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AN ERROR ESTIMATE FOR GAUSS-JACOBI QUADRATURE FORMULA WITH THE HERMITE WEIGHT $w(x) = \exp(-x^2)$

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The purpose of this paper is to give an estimate of the error in approximating the integral

$$\int_{-\infty}^{\infty} f(x) \exp(-x^2)\, dx$$

by the Gauss-Jacobi quadrature formula $Q_n(w;f)$, assuming that f is an entire function satisfying a certain growth condition which depends on the Hermite weight function $w(x) = \exp(-x^2)$.

1. INTRODUCTION

Let $d\alpha$ be a non-negative measure supported in the interval (a, b), $-\infty \leq a < b \leq \infty$. Let the support of $d\alpha$ contain infinitely many points and let

$$\int_a^b x^n \, d\alpha(x) < \infty, \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Then there exists a uniquely determined sequence of orthonormal polynomials $\{p_n(d\alpha; x)\}$ generated by this measure (see e.g. [1; CH. I], [5; CH. II]); they are determined by the properties

(a) $p_n(d\alpha; x) = \gamma_n x^n + \ldots$ is a polynomial of degree n and $\gamma_n > 0$.

(b) $$\int_a^b p_n(d\alpha)\, p_m(d\alpha)\, d\alpha = \begin{cases} 0 & ; \text{ if } m \neq n \\ 1 & ; \text{ if } m = n \end{cases} .$$

It is well known that all zeros x_{kn} $(k = 1, 2, \ldots, n)$ of $p_n(d\alpha; x)$ are real, simple and are contained in (a, b). We shall assume, as usual, that

$$x_{1n} > x_{2n} > \ldots > x_{nn} .$$

If, in addition, $d\alpha$ is an absolutely continuous measure, then $d\alpha(x) = \alpha'(x)\,dx$ and $\alpha'(x)$ is called a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $p_n(d\alpha)$ by $p_n(w)$.

If f is an arbitrary function defined in (a, b), the Gauss-Jacobi quadrature formula is defined by the interpolatory quadrature formula

$$Q_n(d\alpha;\ f) = \sum_{k=1}^{n} \lambda_n(d\alpha;\ x_{kn})\ f(x_{kn})$$

and it has the property that for every polynomial π having degree $2n - 1$, at most, we have

$$Q_n(d\alpha,\ \pi) = \int_a^b \pi\, d\alpha.$$

The coefficients $\lambda_n(d\alpha;\ x_{kn})$ in this formula for $Q_n(d\alpha)$ are called the Christoffel numbers and are the values of the function (see [4]). $\lambda_n^{-1}(d\alpha;\ x) = \sum_{\nu=0}^{n-1} p_\nu^2(d\alpha;\ x)$ at $x = x_{kn} (k = 1, 2, \ldots, n)$.

The nodes x_{kn} are called the Gaussian abscissae with respect to $d\alpha$.

2. PRELIMINARY RESULTS

To prove our main result, we are going to use the following three lemmas. Lemmas 1 and 2 are due to G. Freud [2, 3].

Lemma 1. Let $f(z)$ be an analytic function in a domain $\mathcal{D}$ containing the Gaussian abscissae x_{kn} $(k = 1, 2, \ldots, n)$ and $x_{j,n+1}$ $(j = 1, 2, \ldots, n + 1)$. If $p_n(d\alpha;\ x) = \gamma_n x^n + \ldots$ is the orthonormal polynomial of degree n associated with the measure $d\alpha$, we have

$$Q_{n+1}(d\alpha;\ f) - Q_n(d\alpha;\ f) = \frac{\gamma_{n+1}}{\gamma_n}\cdot\frac{1}{2\pi i}\cdot\oint_{C_n}\frac{f(z)\,dz}{p_n(d\alpha;z)\ p_{n+1}(d\alpha;z)}$$

where $C_n \subset \mathcal{D}$ is a simple closed curve containing the zeros of $p_n(d\alpha)$ and $p_{n+1}(d\alpha)$ in its interior and the error term of the quadrature formula is

$$\int_a^b f\,d\alpha - Q_n(d\alpha;f) = \sum_{\nu=n}^{\infty}\frac{\gamma_{\nu+1}}{\gamma_\nu}\cdot\frac{1}{2\pi i}\cdot\oint_{C_\nu}\frac{f(z)\,dz}{p_\nu(d\alpha;z)\ p_{\nu+1}(d\alpha;z)} \tag{2.1}$$

Proof. See [2].

