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Generalized Functions and Operational Calculus

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Generalized Functions and Operational Calculus

I. Introduction. The subject of this lecture comes from two streams of mathematics that have seen much activity since their beginnings. Operational calculus has its origin almost a century ago when Heaviside [1] solved certain second order partial differential equations by treating the differential operator as an algebraic quantity.

To find the potential V at the start of a cable, he solved the problem:

$$\frac{\partial^2 V}{\partial x^2} = \frac{S}{R}\frac{\partial V}{\partial t}, \quad S, R \text{ constants} \tag{1}$$

$$V = 0 \text{ when } t = 0 \tag{2}$$

$$E - V = z\frac{\partial V}{\partial x} \text{ when } x = 0, \quad E, z \text{ constants} \tag{3}$$

in the following manner. Taking roots of (1) and applying condition (3), he obtained

$$E - V = z\left(\frac{S}{R}\frac{\partial}{\partial t}\right)^{\frac{1}{2}} V.$$

Hence $$V = \frac{1}{1 + \left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}} E \quad \text{where} \quad a = z^2\frac{S}{R}.$$

$$= \left\{1 - \left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}} + a\frac{\partial}{\partial t} - \left(a\frac{\partial}{\partial t}\right)^{\frac{3}{2}} + \left(a\frac{\partial}{\partial t}\right)^{2} - \ldots\right\} E$$

$$= E\left[1 - \left(\frac{a}{\pi t}\right)^{\frac{1}{2}}\left\{1 - \frac{a}{2t} + 1\cdot 3\cdot\left(\frac{a}{2t}\right)^2 - \ldots\right\}\right] \text{ for large } t. \tag{4}$$

Or $$V = \frac{1}{\left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}}\left\{1 + \frac{1}{\left(a\frac{\partial}{\partial t}\right)^{\frac{1}{2}}}\right\}^{-1} E$$

$$= 2E\left(\frac{t}{a\pi}\right)^{\frac{1}{2}}\left\{1 + \frac{2t}{3a} + \frac{1}{3\cdot 5}\left(\frac{2t}{a}\right)^2 + \ldots\right\} - E\left(e^{t/a} - 1\right) \text{ for small } t. \tag{5}$$

Here, he used Euler's fractional differentiation formula [2]:

$$(\frac{d}{dt})^{\alpha}\ t^{p} = \frac{\Gamma(p+1)}{\Gamma(\alpha - p + 1)}\ t^{p-\alpha} \quad , \quad \alpha > 0.$$

Equations (4) and (5) are quite correct. They both represent the solution $E\ \{1 - \frac{2}{\sqrt{\pi}}\ e^{t/a} \int_{(t/a)^{1/2}}^{\infty} e^{-s^2}\ ds\}$; (4) being the asymptotic series as $t \to \infty$ and (5) the convergent series near $t = 0$.

Unfortunately, mathematicians found the procedure rather objectionable; such steps as root extraction and expansion in series of a differential operator were hard to explain. One explanation for the Heaviside calculus was provided by the Laplace transformation defined by

$$\mathcal{L}[f(t)] \equiv \int_0^{\infty} e^{-pt}\ f(t)dt. \tag{6}$$

The connection between the two is given by the formula

$$\mathcal{L}[\frac{df}{dt}] = p\mathcal{L}[\ f(t)] - f(0). \tag{7}$$

Thus, when $f(0) = 0$, the operator $\frac{d}{dt}$ corresponds to multiplication by p in the transform domain.

Generalized functions or distributions are of a more recent vintage. In 1926, Dirac [3] introduced the δ-function as a function defined and continuous on the real line $\mathbb{R}$ which satisfies the following properties.

(i) $\delta(x) = 0$ for $x \neq 0$. (8)

(ii) $\int_{-\infty}^{\infty} \delta(x)dx = 1$. (9)

(iii) $\forall\ f(x) \in C(\mathbb{R})$, $\int_{-\infty}^{\infty} f(x)\delta(a-x)dx = f(a),\ a \in \mathbb{R}$. (10)

These properties clearly violate the classical meanings of a function and the integral. To be sure, there are other singular or improper functions that have become part of the mathematical physicist's tools. Some of these are derivatives of Dirac delta functions, Hadamard's [4] pseudofunctions $\mathrm{pf}\, x^{\alpha}$ in the study of hyperbolic equations, and Cauchy's principal value of a divergent integral. It was natural to consider these as generalized functions and early attempts to formalize this idea were made by Bochner [5] in 1932 and Sobolev [6] in 1936.

In the 1950s, two events produced renewed interest on our subject. One was the appearance of L. Schwartz's Theory of Distributions [7] which gave an elegant basis for the above "functions" and increased the source of possible solutions for partial differential equations. Of necessity, he developed the Fourier and Laplace transformations of distributions as well. The other event was the work of J. Mikusinski [8] who gave a purely algebraic approach to operational calculus closely related to Heaviside's original idea. He also accounted for generalized functions and operators.

In this lecture, we shall give an outline of these innovative ideas and indicate some of the developments since.

II. Mikusinski's Calculus. The central idea here is the extension of a ring of functions to a field of convolution quotients analogous to the extension of integers to rational numbers.

