An Error Estimate for Gauss-Jacobi Quadrature Formula with the Hermite Weight

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AN ERROR ESTIMATE FOR GAUSS-JACOBI QUADRATURE
FORMULA WITH THE HERMITE WEIGHT \( w(x) = \exp(-x^2) \)

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The purpose of this paper is to give an estimate of the error in approximating the integral

\[
\int_{-\infty}^{\infty} f(x) \exp(-x^2) \, dx
\]

by the Gauss-Jacobi quadrature formula \( Q_n(w; f) \), assuming that \( f \) is an entire function satisfying a certain growth condition which depends on the Hermite weight function \( w(x) = \exp(-x^2) \).

1. INTRODUCTION

Let \( da \) be a non-negative measure supported in the interval \( (a, b) \), \(-\infty < a < b < \infty\). Let the support of \( da \) contain infinitely many points and let

\[
\int_a^b x^n \, da(x) < \infty, \text{ for } n = 0, 1, 2, \ldots .
\]

Then there exists a uniquely determined sequence of orthonormal polynomials \( \{p_n(da; x)\} \) generated by this measure (see e.g. [1; CH. 1], [5; CH. II]); they are determined by the properties

(a) \( p_n(da; x) = \gamma_n x^n + \ldots \) is a polynomial of degree \( n \) and \( \gamma_n > 0 \).

(b) \[
\int_a^b p_n(da) \, p_m(da) \, da = \begin{cases} 
0 & \text{if } m \neq n \\
1 & \text{if } m = n 
\end{cases}
\]

It is well known that all zeros \( x_{kn} \) \( (k = 1, 2, \ldots, n) \) of \( p_n(da; x) \) are real, simple and are contained in \( (a, b) \). We shall assume, as usual, that

\[
x_{1n} > x_{2n} > \ldots > x_{nn}.
\]
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If, in addition, \( dx \) is an absolutely continuous measure, then
\( da(x) = a'(x) \, dx \) and \( a'(x) \) is called a weight function. In this case,
a'(x) will be denoted by \( w(x) \) and \( p_n(da) \) by \( p_n(w) \).

If \( f \) is an arbitrary function defined in \((a, b)\), the Gauss-Jacobi quadrature formula is defined by the interpolatory quadrature formula

\[
Q_n(da; f) = \sum_{k=1}^{n} \lambda_n(da; x_{kn}) f(x_{kn})
\]

and it has the property that for every polynomial \( P \) having degree \( 2n - 1 \), at most, we have

\[
Q_n(da, w) = \int_{a}^{b} P dx.
\]

The coefficients \( \lambda_n(da; x_{kn}) \) in this formula for \( Q_n(da) \) are called the Christoffel numbers and are the values of the function (see [4]). \( \lambda^{-1}_n(da; x) = \sum_{\nu=0}^{n-1} p_\nu^2(da; x) \) at \( x = x_{kn}(k = 1, 2, \ldots, n) \).

The nodes \( x_{kn} \) are called the Gaussian abscissae with respect to \( da \).

2. PRELIMINARY RESULTS

To prove our main result, we are going to use the following three lemmas. Lemmas 1 and 2 are due to G. Freud [2, 3].

Lemma 1. Let \( f(z) \) be an analytic function in a domain \( \mathcal{D} \) containing the Gaussian abscissae \( x_{kn}(k = 1, 2, \ldots, n) \) and \( x_{j,n+1}(j = 1, 2, \ldots, n+1) \). If \( p_n(da; x) = \gamma_n x^n + \ldots \) is the orthonormal polynomial of degree \( n \) associated with the measure \( da \), we have
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\[ Q_{n+1}(da; f) - Q_n(da; f) = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2^m} \cdot \oint_{C_n} \frac{f(z) \, dz}{p_n(da; z) \, p_{n+1}(da; z)} \]

where \( C_n \subset J \) is a simple closed curve containing the zeros of \( p_n(da) \) and \( p_{n+1}(da) \) in its interior and the error term of the quadrature formula is

\[ \int_a^b f(da) - Q_n(da; f) = \sum_{v=n}^{\infty} \frac{\gamma_{v+1}}{\gamma_v} \cdot \frac{1}{2^{v+1}} \cdot \oint_{C_v} \frac{f(z) \, dz}{p_v(da; z) \, p_{v+1}(da; z)} \]

(2.1)

Proof. See [2].