Lemma 2. For every even weight function $w(x)$, we have

$$\max_{1\le k\le n-1}\frac{\gamma_{k-1}}{\gamma_k} \le x_{1n} \le 2\max_{1\le k\le n-1}\frac{\gamma_{k-1}}{\gamma_k} \tag{2.2}$$

Proof. See [3].

Lemma 3. Let $w(x)$ be an even weight function. Then we have

$$\sum_{k=1}^{[\frac{n}{2}]^*} x_{kn}^2 = \sum_{k=1}^{n-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 \tag{2.3}$$

Proof. From the fact that w is an even weight function, it follows that (see e.g. [5; §2.3(2)]) $p_n(w;x)$ is an even or an odd polynomial according as n is even or odd. Hence, we can write

$$p_n(w;x) = \gamma_n x^n - \beta_n x^{n-2} + \ldots \tag{2.4}$$

Recalling the recursion formula for orthogonal polynomials generated by an even weight function (see [5; §3.2(1)] or [1; §I.2]), we have

$$xp_n(w;x) = \frac{\gamma_n}{\gamma_{n+1}}\,p_{n+1}(w;x) + \frac{\gamma_{n-1}}{\gamma_n}\,p_{n-1}(w;x) \tag{2.5}$$

Combining (2.4) and (2.5), we get by comparing the coefficients

* Here, [ ] is the greatest integer function.

of x^{n-1} in both sides of (2.5)

$$-\beta_n = -\frac{\beta_{n+1}\gamma_n}{\gamma_{n+1}} + \frac{\gamma_{n-1}^2}{\gamma_n}$$

i.e.,

$$\frac{\beta_{n+1}}{\gamma_{n+1}} = \frac{\beta_n}{\gamma_n} + \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2$$

which implies that

$$\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 .$$

Since it is easy to see that $\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2$, the proof of the lemma is completed.

3. THE MAIN RESULT

THEOREM. Let $f(z)$ be an entire function satisfying the condition

$$\beta = \limsup_{R \to \infty} \frac{\max_{|z|=R} (\log|f(z)|)}{R^2} < \rho \tag{3.1}$$

where ρ (= .70541786) is such that

$$\rho = \frac{(3\varepsilon_0 + 1)(1 - \varepsilon_0)}{8\varepsilon_0}$$ and ε_0 is the solution of

$\frac{1-x}{4} \exp\left(\frac{1-x}{2x}\right) = 1$. Then we have

$$\limsup_{n \to \infty} \left| \int_{-\infty}^{\infty} f(x)\, w(x)\, dx - Q_n(w;\, f)\right|^{\frac{1}{n}} < 1$$

where $w(x) = \exp(-x^2)$.

Proof. Since $w(x) = \exp(-x^2)$, it is well known that the nth orthonormal polynomial generated by this weight function is the nth Hermite polynomial $h_n(x)$ and it is also well known (see e.g. [5; §5.5]) that

$$\gamma_n^2 = \frac{2^n}{\sqrt{\pi}\; n!} \tag{3.2}$$

which easily implies

$$\frac{\gamma_{n-1}}{\gamma_n} = \sqrt{\frac{n}{2}}\,. \tag{3.3}$$

By combining (3.3) and (2.2), we get

$$x_{1,n+1} \le \sqrt{2n}. \tag{3.4}$$

By combining (3.3) and (2.3), we find that

$$\sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2 = \frac{n(n-1)}{4}, \quad n = 2, 3, 4, \ldots \tag{3.5}$$

To prove this theorem we are going to use (2.1). First we will find an inequality for $|h_n(z)|^{-1}$. Since w is an even weight function, it follows that (see [5; §2.3(2)])

$$h_n(z) = \gamma_n\, z^{n-2[\frac{n}{2}]} \prod_{k=1}^{[\frac{n}{2}]} (z^2 - x_{kn}^2)$$

