King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 021

February 1981

An Elementary Proof of the Brouwer Fixed Point Theorem

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Abstract

This paper presents an elementary proof of Brouwer's fixed point theorem. The proof relies heavily on linear algebra and can therefore be used by students who are not yet prepared to understand either the Sperner's Lemma proof or a homology proof.

Introduction

Brouwer's fixed point theorem states that each continuous mapping of an n-cell into itself has at least one fixed point. Since an n-cell is homeomorphic to a Euclidean n-simplex and a continuous mapping from a Euclidean n-simplex into itself may be approximated by a simplicial mapping, it suffices to consider the case of a simplicial mapping from a barycentric subdivision of a Euclidean n-simplex into itself [1,p64]. This proof of the Brouwer fixed point theorem is therefore elementary in the sense that it uses neither the Sperner's Lemma approach nor the homology approach.

<u>Lemma 1</u>:

Let B be a Euclidean n-simplex in R^n, B_j an $(n-1)$ face of B, H the hyperplane containing B_j and L a ray from the barycenter of B_j into the halfspace containing B. If $L \not\subset H$, then $L \cap B_i \neq \phi$ for some $(n - 1)$ face B_i of B different from B_j.

<u>Proof</u>:

Let $b_0, b_1, \ldots b_n$ denote the vertices of B, then since the set $\{b_i\}_{i=0}^n$ is affinely independent, any point $p \in R^n$ can be uniquely represented by $p = \sum_{i=0}^n \lambda_i b_i$, where $\lambda_i \in R$ and $\sum_{i=0}^n \lambda_i = 1$. Thus we may write:

$$B = \{x/x = \sum_{i=0}^n \lambda_i b_i, \lambda_i \in [0, 1], \sum_{i=0}^n \lambda_i = 1\},$$

$$B_j = \{x/x = \sum_{i=0}^n \lambda_i b_i, \lambda_i \in [0, 1], \lambda_j = 0, \sum_{i=0}^n \lambda_i = 1\},$$

and

$$H = \{x / x = \sum_{i=0}^{n} \lambda_i\, b_i,\ \lambda_j = 0,\ \lambda_i \in R,\ \sum_{i=0}^{n} \lambda_i = 1\}.$$

Furthermore the halfspace containing B can be written

$$H(+) = \{x / x = \sum_{i=0}^{n} \lambda_i\, b_i,\ \lambda_j = 0,\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1\}.$$

Now choose a point p in L but not in B. Then $p = \sum_{i=0}^{n} \mu_i\, b_i$ uniquely and $\sum_{i=0}^{n} \mu_i = 1$.

Since $L \subset H(+)$, it follows that $\exists\, \ell$, $0 \leq \ell \leq n$ such that $\mu_\ell > 1$. Since $\sum_{i=0}^{n} \mu_i = 1$, it follows that $\exists\, k$, $0 \leq k \leq n$ such that $\mu_k < 0$ and $k \neq j$.

Recall that the ray L can be expressed as $L = \{b + tp \,/\, t \geq 0\}$. Consider the function $F(t) = b + tp$, $t \geq 0$. $F(t)$ is a continuous function from R into R^{n+1} and is therefore continuous in each coordinate. It follows from the intermediate value theorem that there is a point y in L such that $y = \sum_{i=0}^{n} \nu_i\, b_i$ where $\nu_m = 0$. Thus $y \in B_m$ and therefore $B_m \cap L \neq \phi$.

<u>Lemma 2:</u>

If E is a Euclidean n-simplex and E^α is a barycentric subdivision of E, then an $(n - 1)$ face of an n-simplex in E^α is not common to more than two n-simplexes in E^α.