Lemma 2. For every even weight function \( w(x) \), we have

\[ \max_{1 \leq k \leq n-1} \frac{\gamma_{k-1}}{\gamma_k} < x_1 \leq 2 \max_{1 \leq k \leq n-1} \frac{\gamma_{k-1}}{\gamma_k} \]

(2.2)

Proof. See [3].

Lemma 3. Let \( w(x) \) be an even weight function. Then we have

\[ \sum_{k=1}^{\left[ \frac{n}{2} \right]} x^2 \leq \frac{n-1}{2} \left( \frac{\gamma_{k-1}}{\gamma_k} \right)^2 \]

(2.3)

Proof. From the fact that \( w \) is an even weight function, it follows that (see e.g. [5; §2.3(2)]) \( p_n(w;x) \) is an even or an odd polynomial according as \( n \) is even or odd. Hence, we can write

\[ p_n(w;x) = \gamma_n x^n - \delta_n x^{n-2} + \ldots \]

(2.4)

Recalling the recursion formula for orthogonal polynomials generated by an even weight function (see [5; §3.2(1)] or [1; §1.2]), we have

\[ x p_n(w;x) = \frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(w;x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(w;x) \]

(2.5)

Combining (2.4) and (2.5), we get by comparing the coefficients

* Here, \([ \cdot ]\) is the greatest integer function.
of $x^{n-1}$ in both sides of (2.5)

$$-\beta_n = -\frac{\beta_{n+1} \gamma_n}{\gamma_{n+1}} + \frac{\gamma_n^2}{\gamma_n}$$

i.e.,

$$\frac{\beta_{n+1}}{\gamma_{n+1}} = \frac{\beta_n}{\gamma_n} + \left(\frac{\gamma_n - 1}{\gamma_n}\right)^2$$

which implies that

$$\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2.$$

Since it is easy to see that $\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{n-1} x_{kn}^2$, the proof of the lemma is completed.

3. THE MAIN RESULT

THEOREM. Let $f(z)$ be an entire function satisfying the condition

$$\beta = \limsup_{R \to +} \max_{|z| = R} \frac{|f(z)|}{R} < \rho$$

(3.1)

where $\rho = .70541786$ is such that

$$\rho = \frac{(3\epsilon_0 + 1) (1 - \epsilon_0)}{8\epsilon_0}$$

and $\epsilon_0$ is the solution of

$$\frac{1 - x}{4} \exp\left(\frac{1 - x}{2x}\right) = 1.$$ Then we have

$$\limsup_{n \to \infty} \left| \int_{-\infty}^{\infty} f(x) w(x) \, dx - Q_n(w, f) \right|^n < 1$$

where $w(x) = \exp(-x^2)$.
Proof. Since $w(x) = \exp(-x^2)$, it is well known that the $n$th orthonormal polynomial generated by this weight function is the $n$th Hermite polynomial $h_n(x)$ and it is also well known (see e.g. [5; 5.5]) that
\[ y_n^2 = \frac{2^n}{\sqrt{n} \cdot n!} \]  \hspace{1cm} (3.2)
which easily implies
\[ \frac{y_{n-1}}{y_n} = \sqrt{\frac{n}{2}}. \]  \hspace{1cm} (3.3)

By combining (3.3) and (2.2), we get
\[ x_{1,n} \leq \sqrt{2n}. \]  \hspace{1cm} (3.4)

By combining (3.3) and (2.3), we find that
\[ \sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2 = \frac{n(n-1)}{4}, \quad n = 2, 3, 4, \ldots \]  \hspace{1cm} (3.5)

To prove this theorem we are going to use (2.1). First we will find an inequality for $|h_n(z)|^{-1}$. Since $w$ is an even weight function, it follows that (see [5; §2.3(2)])
\[ h_n(z) = y_n z^{n-2[n/2]} \prod_{k=1}^{[n/2]} (z^2 - x_{kn}^2) \]
Hence,
\[ |h_n(z)| = y_n |z|^{n-2[n/2]} \prod_{k=1}^{[n/2]} |z^2 - x_{kn}^2| \]
\[ = y_n |z|^n \exp \left( \sum_{k=1}^{[n/2]} \log \left| 1 - \frac{x_{kn}^2}{z^2} \right| \right) \]
\[ \gamma_n |z|^n \exp \left\{ \sum_{k=1}^{\frac{n}{2}} \log \left( 1 - \frac{x_{kn}^2}{|z|^2} \right) \right\} \]

