Y})) = X(\tau\eta(\underline{Y}),\underline{P}))\underline{P}$, *where* $\underline{P}' = [\Gamma(\underline{P}_1)|\Gamma(\underline{P}_2)]$ *is the* Γ*-closed admissible vector consisting of the first two components of the partitioning of* $\underline{P}_0$ *into* Γ*-closed admissible triple. Then*

$$E_{\Omega,\Gamma}(\underline{\mu}'\underline{P}^0_{1,\eta,\tau}) = \underline{\mu}'(\Gamma(\underline{P}_1)),$$

where $E_{\Omega,\Gamma}$ *is defined by*

$$(4.6) \quad E_{\Omega,\Gamma}(\underline{\mu}'P^0_{1,\eta,\tau}) = \frac{1}{\prod(s_i!)^{n_i}} \frac{1}{|\Gamma|} \sum_{\substack{\omega\in\Omega \\ \delta\in\Gamma}} E_G(\underline{\mu}'\underline{P}^0_{1,\omega,\delta}|\delta\omega(\underline{Y})),$$

G is the probability distribution of the random variable $\delta\omega(\underline{Y})$ and $\underline{\mu}$ is a $|\Gamma(P_1)| \times 1$ column vector such that $[\underline{\mu}' | \underline{0}']$ lies in the row space of the design matrix

$$[X(\delta\omega(\underline{Y}), \Gamma(\underline{P}_1)) \mid X(\delta\omega(\underline{Y}), \Gamma(\underline{P}_2))]$$ *for each* ω *in* Ω *and* δ *in* Γ.

PROOF. Substituting equation (4.5) in (4.6) $E_{\Omega,\Gamma}$ reduces to a sum over three separate expressions. By Lemma 4.1 the sum over the first expression reduces to $\underline{\mu}'(\Gamma(\underline{P}_1))$. By a property of generalized inverse the sum over the second expression reduces to the zero matrix. By Lemma 4.2 the sum over the third expression reduces to the zero matrix as well, completing the proof.

5. *Some applications*. Parametric sets which are Δ-closed admit many practical settings, e.g. odd and even resolution cases. Similarly, admissible parametric sets cover a wide range of applicability. As already stated in section 1, if P is an admissible set of parameters then $\Gamma(P)$ is admissible and $\Gamma(P)$ is also Γ-closed. In the papers by Srivastava, Raktoe and Pesotan (1976) and Pesotan and Raktoe (1981) applications were given for the odd and even resolution

cases respectively. In these cases the parametric vector $\underline{P}_1$ is both admissible and Δ-closed. The class $\Omega(D)$ of designs was obtained by the action of Ω on a given initial design of desired resolution. Indeed, one may start with an optimal initial design D, and as long as the optimality criterion is a functional on the spectrum of the information matrix, all the designs in the generated class $\Omega(D)$ will be optimal also. In the Srivastava-Raktoe-Pesotan paper a minimal d-optimal resolution III design of the 2^4 factorial was selected as the initial design in order to illustrate the various applications of the class $\Omega(D)$, while in the Pesotan-Raktoe paper a minimal resolution IV design of the 2^3 factorial was the initial design to generate the class $\Omega(D)$.

In terms of applications of the present development our interest lies in the class of designs $(\Omega\circ\Delta)(D)$ generated by the action of the composition $\Omega\circ\Delta$ on the initial design D, which will be taken to be of odd (even) resolution. This means that the row-space condition is not needed since the information matrix of a design of odd (even) resolution is non-singular and this is preserved under $\Omega\circ\Delta$ as implied by the invariance theorem. Hence $(\Omega\circ\Delta)(D)$ will be a class of genuine designs of odd (even) resolution. As said above, an optimality criterion based on the spectrum of the information matrix can be imposed on the initial design D so that the resulting class $(\Omega\circ\Delta)(D)$ will be a class of optimal designs. In any case, the randomization theorem tells us that if a design is selected at random from the class $(\Omega\circ\Delta)(D)$, then an unbiased estimator of $\underline{P}_1$ is obtained even if $\underline{P}_3$ is not equal to $\underline{0}$ in both the odd and even resolution case. If $\underline{P}_3$ is equal to $\underline{0}$ (and there are examples of this in practice for both settings) then a further criterion, not necessarily based on the spectrum of the information matrix (e.g. cost) can be used to select a cost-optimal design from $(\Omega\circ\Delta)(D)$. As a final application the composition $\Omega\circ\Delta$ can be used to reduce the catalogue of the set of all possible designs of given resolution and cardinality. This will lead to an improvement over the catalogues which can be developed from the results of Pesotan, Raktoe and Federer (1975), who used the sub-group of translations of the group Ω.

Let us now illustrate the three applications mentioned above with a specific example. Consider the minimal d-optimal resolution III design $D^* = \{(000), (110), (101), (011)\}$ of the 2^3 factorial. The vector $\underline{P}_1$ of main effects and the mean is both admissible and

Δ-closed. Under the action of $\Omega \circ \Delta$ the class $(\Omega \circ \Delta)(D^*)$ contains only two distinct designs, namely $D_1 = D^*$ and $D_2 = \{(111), (001), (010), (100)\}$. If a design is randomly selected then the main effects and the mean can be estimated unbiasedly even though two and higher interactions may not be zero. If the two and higher order interactions are zero then under the cost function: $c(000) = 1$, $c(100) = 2$, $c(010) = 2$, $c(001) = 2$, $c(110) = 4$, $c(101) = 4$, $c(011) = 4$, $c(111) = 6$, we see that $c(D_1) = 12$ and $c(D_2) = 13$. Hence the cost-optimal and d-optimal design is D_1. Of the possible $(8!)|(4')(4!) = 70$ designs, 12 are singular and 58 are non-singular resolution III designs. One may provide a catalog of 14 designs, which generate under the action of Ω all the 70 designs. A reduction from 14 to 6 designs is made if the group $\Omega \circ \Delta$ is used rather than Ω itself. The resulting 6 designs with their corresponding determinants of the information matrix (under the usual (-1,1)-model) are:

	D_1	D_2	D_3	D_4	D_5	D_6
	000	000	000	000	000	000
	011	100	100	101	001	001
	101	101	101	110	010	110
	110	110	111	111	011	111
$\lvert X'X \rvert$:	256	64	64	64	0	0

As a final note, the cardinalities of the three sets of designs of different determinants and exhausting the 70 designs are given by: $|(\Omega\circ\Delta)(D_1)| = 2$, $|(\Omega\circ\Delta)(D_2) \cup (\Omega\circ\Delta)(D_3) \cup (\Omega\circ\Delta)(D_4)| = 56$ and $|(\Omega\circ\Delta)(D_5) \cup (\Omega\circ\Delta)(D_6)| = 12$.

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