<u>Proof:</u>

Let A be an $(n - 1)$ face of an n-simplex in E^α and $a_0, a_1, \ldots, a_{n-1}$ denote the vertices of A and let H be the hyperplane determined by A. Suppose that A is common to more than two n-simplexes in E^α, then at least two n-simplexes, say B and C, must be contained in one of the halfspaces determined by H. Denote the vertices of B by $b, a_0, a_1, \ldots, a_{n-1}$ and the vertices of C by $c, a_0, a_1, \ldots, a_{n-1}$. Let $H(+) = \{x/x = \sum_{i=0}^{n-1} \lambda_i a_i + \lambda_n b, \sum_{i=0}^{n} \lambda_i = 1, \lambda_n \geq 0\}$ denote the halfspace containing B and C. Consider the segment L from b to the barycenter a of A. By lemma 1, L intersects a face of C different from A. Therefore L contains an interior point of C and hence $B \cap (\text{Interior } C) \neq \phi$ since $L \subset B$. But B and C are in E^α and thus $B \cap C$ is a face common to both B and C. The only such face containing an interior point of C is C itself, hence B = C. Therefore A is not a face of more than two n-simplexes in E^α.

Lemma 3:

If E is a Euclidean n-simplex in R^n, E^α is a barycentric subdivision of E, $B \subset E^\alpha$ is an n-simplex and B_j is an (n - 1) face of B, then exactly one of the following statements is true:

(1) B_j is in the boundary ∂E^α of E^α, or

(2) B_j is common to exactly two n-simplexes in E^α.

Proof:

The proof follows directly from the definition of boundary and Lemma 2.

Theorem 1:

There is no simplicial retraction of a barycentric subdivision E^α of a closed Euclidean n-simplex $E \subset R^n$ onto its boundary ∂E^α.

Proof:

Suppose that $f : E^\alpha \to \partial E^\alpha$ is a simplicial retraction. Note that f has the following properties:

a) f is continuous.

b) f is linear in terms of barycentric coordinates.

c) f maps vertices onto vertices.

d) simplexes are preserved under f, i.e., if $(a_0, a_1, \ldots a_\ell)$, $0 \le \ell \le n$ is an ℓ-simplex in E^α, then $(f(a_0), f(a_1), \ldots, f(a_\ell))$ is a k-simplex in ∂E^α, $0 \le k \le \ell$, $k \le (n - 1)$.

e) f restricted to ∂E^α is the identity mapping.

The proof will be given in two parts:

Part 1: Any n-simplex $B \subset E^\alpha$ mapping onto an $(n - 1)$ simplex $A \subset \partial E^\alpha$ has precisely two $(n - 1)$ faces mapping onto A.

Part 2: If A is an $(n - 1)$ simplex in ∂E^α, then there is a reversible chain $\{B_m\}_{m=1}^{m=k}$ of n-simplexes in E^α, i.e., a reversible function from the first k positive integers into the n-simplexes of E^α, such that:

(1) B_m maps onto A, $m = 1, 2, \ldots, k$.

(2) If $|q - \ell| = 1$, $1 \le q, \ell \le k$, then B_q and B_ℓ have a common $(n - 1)$ face. Otherwise, $B_q \cap B_\ell = \phi$.

(3) B_m has exactly two $(n - 1)$ faces mapping onto A, $m = 1,2,\ldots,k$.

(4) B_k has an $(n - 1)$ face in ∂E^α which is distinct from A and which maps onto A.

The existence of such a chain proves the theorem by contradicting the identity mapping property of f when restricted to E^α.

Proof of Part 1:

Let $B \subset E^\alpha$ be an n-simplex mapping onto an (n - 1)-simplex $A \subset \partial E^\alpha$. Such a B exists since each (n - 1) face in ∂E^α maps onto itself and is a face of some n-simplex in E^α.

Let $a_0, a_1, \ldots, a_{n-1}$ denote the vertices of A and $b_0, b_1, \ldots, b_n$ denote the vertices of B. Define the i-th face of B as the set B_i of points in B such that the i-th barycentric coordinate is zero. Since $f : B \to A$ is onto, n vertices map onto (n - 1) vertices and hence, by the pigeon-hole principle, two vertices of B, say b_k and b_ℓ map onto the same vertex of A, i.e., $f(b_k) = f(b_\ell)$. If i is neither k nor ℓ, then $f(b_i) \neq (f(b_j)$ for $i \neq j$ and i, j integers in $[0, n]$.