\[ \geq \gamma_n |z|^n \exp \left\{ - \sum_{k=1}^{\frac{n}{2}} \frac{x_{kn}^2}{(|z|^2 - x_{kn}^2)} \right\}, \]

for every complex number \( z \) such that \( x_{1n} < |z| \). We have next

\[ \frac{1}{|z|^2 - x_{kn}^2} \leq \frac{1}{|z|^2 - x_{1n}^2} \]

and so

\[ |h_n(z)| \geq \gamma_n |z|^n \exp \left\{ 1 - \frac{1}{|z|^2 - x_{1n}^2} \sum_{k=1}^{\frac{n}{2}} x_{kn}^2 \right\}. \]

Using (3.5), we find that

\[ |h_n(z)| \geq \gamma_n |z|^n \exp \left\{ - \frac{n(n-1)}{4(|z|^2 - x_{1n}^2)} \right\}. \]

Therefore,

\[ \frac{1}{|h_n(z)|} \leq \frac{1}{\gamma_n |z|^n} \exp \left\{ \frac{n(n-1)}{4(|z|^2 - x_{1n}^2)} \right\}. \]

And so, for \( x_{1,n+1} < |z| \), it follows that

\[ \frac{1}{|h_n(z)h_{n+1}(z)|} \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp \left\{ \frac{n(n-1)}{4(|z|^2 - x_{1n}^2)} + \frac{n(n+1)}{4(|z|^2 - x_{1,n+1}^2)} \right\} \]

\[ \leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp \left\{ \frac{n^2}{2(|z|^2 - x_{1,n+1}^2)} \right\}. \]

\[ \max \left( \log |f(z)| \right) \]

Since \( \beta = \lim_{R \to \infty} \sup_{R^2} \left( \frac{|z|^2}{R} \right) \), for any \( \sigma > 0 \), we can find \( N_0 \) such that

\[ |f(z)| \leq \exp \left\{ ((\beta + \sigma)|z|^2) \right\}, \text{ for all } |z| \geq N_0. \]
Denoting by \( I_n \), the expression

\[
\frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} f(z) \frac{dz}{h_n(z) h_{n+1}(z)},
\]

taking the path of integration to be the circle \(|z| = R\), where

\[
R^2 \geq \frac{\lambda_{1,n+1}^2}{1 - \varepsilon}, \quad (0 < \varepsilon < 1)
\]

(3.7)

we find, on \(|z| = R\), that

\[
\left| \frac{1}{h_n(z)} \right| \leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp \left\{ \frac{n^2}{2(R^2 - \lambda_{1,n+1}^2)} \right\}
\]

\[
\leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp \left\{ \frac{n^2}{2(R^2 - (1 - \varepsilon)R^2)} \right\}
\]

\[
\leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp \left\{ \frac{n^2}{2\varepsilon R^2} \right\}
\]

Using this last inequality, (3.2) and (3.6), we conclude that for \( R \geq R_0 \),

\[
|I_n| \leq \frac{\sqrt{n} \lambda_{1,n+1}}{2^n} \max_{|z|=R} |f(z)| \exp \left\{ \frac{n^2}{2\varepsilon R^2} \right\}
\]

\[
\leq \frac{\sqrt{n} \lambda_{1,n+1}}{2^n} \cdot \frac{1}{R^{2n}} \exp \left\{ ((\beta + \sigma) R^2 + \frac{n^2}{2\varepsilon R^2}) \right\}
\]

\( R \) will be chosen next so as to minimize the right hand side of this inequality, and at the same time, to satisfy (3.7).

Consider the function

\[
h(R) = \frac{1}{R^{2n}} \exp \left\{ ((\beta + \sigma) R^2 + \frac{n^2}{2\varepsilon R^2}) \right\}.
\]

By differentiating \( h(R) \) and setting \( h'(R) = 0 \), we get

\[
2(\beta + \sigma) \varepsilon R^4 - 2n \varepsilon R^2 - n^2 = 0.
\]

(3.8)

If we denote by \( R_n \) the positive solution to this equation, we find that
\[
R_n^2 = \frac{1 + \left(1 + \frac{2(\beta + \sigma)}{c}\right)^{\frac{1}{2}}}{2(\beta + \sigma)} \cdot n
\]

(We can easily check that \( f(R) \) attains its minimum value at \( R = R_n \)).