Hence,

$$|h_n(z)| = \gamma_n\, |z|^{n-2[\frac{n}{2}]} \prod_{k=1}^{[\frac{n}{2}]} |z^2 - x_{kn}^2|$$

$$= \gamma_n\, |z|^n \exp\Big(\sum_{k=1}^{[\frac{n}{2}]} \log\Big|1 - \frac{x_{kn}^2}{z^2}\Big|\Big)$$

$$\geq \gamma_n |z|^n \exp\{\sum_{k=1}^{[\frac{n}{2}]} \log(1 - \frac{x_{kn}^2}{|z|^2})\}$$

$$\geq \gamma_n |z|^n \exp\{ - \sum_{k=1}^{[\frac{n}{2}]} \frac{x_{kn}^2}{(|z|^2 - x_{kn}^2)}\},$$

for every complex number z such that $x_{1n} < |z|$. We have next

$$\frac{1}{|z|^2 - x_{kn}^2} \leq \frac{1}{|z|^2 - x_{1n}^2}$$

and so

$$|h_n(z)| \geq \gamma_n |z|^n \exp\{ - \frac{1}{|z|^2 - x_{1n}^2} \sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2\}.$$

Using (3.5), we find that

$$|h_n(z)| \geq \gamma_n |z|^n \exp\{- \frac{n(n-1)}{4(|z|^2 - x_{1n}^2)}\}.$$

Therefore,

$$\frac{1}{|h_n(z)|} \leq \frac{1}{\gamma_n |z|^n} \exp\{\frac{n(n-1)}{4(|z|^2 - x_{1n}^2)}\}.$$

And so, for $x_{1,n+1} < |z|$, it follows that

$$\frac{1}{|h_n(z)\, h_{n+1}(z)|} \leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{|z|^{2n+1}} \exp\{\frac{n(n-1)}{4(|z|^2 - x_{1n}^2)} + \frac{n(n+1)}{4(|z|^2 - x_{1,n+1}^2)}\}$$

$$\leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp\{\frac{n^2}{2(|z|^2 - x_{1,n+1}^2)}\}.$$

Since $\beta = \limsup_{R \to \infty} \frac{\max_{|z|=R} (\log|f(z)|)}{R^2}$, for any $\sigma > 0$, we can find N_σ such that

$$|f(z)| \leq \exp\{(\beta + \sigma)|z|^2\}, \text{ for all } |z| \geq N_\sigma. \tag{3.6}$$

Denoting by I_n, the expression

$$\frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{h_n(z)\,h_{n+1}(z)}\,,$$ taking the path of integration

to be the circle $|z| = R$, where

$$R^2 \geq \frac{x_{1,n+1}^2}{1-\varepsilon}\,, \quad (0 < \varepsilon < 1) \tag{3.7}$$

we find, on $|z| = R$, that

$$\frac{1}{|h_n(z)\ h_{n+1}(z)|} \leq \frac{1}{\gamma_n\ \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2(R^2-x_{1,n+1}^2)}\right\}$$

$$\leq \frac{1}{\gamma_n\ \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2(R^2-(1-\varepsilon)R^2)}\right\}$$

$$\leq \frac{1}{\gamma_n\ \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2\varepsilon R^2}\right\}.$$

Using this last inequality, (3.2) and (3.6), we conclude that for $R \geq N_\sigma$,

$$|I_n| \leq \frac{\sqrt{\pi}\,n!}{2^n} \frac{\max_{|z|=R} |f(z)|}{R^{2n}} \exp\left\{\frac{n^2}{2\varepsilon R^2}\right\}$$

$$\leq \frac{\sqrt{\pi}\ n!}{2^n} \cdot \frac{1}{R^{2n}} \exp\left\{(\beta + \sigma)\ R^2 + \frac{n^2}{2\varepsilon R^2}\right\}$$

R will be chosen next so as to minimize the right hand side of this inequality, and at the same time, to satisfy (3.7).

Consider the function

$$h(R) = \frac{1}{R^{2n}} \exp\left\{(\beta + \sigma)\ R^2 + \frac{n^2}{2\varepsilon R^2}\right\}.$$

By differentiating $h(R)$ and setting $h'(R) = 0$, we get

$$2(\beta + \sigma)\ \varepsilon\ R^4 - 2n\,\varepsilon\,R^2 - n^2 = 0. \tag{3.8}$$

If we denote by R_n the positive solution to this equation, we find that

$$R_n^2 = \frac{1 + \{1 + \frac{2(\beta+\sigma)}{\varepsilon}\}^{\frac{1}{2}}}{2(\beta+\sigma)} \cdot n$$

(We can easily check that $f(R)$ attains its minimum value at $R = R_n$).