Now consider f restricted to B_k. If $x \in B_k$ then $x = \sum_{\substack{i=0 \\ i \neq k}}^{n} \lambda_i b_i$ where $\lambda_i \geq 0$ and $\sum_{\substack{i=0 \\ i \neq k}}^{n} \lambda_i = 1$. Thus $f(x) = \sum_{\substack{i=0 \\ i \neq k}}^{n} \lambda_i f(b_i)$.

If y is a point in A, then $y = \sum_{i=0}^{n-1} \mu_i a_i$.

Let $b_{u(i)}$ be the vertices of B_k which maps onto a_i, i.e. $f(b_{u(i)}) = a_i$, and let $z \in B_k$ be the point such that $z = \sum_{i=0}^{n-1} \mu_i b_{u(i)}$. Then $f(z) = \sum_{i=0}^{n-1} \mu_i f(b_{u(i)}) = \sum_{i=0}^{n-1} \mu_i a_i = y$. Therefore f restricted to B_k is onto. Furthermore, by a similar argument, f restricted to B_ℓ is onto.

Now consider f restricted to B_j, $k \neq j \neq \ell$, $j \in [0, n]$. If x is a point in B_j, then $x = \sum_{\substack{i=0 \\ i \neq j}}^{n} \nu_i b_i$ and $f(x) = \sum_{\substack{i=0 \\ i \neq j}}^{n} \nu_i f(b_i)$. But $f(b_k) = f(b_\ell)$ and therefore the n vertices of B_j are mapped onto $(n - 1)$ vertices of A. Consequently one vertex of A is not in the range of f restricted to B_j and hence f restricted to B_j is not onto. We therefore conclude that precisely two $(n - 1)$ faces of B map onto A.

Proof of Part 2:

Let A be an arbitrary but fixed $(n - 1)$ simplex in E^α and let C be the collection of chains of n-simplexes in E^α which satisfy conditions (1), (2) and (3) above. Partially order C as follows: If $D, E \in C$, $D \leq E$ iff there exists an integer $q > 0$ such that $D_i = E_i$ for $1 \leq i \leq q$. C is not empty since A is an $(n - 1)$ face of an n-simplex in E^α. E^α consists of a finite set of n-simplexes and therefore every chain in C is bounded above. This being the case, by Zorn's lemma, C has a maximal element, say $M = \{M_i\}_{i=1}^{k}$. Suppose M does not satisfy condition (4). M_k has exactly two $(n - 1)$ faces which map onto A. One of these faces is common to M_{b-1}. The second face cannot be common to another n-simplex M_j in M, for then M_j has three $(n - 1)$ faces which map onto A, a contradiction. Therefore, by lemma 3, this second face of M_k is either in ∂E^α or is common to a n-simplex F in E^α, $F \notin M$. Suppose the latter is true. Since an $(n - 1)$ face of F maps onto A, F maps onto A and we thus extend $\{M_i\}_{i=1}^{k}$ to a chain $\{M_i\}_{i=1}^{k+1}$. This contradicts M being a maximal chain in E^α and implies that the second face of M_k which maps onto A is in ∂E^α. This in turn contradicts the hypothesis that f restricted to ∂E^α is the identity mapping and proves the theorem.

<u>Theorem 2</u>: <u>Brouwer's Theorem</u>.

If E is a Euclidean n-simplex and f is a continuous mapping of E into itself, then there is a point $p \in E$ for which $f(p) = p$.

<u>Proof</u>:

Suppose $f : E \to E$ is a mapping such that $f(x) \neq x$ for each point $x \in E$. Define $r(x)$ as the point in the boundary of E such that x lies on the line segment from $f(x)$ to $r(x)$. Given $x \in E$ suppose $r(x)$ is not unique. Then there are two points y and z in the boundary of E such that $y = f(x) + t[x - f(x)]$ for some $t \geq 1$ and $z = f(x) + s\,[x - f(x)]$ for some $s \geq 1$.

Solving each of the above equations for $f(x)$ and subtracting gives

$$f(x) - f(x) = y - z + s\,[x - f(x)] - t\,[x - f(x)] =$$

$$= y - z + [x - f(x)]\,[s - t] = 0$$

Each of x, $f(x)$, y and z are points in E and may therefore be expressed in barycentric coordinates with respect to $C_0, C_1, \ldots, C_n$, the vertices of E.