Let L be the set of locally integrable functions on $(0, \infty)$, i.e., $f \in L$ if f is measurable and $\int_K |f| dt < \infty$ for each compact subset K of $(0, \infty)$. For two functions f and g in L, define their convolution $f*g$ by

$$(f*g)(t) = \int_0^t f(t-x)g(x)dx. \tag{11}$$

The convolution is well-defined and is also in L. Under the usual addition of functions and convolution as multiplication, L is a commutative ring. Indeed, it is easy to show that L is an Abelian group under addition. That multiplication is commutative follows from a simple change of variable:

$$(f*g)(t) = \int_0^t f(t-x)g(x)dx = \int_0^t f(u)g(t-u)du = (g*f)(t).$$

Associativity is shown as follows:

$$\begin{aligned}[(f*g)*h](t) &= \int_0^t \left(\int_0^\tau f(u)g(\tau-u)du\right)h(t-\tau)d\tau \\ &= \int_0^t \int_u^t f(u)g(\tau-u)h(t-\tau)d\tau du \\ &= \int_0^t f(u) \int_0^{t-u} g(t-u-x)h(x)dxdu \\ &= [f*(g*h)](t).\end{aligned}$$

A crucial theorem of Titchmarsh ([8] Chap. 2) guarantees that the ring L has no zero divisors, i.e. if $f, g \in L$ and $f*g = 0$, then $f = 0$ or $g = 0$. Thus, L is indeed an integral domain and may be extended into a field M consisting of equivalence classes of ordered pairs (f, g) where $f, g \in L$ and $g \neq 0$. Two pairs (f_1, g_1) and (f_2, g_2) are said to be equal if $f_1*g_2 = f_2*g_1$. We shall denote an equivalence class of equal ordered pairs by f/g where (f, g) is any pair in the class.

The following operations and properties of M are clear:

1. Equality: $\frac{a}{b} = \frac{c}{d}$ iff $a*d = b*c$.
2. Cancellation law holds: $\frac{a*c}{b*c} = \frac{a}{b}$.

3. L can be embedded in M by identifying $f \in L$ with $\frac{f*a}{a}$ for any $a \neq 0$.

4. $\mathbb{R}$ can be embedded in M by identifying $\alpha \in \mathbb{R}$ with $\frac{\alpha a}{a}$ for any $a \neq 0$. (Note that $\alpha a \neq \alpha * a = \alpha \int_0^t a(t)dt$.)

5. The unit element in M is $\frac{a}{a} = 1$, $a \neq 0$. The zero element in M is $\frac{0}{a} = 0$, $a \neq 0$.

6. Addition: $\frac{a}{b} + \frac{c}{d} = \frac{a*d + b*c}{b*d}$

7. Scalar multiplication: $\alpha \frac{a}{b} = \frac{\alpha a}{b}$, $\alpha \in \mathbb{R}$.

8. Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a*c}{b*d}$

9. Every nonzero element $\frac{a}{b} \in M$ has a unique inverse, $(\frac{a}{b})^{-1} = \frac{b}{a}$. Consequently, if $f, g \in M$ and $f \neq 0$, then the solution of $f*u = g$ is unique in M, viz. $u = f^{-1}g$.

In view of the property of the δ-function (10), i.e. $\delta * f = f$, we may regard the unit element 1 in M as $\delta(t)$. This, together with the embedding of L and $\mathbb{R}$ in M, suggests that convolution quotients may be regarded as generalized functions. Mikusinski calls them "operators" for indeed M includes differential and integral operators.

Let $\{f(t)\}$ denote the function $f(t)$ restricted to $(0, \infty)$. Let $\ell = \{1\} \equiv$ the unit function on $(0, \infty)$. For $f \in L$,

$$\ell * f = \int_0^t f(x)dx, \tag{12}$$

i.e. the integration operator is represented as a convolution with ℓ. By induction, it is easy to show that the n-order integration operator is given by

$$\ell^n = \{\frac{t^{n-1}}{(n-1)!}\}, \quad n = 1, 2, \ldots. \tag{13}$$

A natural extension of (13) to fractional or complex order integration is given by

$$\ell^\alpha = \{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\}, \quad \alpha \in \mathbb{C}, \quad \alpha \neq -1, -2, \ldots. \tag{14}$$

This turns out to be the Riemann-Liouville integral operator of order α (see [2]).

To define a differential operator in M, consider $f \in C^1(\mathbb{R}_+)$ and note that

$$f(t) = f(0) + \int_0^t f'(x)dx = f(0)\ell + \ell * f'(t). \tag{15}$$

Hence, if we let $s = \ell^{-1}$ and multiply (15) by s:

$$sf = f(0)1 + f'. \tag{16}$$

Equation (16) is reminiscent of equation (7). Thus, Mikusinski's theory not only introduces generalized functions but also does away with the Laplace transformation. The operator s is now interpreted as a generalized differentiation. It can be shown by induction that for $f \in C^n(\mathbb{R})$,

$$s^n f = f^{(n)} + f^{(n-1)}(0)1 + f^{(n-2)}(0)s + \cdots + f(0)s^{n-1}. \tag{17}$$

Using equations (16) and (17), we obtain the following operational formulas:

$$\frac{1}{s-\alpha} = \{e^{\alpha t}\}. \tag{18}$$

$$\frac{1}{s^2+\alpha^2} = \frac{1}{\alpha}\{\sin\alpha t\}. \tag{19}$$

$$\frac{s}{s^2+\alpha^2} = \{\cos\alpha t\}. \tag{20}$$

$$\frac{1}{(s-\alpha)^n} = \{\frac{e^{\alpha t}t^{n-1}}{(n-1)!}\}. \tag{21}$$

These formulas enable one to solve ordinary differential equations with constant coefficients, systems of equations, and certain integral equations. We give two simple examples:

(i) Solve $\frac{d^2u}{dt^2} + \alpha^2 u = f(t)$ on $0 < t < \infty$ if $f \in L$.