For \( n > N \), for a suitable \( N > 0 \), we will have \( R_n > N \). Also, from (3.4), it follows that

\[
R_n^2 \geq \frac{1 + \left(1 + \frac{2(\beta + \sigma)}{c}\right)^{\frac{1}{2}}}{4(\beta + \sigma)} \cdot n^2_{1,n+1}
\]

and consequently, condition (3.8) will be satisfied if

\[
\frac{4(\beta + \sigma)}{1 + \left(1 + \frac{2(\beta + \sigma)}{c}\right)^{\frac{1}{2}}} = 1 - \epsilon.
\]

Since \( R_n \) satisfies equation (3.8), we find that

\[
(\beta + \sigma) R_n^2 = n + \frac{n^2}{2\epsilon R_n^2}
\]

and it follows that

\[
|I_n| \leq \sqrt{\pi} \frac{n!}{2^n} \cdot \frac{1}{R_n^{2n}} \exp \left( n + \frac{n^2}{2\epsilon R_n^2} \right)
\]

\[
\leq \sqrt{\pi} \cdot \frac{n!}{2^n} \cdot \left( \frac{2(\beta + \sigma)}{1 + (1 + \frac{2(\beta + \sigma)}{c})^{\frac{1}{2}}} \right)^n
\]

\[
\cdot \exp \left( n + \frac{2(\beta + \sigma)n}{c(1 + (1 + \frac{2(\beta + \sigma)}{c})^{\frac{1}{2}})} \right)
\]

Using (3.9), we find that

\[
|I_n| \leq \sqrt{\pi} \frac{n!}{2^n n^n} \left( \frac{1 - \epsilon}{2} \right)^n \exp \left( n + \frac{(1 - \epsilon)n}{2\epsilon} \right)
\]

Using the Stirling formula \( n! = \sqrt{2\pi n} n^n e^{-n} \), we find that
\[ |I_n| \leq K \sum_{k=0}^{n} \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right)^k, \quad K = \text{constant and } n \text{ in sufficiently large.} \]

It is easy to see that \( g(\epsilon) = \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right) \) is a decreasing function on \( (0, 1) \). Consequently, if \( \epsilon_0 \) is the unique solution of \( g(\epsilon) = 1 \), then for \( \epsilon_0 < \epsilon < 1 \), we have

\[ 0 < g(\epsilon) < 1. \]

Thus, \( \sum_{k=1}^{\infty} |I_k| \) is a convergent series.

If \( \Lambda_n = \sum_{k=1}^{n} I_k \), we have

\[ |\Lambda_n| \leq \sum_{k=1}^{n} |I_k| \leq K \sum_{k=1}^{n} \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right)^k. \]

Since \( \sum_{k=1}^{n} kx^k \leq \frac{(n + 2)x^n}{(1 - x)^2} \) for \( 0 < x < 1 \), it follows that

\[ |\Lambda_n| \leq \frac{K(n + 3)}{1 - \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right)} \left( \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right) \right)^n \]

and so

\[ \limsup_{n \to \infty} \frac{1}{|\Lambda_n|^n} \leq \frac{1 - \epsilon}{4} \exp \left( \frac{1 - \epsilon}{2 \epsilon} \right) < 1 \text{ for every } \epsilon_0 < \epsilon < 1. \]

Finally, it remains to justify the assumption (3.1).

From (3.9) and the choice of \( \epsilon_0 < \epsilon < 1 \), we see that

\[ \frac{4(\beta + \sigma)}{1 + (1 + 2(\beta + \sigma))^{4/5}} \frac{4(\beta + \sigma)}{1 + (1 + 2(\beta + \sigma))^{4/5}} = 1 - \epsilon < 1 - \epsilon_0 \]

and therefore,
\[ \frac{4(\beta + \sigma)}{1 + (1 + \frac{2(\beta + \sigma)}{\epsilon_0})^{1/2}} < 1 - \epsilon_0. \]

Solving this inequality, we find that
\[ 0 < \beta + \sigma < \frac{(3\epsilon_0 + 1)(1 - \epsilon_0)}{8\epsilon_0} = \rho \quad (\approx 0.70541786) \]

Hence, we must have \( 0 < \beta < \rho \), which completes the proof of the theorem.

REFERENCES