Let $$x = \sum_{i=0}^{n} \lambda_i C_i\,, \quad \lambda_i \geq 0, \quad \sum_{i=0}^{n} \lambda_i = 1$$

$$f(x) = \sum_{i=0}^{n} \mu_i C_i, \; \mu_i \geq 0, \quad \sum_{i=0}^{n} \mu_i = 1$$

$$y = \sum_{i=0}^{n} \nu_i C_i\,, \quad \nu_i \geq 0, \quad \sum_{i=0}^{n} \nu_i = 1$$

$$z = \sum_{i=0}^{n} \eta_i C_i, \quad \eta_i \geq 0, \quad \sum_{i=0}^{n} \eta_i = 1.$$

Let $s - t = w$. Then

$$y - z + [x - f(x)]\, w = \sum_{i=0}^{n} (\nu_i - \eta_i)\, C_i + w \sum_{i=0}^{n} (\lambda_i - \mu_i)\, C_i =$$

$$= \sum_{i=0}^{n} [\nu_i - \eta_i + w(\lambda_i - \mu_i)]\; C_i = 0$$

$C_0, C_1, \ldots, C_n$ are affinely independent and $\sum_{i=0}^{n} [\nu_i - \eta_i + w(\lambda_i - \mu_i)]\, C_i = 0$, so $\nu_i - \eta_i + w(\lambda_i - \mu_i) = 0$ for $i = 0, 1, \ldots, n$. Hence

$$w = \frac{\eta_i - \nu_i}{\lambda_i - \mu_i} \quad \text{for } i = 0, 1, \ldots, n.$$

But y and z are in the boundary of E and therefore $\nu_i = \eta_i = 0$ for some i, $0 \leq i \leq n$. It follows that $w = 0$ and $y = z$. Hence $r(x)$ is unique.

Now let $\varepsilon > 0$ be given. Consider

$$F(t) = f(x) + t[x - f(x)] \quad \text{and}$$

$$G(t) = f(z) + t[z - f(z)]\,. \quad \text{Then}$$

$$|F(t) - G(t)| = |f(x) - f(z) + t[x - f(x)] - t[z - f(z)]|$$

$$\leq |f(x) - f(z)| + |t[f(z) - f(x)] + t(x - z)|$$

$$\leq |f(x) - f(z)| + |t|\,|f(z) - f(x)| + |t|\; |x - z|$$

Recall that $r(x) = f(x) + t(x - f(x))$ for some $t \geq 1$. Hence $t = \dfrac{r(x) - f(x)}{x - f(x)}$ and $|t| = \dfrac{|r(x) - f(x)|}{|x - f(x)|}$.

By definition, $|r(x) - f(x)| = [\sum_{i=0}^{n} (\lambda_i - \nu_i)^2]^{\frac{1}{2}}$ and the diameter of a simplex is the length of its longest edge, hence the maximum value of $|r(x) - f(x)|$ occurs when $\lambda_i = 1$, $\nu_j = 1$ and $i \neq j$, i.e., $|r(x) - f(x)| \leq \sqrt{2}$. Therefore

$|t| \leq \dfrac{\sqrt{2}}{|x - f(x)|} < \infty$ since $f(x) \neq x$.

f is continuous, so if $\alpha = \frac{\varepsilon}{3|t|}$, then there exists a $\beta > 0$ such that $|x - z| < \beta$ implies $|f(x) - f(z)| < \alpha$. Let $\sigma = \min\{\frac{\varepsilon}{3|t|}, \beta\}$, then $|x - z| < \sigma$ gives

$$|F(t) - G(t)| \leq |f(x) - f(z)| + |t| \; |f(z) - f(x)| + |t| \; |x - z|$$

$$\leq \frac{\varepsilon}{3|t|} + \frac{|t|\varepsilon}{3|t|} + \frac{|t|\sigma}{3|t|} \leq \varepsilon$$ since $|t| \geq 1$. Therefore $r(x)$ is continuous and is a retraction of E onto its boundary. This contradicts theorem 1 and proves the Brouwer fixed point theorem.

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