From (17) we have

$$s^2u - u'(0) - u(0)s + \alpha^2 u = f.$$

Hence $u = \frac{1}{s^2+\alpha^2}\cdot f + \frac{u'(0) + u(0)s}{s^2+\alpha^2}$

$$= \{\frac{\sin\alpha t}{\alpha}\} * \{f(t)\} + \{\frac{u'(0)\sin\alpha t}{\alpha}\} + \{u(0)\cos\alpha t\}$$

$$\therefore \quad u(t) = \frac{1}{\alpha}\int_0^t \sin\alpha(t-x)f(x)dx + \frac{u'(0)}{\alpha}\sin\alpha t + u(0)\cos\alpha t$$

which is locally integrable.

(ii) The integral equation

$$\int_0^t \sin(t-x)u(x)dx = \alpha t$$

becomes, in terms of convolution quotients:

$$\{\sin t\} * \{u(t)\} = \{\alpha t\}.$$

Hence $\frac{1}{s^2+1} u = \alpha \ell^2$.

$$\therefore\ u = (s^2+1)\cdot\alpha\ell^2 = \alpha 1 + \alpha\ell^2$$

$$= \alpha\delta(t) + \{\alpha t\},$$

which is not an ordinary function but a generalized function involving $\delta(t)$, i.e., ε M.

Minkusinski's calculus may be applied to partial differential equations. However, we shall need the concepts of operator functions, their convergence, differentiation and integration. Interested persons should refer to [8] or [9].

III. Distribution Theory in a Nutshell. Distributions as limits or classes of fundamental sequences have been developed by a number of people including Korevaar [10], Temple [11], Lighthill [12] and Antosik, Mikusinski and Sikorski [13]. Though simple, they are not quite as powerful or elegant as the functional approach. We shall therefore restrict our talk to the theory according to Schwartz [7], [14], [15].

1. The space $\mathcal{D}$. Let $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$, $|t| = \sum_{i=1}^{n} t_i$ and $D^k\phi = \frac{\partial^{|k|}}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}$. A function $\phi(t)$ is said to be smooth if it has partial derivatives of all orders in $\mathbb{R}^n$. The support of $\phi(t)$ is the closure of the set of points in $\mathbb{R}^n$ for which $\phi(t) \neq 0$. $\mathcal{D}$ is the linear space of all complex-valued smooth functions $\phi(t)$ which have compact support. Members of $\mathcal{D}$ are called test functions; an example is the function

$$\xi(t) = \begin{cases} \exp\left(\frac{1}{|t|^2 - 1}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}.$$

A sequence of functions $\{\phi_n\}$ is said to converge to zero in $\mathcal{D}$ if (i) $\phi_n \in \mathcal{D}\ \forall n$, (ii) there is a fixed bounded set K which contains the support of $\phi_n\ \forall n$, and (iii) $D^k \phi_n \to 0$ uniformly in $\mathbb{R}^n$ for each k as $n \to \infty$. For example $\{\frac{1}{n}\xi(t)\} \to 0$ in $\mathcal{D}$ but $\{\frac{1}{n}\xi(\frac{t}{n})\} \not\to$ in $\mathcal{D}$ since $\operatorname{supp}[\frac{1}{n}\xi(\frac{t}{n})]$ is not fixed. The space $\mathcal{D}$ turns out to be closed under this convergence concept.

2. Distributions. A distribution is a continuous linear functional on $\mathcal{D}$. If T is a distribution and $\phi \in \mathcal{D}$, T assigns to ϕ a complex number denoted by $\langle T, \phi\rangle$ such that T is linear, i.e. $\langle T, \phi_1 + \lambda\phi_2\rangle = \langle T, \phi_1\rangle + \lambda\langle T, \phi_2\rangle$, $\lambda \in \mathbb{C}$, and continuous, i.e. $\langle T, \phi_n\rangle \to 0$ whenever $\phi_n \to 0$ in $\mathcal{D}$. The set of distributions on $\mathcal{D}$ is called the dual of $\mathcal{D}$ and is denoted by $\mathcal{D}'$. $\mathcal{D}'$ is a linear space with addition and scalar multiplication defined by

$$\langle T_1 + T_2, \phi\rangle = \langle T_1, \phi\rangle + \langle T_2, \phi\rangle \qquad \forall \phi \in \mathcal{D}$$

$$\langle \lambda T, \phi\rangle = \lambda\langle T, \phi\rangle \qquad \forall \phi \in \mathcal{D},\ \lambda \in \mathbb{C}.$$

Some examples of distributions are:

(a) A locally integrable function $f(t)$ on $\mathbb{R}^n$ defines a distribution T_f by $\langle T_f, \phi\rangle = \int_{\mathbb{R}^n} f(t)\,\phi(t)\,dt$. If f and g are two such functions, then $T_f = T_g$ if and only if $f = g$ almost everywhere. Thus, no distinction is made between f and T_f.