For $n \geq N$, for a suitable $N > 0$, we will have $R_n \geq N_0$. Also, from (3.4), it follows that

$$R_n^2 \geq \frac{1 + \{1 + \frac{2(\beta+\sigma)}{\varepsilon}\}^{\frac{1}{2}}}{4(\beta+\sigma)} \cdot x_{1,n+1}^2$$

and consequently, condition (3.8) will be satisfied if

$$\frac{4(\beta+\sigma)}{1 + \{1 + \frac{2(\beta+\sigma)}{\varepsilon}\}^{\frac{1}{2}}} = 1 - \varepsilon. \tag{3.9}$$

Since R_n satisfies equation (3.8), we find that

$$(\beta + \sigma)\ R_n^2 = n + \frac{n^2}{2\varepsilon R_n^2}$$

and it follows that

$$|I_n| \leq \sqrt{\pi}\ \frac{n!}{2^n} \cdot \frac{1}{R_n^{2n}} \exp\ \{n + \frac{n^2}{\varepsilon R_n^2}\}$$

$$\leq \sqrt{\pi} \cdot \frac{n!}{2^n}\ \{\frac{2(\beta+\sigma)}{1 + (1 + \frac{2(\beta+\sigma)}{\varepsilon})^{\frac{1}{2}}}\}^n \cdot$$

$$\cdot \exp\ \{n + \frac{2(\beta+\sigma)\,n}{\varepsilon(1 + (1 + \frac{2(\beta+\sigma)}{\varepsilon})^{\frac{1}{2}})}\}$$

Using (3.9), we find that

$$|I_n| \leq \sqrt{\pi}\ \frac{n!}{2^n n^n}\ (\frac{1 - \varepsilon}{2})^n \exp\ \{n + \frac{(1-\varepsilon)n}{2\varepsilon}\}$$

Using the Stirling formula $n! \approx \sqrt{2\pi n}\ n^n\ e^{-n}$, we find that

$$|I_n| \le K\sqrt{n}\,\left(\frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right)^n, \quad K = \text{constant and } n \text{ is}$$

sufficiently large.

It is easy to see that $g(\varepsilon) = \frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)$ is a decreasing function on (0, 1). Consequently, if ε_0 is the unique solution of $g(\varepsilon) = 1$, then for $\varepsilon_0 < \varepsilon < 1$, we have

$$0 < g(\varepsilon) < 1.$$

Thus, $\sum_{k=1}^{\infty} |I_k|$ is a convergent series.

If $\Delta_n = \sum_{k=n}^{\infty} I_k$, we have

$$|\Delta_n| \le \sum_{k=n}^{\infty} |I_k| \le K \sum_{k=n}^{\infty} \sqrt{k}\,\left(\frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right)^k.$$

Since $\sum_{k=n}^{\infty} kx^k \le \frac{(n+2)x^n}{(1-x)^2}$ for $0 < x < 1$, it follows that

$$|\Delta_n| \le \frac{K(n+3)}{\left\{1 - \frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right\}^2}\left(\frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right)^n$$

and so

$$\limsup_{n\to\infty} |\Delta_n|^{\frac{1}{n}} \le \frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) < 1 \quad \text{for every } \varepsilon_0 < \varepsilon < 1.$$

Finally, it remains to justify the assumption (3.1).

From (3.9) and the choice of $\varepsilon_0 < \varepsilon < 1$, we see that

$$\frac{4(\beta+\sigma)}{1+\left\{1+\frac{2(\beta+\sigma)}{\varepsilon_0}\right\}^{\frac{1}{2}}} < \frac{4(\beta+\sigma)}{1+\left\{1+\frac{2(\beta+\sigma)}{\varepsilon}\right\}^{\frac{1}{2}}} = 1 - \varepsilon < 1 - \varepsilon_0$$

and therefore,

$$\frac{4(\beta+\sigma)}{1+\left(1+\frac{2(\beta+\sigma)}{\epsilon_0}\right)^{\frac{1}{2}}} < 1-\epsilon_0 .$$

Solving this inequality, we find that

$$0 < \beta+\sigma < \frac{(3\epsilon_0+1)\,(1-\epsilon_0)}{8\epsilon_0} = \rho \quad (\approx .70541786)$$

Hence, we must have $0 < \beta < \rho$, which completes the proof of the theorem.

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