(b) The Dirac δ-function is indeed a distribution δ defined by $\langle\delta, \phi\rangle = \phi(0)$.

(c) The integral $\int_0^\infty \frac{\ln t}{t}\,\phi(t)dt$ is generally divergent for $\phi \in \mathcal{D}(\mathbb{R})$ because of a singularity at $t = 0$. If we set $\phi(t) = \phi(0) + t\psi(t)$ with ψ continuous on $[0,\infty)$, then $\int_0^\infty \frac{\ln t}{t}\,\phi(t)dt = \lim_{\varepsilon \to 0^+} \{ \int_\varepsilon^b \psi(t)\ln t dt + \frac{\phi(0)}{2}[(\ln b)^2 - (\ln\varepsilon)^2]\}$ where b = upper bound for support ϕ. The first two terms on the right are finite. Thus, a distribution, pf $\frac{1_+(t)\ln t}{t}$, is defined by

$$<\text{pf}\,\frac{1_+(t)\ln t}{t}\,\phi> \;=\; \lim_{\varepsilon \to 0^+}[\int_\varepsilon^\infty \frac{\ln t}{t}\,\phi(t)dt + \frac{\phi(0)}{2}(\ln\varepsilon)^2]$$

which corresponds to Hadamard's finite part of $\frac{\ln t}{t}$. The symbol pf stands for "pseudofunction," a term used by Schwartz to describe these singular distributions. $1_+(t)$ is 1 for $t > 0$ and 0 for $t < 0$, also called Heaviside's function.

3. Operations on distributions. We have seen that locally integrable functions are examples of distributions. It is therefore natural to define operations on distributions that will remain valid for integrable functions. Let $f, g \in \mathcal{D}'$, $\alpha \in \mathbb{C}$, $t, \tau \in \mathbb{R}^n$ and $a \in (0, \infty)$. Then for every $\phi \in \mathcal{D}$, the following operations are defined.

a) Addition: $<f + g, \phi> = <f, \phi> + <g, \phi>$.

b) Scalar multiplication: $<\alpha f, \phi> = <f, \alpha\phi>$

c) Translation: $<f(t - \tau), \phi(t)> = <f(t), \phi(t + \tau)>$

d) Dilation: $<f(at), \phi> = <f, a^{-n}\,\phi(t/a)>$

e) Multiplication by $a(t) \in C^\infty(\mathbb{R}^n)$: $<af, \phi> = <f, a(t)\phi(t)>$
   e.g. $<a(t)\delta(t), \phi> = a(0)\phi(0)$.

f) Differentiation: $\langle D^k f, \phi\rangle = (-1)^k \langle f, D^k\phi\rangle$

e.g. $\langle 1_+'(t), \phi\rangle = -\langle 1_+(t), \phi'\rangle = -\int_0^\infty \phi'(t)d(t) = \phi(0)$

$\therefore \quad 1_+'(t) = \delta(t)$.

In other words, the derivative of the Heaviside function is Dirac's δ-function which was one of the properties of the δ-function deduced by Dirac.

An important consequence of the definition is that distributions have derivatives of all orders. This fact tells us that continuous and locally integrable functions are now infinitely differentiable in the distributional (or weak) sense. A main attraction of distribution theory is the relief we get from difficulties that arise with nondifferentiable functions.

g) Convergence: $\{f_n\} \to f$ in $\mathcal{D}'$ if $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ for every $\phi \in \mathcal{D}$. This is called weak convergence and f is the weak limit. e.g. $\lim_{n\to\infty} 1_+(t) n \sin nt = \delta(t)$. Under this convergence, $\mathcal{D}'$ turns out to be a complete space.

4. Tensor product and convolution of two distributions. Let E^m and E^n be Euclidean spaces of dimensions m and n respectively. Let the product $E^m \times E^n$, denoted by E^{m+n}, be the set of points $(t, \tau) = (t_1, \ldots, t_m, \tau_1, \ldots, \tau_n)$. Let $\mathcal{D}_t$, $\mathcal{D}_\tau$ and $\mathcal{D}_{t\tau}$ be the test function spaces on E^m, E^n and E^{m+n}. The tensor product $f(t) \otimes g(\tau)$ of distributions $f \in \mathcal{D}_t'$ and $g \in \mathcal{D}_\tau'$ is defined by

$$\langle f(t) \otimes g(\tau), \phi(t,\tau)\rangle = \langle f(t), \langle g(\tau), \phi(t,\tau)\rangle\rangle \quad \forall \phi(t,\tau) \in \mathcal{D}_{t\tau}.$$

The product $f(t) \otimes g(\tau)$ is again a distribution belonging to $\mathcal{D}'_{t\tau}$. It is commutative and associative when extended to any finite number of distributions.

The convolution of two locally integrable functions f and g on $\mathbb{R}^n$ is given by

$$f*g = \int_{\mathbb{R}^n} f(\tau)g(t-\tau)d\tau.$$

Since it is again locally integrable, we have

$$\begin{aligned} \langle f*g, \phi\rangle &= \int_{\mathbb{R}^n} \phi(t) \int_{\mathbb{R}^n} f(\tau)g(t-\tau)d\tau dt \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(y)\phi(x+y)dxdy \\ &= \langle f(x) \otimes g(y), \phi(x+y)\rangle. \end{aligned}$$

The convolution of two distributions may be defined in the same manner. But $\phi(x + y)$ need not have bounded support even if $\phi(x) \in \mathcal{D}_x$. To make the definition meaningful, we can put restrictions on f and g such that the support of $f \otimes g$ intersects the support of $\phi(x + y)$ in a bounded set.

The <u>support</u> of a distribution f is the complement of the union of open sets Ω on which f vanishes. ($f = 0$ on Ω if $\langle f, \phi\rangle = 0$ for each $\phi \in \mathcal{D}$ with support of $\phi \subset \Omega$.) If f or g has bounded support, then the <u>convolution</u> $f*g$ is well-defined and given by

$$\langle f*g, \phi\rangle = \langle f(t) \otimes g(\tau), \lambda(t, \tau)\ \phi(t + \tau)\rangle$$

where $\lambda(t, \tau)$ is in $\mathcal{D}_{t\tau}$ and equal to 1 on a neighborhood containing $\operatorname{supp}(f(t) \otimes g(\tau)) \cap \operatorname{supp} \phi(t + \tau)$.

It is easy to verify that $\delta^{(m)} * f = f^{(m)}$. Thus, if $F(\delta)$ is a polynomial in δ, then $F(\delta)*y = v$ is an ordinary differential equation. Another property of the convolution is that if f or g has compact support, then

$$D^k(f*g) = (D^k f)*g = f*(D^k g)$$

for every multi-index k.

These properties give a simple proof of the existence theorem for linear partial differential equations with constant coefficients:

$$P(D)u = v. \tag{19}$$

Malgrange and Ehrenpreis (see [16]) have proved the existence of fundamental solutions of such equations, i.e., E exists such that $P(D)E = \delta$. If v has compact support, then $u = E*v$ is a solution of (19). Indeed,

$$P(D)\ (E*v) = (P(D)E)*v = \delta*v = v.$$

IV. Integral Transformations of Generalized Functions.

1. Schwartz extended the Fourier transformation given by

$$(\mathcal{F}\phi)\ (s) = \int_{-\infty}^{\infty} \phi(t)\ e^{-ist}\,dt$$

to distributions. If f is a distribution, he defined the Fourier transformation of f by

$$\langle \mathcal{F}f, \phi \rangle = \langle f, \mathcal{F}\phi \rangle . \tag{20}$$

Since the Fourier transformation of a test function in $\mathcal{D}$ does not itself have compact support, definition (20) needs qualification. He provided one by requiring ϕ to be a smooth function satisfying the condition

$$\gamma_{m,k}(\phi) \equiv \sup_{0<t<\infty} |t^m D^k\phi| < \infty \ , \quad m, k = 0, 1, 2, \ldots \ . \tag{21}$$

Denoting the space of all such functions by $\mathcal{S}$, he showed that the Fourier transformation is an isomorphism on $\mathcal{S}$. Equation (20) is now meaningful for elements of the dual space $\mathcal{S}'$ which he called tempered distributions.

An operational calculus is developed via such transform formulas as

$$\mathcal{F}[P(D)f] = P(is)\,\mathcal{F}f\ , \quad f \in \mathcal{S}'. \tag{22}$$

$$\mathcal{F}[P(-it)f] = P(D)\mathcal{F}f\ , \quad f \in \mathcal{S}'. \tag{23}$$

An alternative approach to (20) is given by Gelfand and Shilov [17] who defined the Fourier transformation of f belonging to $\mathcal{D}'$ by

$$<\mathcal{F}f, \mathcal{F}\phi> \ = 2\pi<f,\ \phi>. \tag{24}$$

This means that $\mathcal{F}$ will be mapping $\mathcal{D}$ onto a space Z of Fourier transforms. The Fourier transform of the distribution f is then an element in Z'. Formulas analogous to (22) and (23) are derived to generate an operational calculus.

Schwartz also defined the Laplace transform of a distribution f as the Fourier transform of $e^{-\sigma t}f(t)$ where σ is restricted so as to make $e^{-\sigma t}f(t)$ a tempered distribution.

2. In recent years, other integral transformations have been extended to generalized functions. The definitive work in this area is the book of Zemanian [18] which gave extensions for the two-sided Laplace, the Mellin, the Hankel, the Meijer, the Weierstrass and the convolution transformations.

In each case, a complete topological vector space Z of infinitely differentiable test functions is constructed. Its topology is generated by a separating family of seminorms called a multinorm. An example of a multinorm is the set $\{\gamma_{m,k}\}$ in (21) for the space $\mathcal{S}$. The transformation is then defined on the dual space Z' either by means of a Parseval relation similar to (20) and (24) or by the application of the generalized function to the appropriate kernel. We distinguish the two by calling the former, the adjoint method and the latter, the kernel method.

Suppose T is an integral transformation with kernel $k(s, t)$ over some subset I of the real line. Classically, this is written

$$(Tf)(s) = \int_I k(s,t)f(t)dt. \tag{25}$$

Under suitable integrability conditions, a Parseval relation is given by

$$\int_I (Tf)(x)\phi(x)dx = \int_I f(x)(T^*\phi)(x)dx \tag{26}$$

or

$$\int_I f(t)\phi(t)dt = \int_I (Tf)(s)(T^*\phi)(s)ds \tag{27}$$

where T^* is the adjoint of T.

The adjoint method then generalizes (26) and (27) to forms similar to (20) and (24) by requiring that T^* be an isomorphism from Z onto itself or onto another space U, say. The generalized transformation is then defined as an automorphism on Z' or as an isomorphism of U' onto Z'. This method was used to extend the Hankel transformation.

In the kernel method, the transform of a generalized function $f \in Z'$ is given by

$$(Tf)\ (s) = \langle f(t),\ k(s,\ t)\rangle \tag{28}$$

which naturally reduces to (25) when $f(t)$ is an ordinary function. The test function space Z is then constructed so as to contain $k(s, t)$ as an element for suitably restricted s. Except for the Hankel transformation, Zemanian extended all the above transformations by the kernel method. Later, Koh and Zemanian [19] also used the kernel method to extend the Hankel and the I-transformations.

Other studies and extensions have appeared since [18]. These involve the Hankel transform by Koh [20], [21], [22], Dube and Pandey [23], Lee [24], [25], and McBride [26]; the fractional transform by Erdélyi and McBride [27], McBride [28], [29], [30], and Braaksma and Schuitman [31]; the Stieltjes transform by Pandey [32], Erdélyi [33], Ghosh [34] and Pathak [35]; the Hardy transform by Pathak and Pandey [36], [37]; the Kontorovich-Lebedev transform by Zemanian [38], Pathak and Pandey [39] and Buggle [40]; the Mehler-Fok transform by Tiwari [41] and Buggle [40]; the Watson transform by Hsu [42] and Braaksma and Schuitman [31], the ${}_1F_1$-transform by Rao [43], the Mellin transform by Komeč [44] and the Hilbert transform by Orton [45].

To be sure, there is a plethora of test function spaces and their dual spaces, each tailored according to the behavior of the integral transformation. The common thread through these is that the Schwartz space $\mathcal{D}$ turns out to be a dense subspace of the constructed test function space and the inversion theorem recovers the original distribution in the sense of convergence in $\mathcal{D}'$. For example, the test function space $\mathcal{J}_{\mu,a}$ [19] is given by

$$\mathcal{J}_{\mu,a} \equiv \{\phi \in C^\infty(0,\ \infty) : \sup_{0<x<\infty} |e^{-ax}\ x^{-\mu-\frac{1}{2}}\ S_\mu^k(\phi)| < \infty,\ k = 0,\ 1,\ 2,\ \ldots\}$$

where $S_\mu^k = (x^{-\mu-\frac{1}{2}} Dx^{2\mu+1} Dx^{-\mu-\frac{1}{2}})^k$. This space contains $\sqrt{xy}\, J_\mu(xy)$ if $\mu \geq -\frac{1}{2}$ and y belongs to the strip $\{y: |\text{Im } y| < a,\ y \neq 0$ or a negative number$\}$. Using the kernel method, the μth order Hankel transformation for $f \in \mathcal{J}'_{\mu,a}$ is defined by

$$(\mathcal{H}_\mu f)(y) = \langle f(x), \sqrt{xy}\ J_\mu(xy)\rangle. \tag{29}$$

Since $\mathcal{D}(0, \infty)$ is a subspace of $\mathcal{J}_{\mu,a}$, the restriction of any $f \in \mathcal{J}'_{\mu,a}$ to $\mathcal{D}(0, \infty)$ is in $\mathcal{D}'(0, \infty)$. If a function $F(y)$ is given by (29) where y is a positive real variable, then in the sense of convergence in $\mathcal{D}'(0, \infty)$,

$$f(x) = \lim_{r \to \infty} \int_0^r F(y) \sqrt{xy}\ J_\mu(xy)\, dx.$$

Thus, the classical inversion theorem for the Hankel transform still holds in a distributional sense.

V. Mikusinski-type Operational Calculi. Following Mikusinski's work, one stream of research saw the development of operational calculi for other differential operators. Ditkin [46] and Ditkin and Prudnikov [47] developed an operational calculus for the Bessel operator DtD. They considered the convolution ring $C^2[0, \infty)$ where the convolution $*$ is defined by

$$\phi * \psi = DtD \int_0^t d\xi \int_0^1 \phi(x\xi)\, \psi[(1-x)(t-\xi)]dx.$$

This ring also has a field extension. Meller [48],[49] generalized Ditkin's calculus to operators $B_\alpha = t^{-\alpha}Dt^{1+\alpha}D$ with $-1 < \alpha < 1$ by a convolution process that reduces to Ditkin's when $\alpha = 0$. Using fractional calculus, Koh [50], [51], [52] extended Meller's calculus to $-1 < \alpha < \infty$. His convolution process is given by

$$\phi*\psi = \frac{1}{\Gamma(\alpha+1)} Dt^{1-\alpha}D^{\alpha+1} \int_0^t \xi^\alpha(t-\xi)^\alpha \int_0^1 \phi(x\xi)\psi[(1-x)(t-\xi)]dxd\xi$$

where D^α is the Riemann-Liouville derivative of order μ (see [2]).

Other generalizations were to operators containing derivatives of order n or $2n$. These include the work of Botashev [53] for the operator $t^{-1}(tD)^n$; Krätzel [54], [55] for $Dt^{\frac{1}{n}-\nu}(t^{1-\frac{1}{n}}D)^{n-1}t^{\nu-1-\frac{2}{n}}$; Dimovski [56], [57] for $t^{\alpha_0}D\, t^{\alpha_1} \dots t^{\alpha_{n-1}}D\, t^{\alpha_n}$; and Ditkin and Prudnikov [58] for $t^{1-n}D^n t^n D^n t^{n-1}$. Inevitably, the convolution process becomes horrendously involved. However, the attempts have strived to include the Bessel operators DtD and B_α as special cases.

A different direction in developing a suitable convolution process for a given operator is by way of similarity transformations between linear operators. Linear operators $A:M \to M$ and $B:N \to N$ are said to be similar if there is a bijective linear map $S:M \to N$ such that $SA = BS$. S is called a similarity. It is easy to show that if $\phi *_B \psi$ is a convolution process for the operator B and S is a similarity from A to B, then $f *_A g = S^{-1}[Sf *_B Sg]$ is a convolution process for A. The problem of finding a suitable convolution process for a given operator A is now reduced to finding a similarity from A to the operator D or D^2 for which Mikusinski has developed his calculus. Recently, Bozhinov and Dimovski [59] gave a similarity between the general second order operator $a(x)D^2 + b(x)D + c(x)$ and the operator $D^2 - q(t)$ where $q(t)$ is a function derived from $a(x)$, $b(x)$ and $c(x)$. They then established a similarity between the latter operator and D^2.

Finally, we note that a Mikusinski calculus may be developed without the Titchmarsh Theorem by treating the operators as convolution quotients [a, b] with b belonging to a multiplicative system, [60].

References

[1] O. Heaviside, Electromagnetic Theory, Vol. I-III, 3rd. ed., Chelsea Publ. Co., New York, 1971.

[2] B. Ross (Ed.), Fractional Calculus and its Applications, Springer-Verlag, Berlin, 1975.

[3] P.A.M. Dirac, The Principles of Quantum Mechanics, 3rd. ed., Univ. Press, Oxford, 1947.

[4] J. Hadamard, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Hermann, Paris, 1932.

[5] S. Bochner, Vorlesungen über Fouriersche Integrale, Akad. - Verlag, Leipzig, 1932.

[6] S.L. Sobolev, Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales, Mat. Sbornik 1, 39-72 (1936).

[7] L. Schwartz, Théorie des distributions, 3rd. ed., Hermann, Paris, 1966.

[8] J. Mikusinski, Operational Calculus, Pergamon Press, Oxford, 1959.

[9] A. Erdélyi, Operational Calculus and Generalized Functions, Holt, Rinehart, Co., New York, 1966.

[10] J. Korevaar, Distributions defined from the point of view of applied mathematicians, Proc. Kon. Ned. Ak. W. ser. A. Amsterdam, 368-383, 463-503, 563-674 (1955).

[11] G. Temple, Theories and applications of generalized functions, J. London Math. Soc. 28, 134-148 (1953).

[12] M.J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge U. Press, Cambridge, 1958.

[13] P. Antosik, J. Mikusinski and R. Sikorski, Theory of Distributions: the Sequential Approach, Elsevier, Amsterdam, 1973.

[14] F. Trèves, Topological Vector Spaces, Distributions, and Kernels, Academic Press, New York, 1967.

[15] A.H. Zemanian, Distribution Theory and Transform Analysis, McGraw-Hill, New York, 1965.

[16] F. Trèves, Linear Partial Differential Equations with Constant Coefficients, Gordon and Breach, Science Publ., New York, 1966.

[17] I.M. Gelfand and G.E. Shilov, Generalized Functions, Vol. I, Academic Press, New York, 1964.

[18] A.H. Zemanian, Generalized Integral Transformations, Interscience, New York, 1968.

[19] E.L. Koh and A.H. Zemanian, The complex Hankel and I-transformations of generalized functions, SIAM J. Appl. Math. 16, 945-957 (1968).

[20] E.L. Koh, A representation of Hankel transformable generalized functions, SIAM J. Math. Anal. 1 (1), 33-36 (1970).

[21] E.L. Koh, The Hankel transformation of negative order for distributions of rapid growth, SIAM J. Math. Anal. 1 (3), 332-327 (1970).

[22] E.L. Koh, The n-dimensional distributional Hankel transformation, Can. J. Math. 27, 423-433 (1975).

[23] L.S. Dube and J.N.Pandey, On the Hankel transformation of distributions, Tohoku Math. J. 27, 337-354 (1973).

[24] W.Y.K. Lee, On spaces of Type H_μ and their Hankel transformations, SIAM J. Math. Anal. 5, 336-347 (1974).

[25] W.Y.K. Lee, On Schwartz's Hankel transformation of certain spaces of distributions, SIAM J. Math. Anal. 6, 427-432 (1975).

[26] A.C. McBride, The Hankel transform of some classes of generalized functions and connections with fractional integration, Proc. Roy. Soc. Edinburgh 81A, 95-117 (1978).

[27] A. Erdélyi and A.C. McBride, Fractional integrals of distributions, SIAM J. Math. Anal. 1, 547-557 (1970).

[28] A.C. McBride, A theory of fractional integration for generalized functions, SIAM J. Math. Anal. 6, 583-599 (1975).

[29] A.C. McBride, A theory of fractional integration for generalized functions II, Proc. Roy. Soc. Edinburgh 77A, 335-349 (1977).

[30] A.C. McBride, Fractional Calculus and Integral Transforms of Generalized Functions, Pitman, San Francisco, 1979.

[31] B.L.J. Braaksma and A. Schuitman, Some classes of Watson transforms and related integral equations for generalized functions, SIAM J. Math. Anal. 7, 771-798 (1976).

[32] J.N. Pandey, On the Stieltjes transform of generalized functions, Proc. Camb. Philo. Soc. 71, 85-96 (1972).

[33] A. Erdélyi, Stieltjes transforms of generalized functions, Proc. Roy. Soc. Edinburgh 77A, 231-249 (1977).

[34] J.D. Ghosh, The Stieltjes transform of distributions, doctoral thesis, Ranchi University, Bihar, India, 1975.

[35] R.S. Pathak, A distributional generalized Stieltjes transformation, Proc. Edinb. Math. Soc. 20, 15-22 (1976).

[36] R.S. Pathak and J.N. Pandey, A distributional Hardy transformations, Proc. Camb. Philo. Soc. 76, 247-262 (1974).

[37] R.S. Pathak and J.N. Pandey, A distributional Hardy transformation II, International J. Math. & Math Sc. 2, 693-701 (1979).

[38] A.H. Zemanian, The Kontorovich-Lebedev transformation on distributions of compact support and its inversion, Proc. Camb. Philo. Soc. 77, 139-143 (1975).

[39] R.S. Pathak and J.N. Pandey, The Kontorovich-Lebedev transformation of distributions, Math Z. 165, 29-51 (1979).

[40] G. Buggle, Die Kontorovich-Lebedev-transformation und die Mehler-Fok-transformation für Klassen verallgemeinerter Funktionen, doctoral thesis, T.H. Darmstadt, Germany, 1977.

[41] U.N. Tiwari, The Mehler-Fok transformation of distributions, unpublished.

[42] Hsing-Yuan Hsu, Distributional Watson transforms, Can. J. Math. 24, 1191-1197 (1972).

[43] G.L.N. Rao, A complex inversion theorem for a distributional ${}_1F_1$-transform, Bull. Cal. Math. Soc. 68, 267-274 (1976).

[44] A.I. Komeč, Equations with homogeneous kernels and the Mellin transform of generalized functions, Teoret. Mat. Fiz. 27, 149-162 (1976).

[45] M. Orton, Functional integrals and Hilbert transforms of distributions, Proc. Roy. Soc. Edinburgh, (to appear).

[46] V.A. Ditkin, Operational calculus theory, Dokl. Akad. Mank. SSSR. 116, 15-17 (1957).

[47] V.A. Ditkin and A.P. Prudnikov, Operational Calculus in Two Variables, Pergamon Press, New York, 1962.

[48] N.A. Meller, On an operational calculus for the operator $B_\alpha = t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}$, Vichisl. Mat. 6, 161-168 (1960).

[49] N.A. Meller, Some applications of operational calculus to problems in analysis, Zh. Vichisl. Mat. i. Mat. Fiz. 3(1), 71-89 (1963).

[50] E.L. Koh, A Mikusinski calculus for the Bessel operator B_μ, Proc. Conf. D.E. at Dundee, Springer-Verl. Lec. Notes # 564, 291-300 (1976).

[51] E.L. Koh, Application of an operational calculus to certain time-varying systems, Int. J. Syst. Sci. 9(15), 595-601 (1978).

[52] E.L. Koh, A direct extension of Meller's Calculus, Univ. Petrol. & Minerals Math. Tech. Report # 13, Nov. 1980.

[53] A.I. Botashev, Operational calculus theory, Issled. Integrodiff. Uravn. Kirgisu 2, 997-1018 (1962).

[54] E. Krätzel, Bemerkungen zur Meijer-Transformation und Anwendungen, Math. Nachr, 30, 327-334 (1965).

[55] E. Krätzel, Differentiationssätze der L-Transformation und Differentialgleichungen nach dem Operator $\frac{d}{dt}\, t^{\frac{1}{n}-\nu}\,(t^{1-\frac{1}{n}}\frac{d}{dt})^{n-1}\, t^{\nu+1-\frac{2}{n}}$, Math. Nachr. 35, 105-114 (1967).

[56] I. Dimovski, On an operational calculus for a differential operator, C.r. Acad. Bul. Sc., 21, 513-516 (1968).

[57] I. Dimovski, Foundations of operational calculi for the Bessel-type differential operator, Serdica, 1, 51-63 (1975).

[58] V.A. Ditkin and A.P. Prudnikov, Operatsionnoe Ischislenie, Byss. Skola, Moscow, 1975.

[59] N.S. Bozhinov and I.H. Dimovski, Operational Calculi for Boundary-Value Problems, Generalized Functions and Operational Calculus, ed. by I. Dimovski, Bulg. Acad. Sc. Publ. Sofia, 1979.

[60] M.A. Goldberg and H. Bowman, Mikusinski operators without the Titchmarsh Theorem, A.M. Monthly, 87, 564-567 (1980).