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LIMIT THEOREMS AND THEIR REMAINDER TERMS IN RENEWAL THEORY

by

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Abstract

The classical and mixing versions of the central limit theorem of renewal variables are discussed both when the number of renewals have positive or zero drift. The emphasis is on obtaining exact explicit remainder terms in these theorems in terms of moment conditions. Analogous results for first passage variables of random walks are also obtained.

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1. Introduction.

Let $\{X_n\}$ be a sequence of independent identically distributed (iid) random variables (rv's) having common distribution F such that $EX_1 = \mu$ and $\text{Var } X_1 = \sigma^2 > 0$. The number of renewals (first passage time) during the time interval $[0, t]$ is:

$$(1.1) \qquad N(t) = \sup \{n : \sum_{i=1}^{n} X_i \leq t\},$$

and

$$(1.2) \qquad U(t) = \inf \{n : \sum_{i=1}^{n} X_i > t\},$$

respectively.

The central limit theorem for $N(t)$ $(U(t))$ is well known and proved in Billingsley (1968) Theorem 17.3 when $X_i \geq 0$ and in Siegmund (1968) in general provided that $\mu > 0$. When $\mu = 0$ the central limit theorem is given in Teicher (1973) and Gut (1974).

The rates of convergence in the central limit theorem are relatively new. Ahmad (1981a) has established equivalence between the rate of convergence of $N(t)$ and of the partial sums, respectively between $U(t)$ and the maximum partial sums of $\{X_n\}$ when $\mu > 0$. On the other hand Englund (1980) established an explicit remainder term in the normal approximation of $N(t)$ (and obviously analogous result is also possible for $U(t)$). Many more avenues of investigation remain, e.g., does a mixing (in the sense of Renyi (1958)) central limit theorem hold for $N(t)$ (or $U(t)$), and if so at what rate? These and other questions led us to systematically discuss the problem and the result is the present multipurpose investigation.

The two cases $\mu = 0$ and $\mu > 0$ have to be dealt with separately. This we do. For each one of the two cases we present the central limit theorem and its rate of convergence. Then we present the conditioned (mixing) central limit theorem and its rate. The remainder terms which we obtain are explicit in their dependence on the moments of $\{X_n\}$.

The organization of the paper is as follows: It includes six sections; after this introduction, Section 2 includes a presentation of the results when $\mu > 0$ and the proofs of these results are in Section 3. Section 4 deals with the results when $\mu = 0$ and their proofs are in Section 5. Finally in Section 6 we present a nonuniform bound in the central limit theorem for maximum sums with zero mean. That is needed in the proofs of Section 5.

2. The positive drift case.

First we deal with the renewal variable $N(t)$. Define

$$W_t(m) = \left|P[N(t) \leq m] - \Phi\left(\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}} \right)\right|, \tag{2.1}$$

and set $W_t = \sup_m W_t(m)$. Firstly we state, for sake of completeness, two known results.

THEOREM 1 [Siegmund (1968)]. For any $m \geq 1$,

$$W_t(m) \to 0, \quad \text{as} \quad t \to \infty. \tag{2.2}$$

The rate in the above theorem is possible under a stronger moment condition, viz.,

THEOREM 2 [Englund (1980)]. Let $\gamma = E^{1/(2+\delta)}|X_1 - \mu|^{2+\delta} < \infty$, $0<\delta\leq 1$,

<u>then there is a positive constant</u> C <u>such that for all</u> $t \geq 0$,

$$W_t \leq C \left(\frac{\gamma}{\sigma}\right)^{2+\delta} \left(\frac{\mu}{t}\right)^{\delta/2}, \quad 0 < \delta \leq 1. \tag{2.3}$$

Next, we give a mixing analogue to the above two results. Let $\{X_n\}$ be defined on the probability space $(S, \mathcal{S}, P)$ and let $B \in \mathcal{S}$ be an event such that $P(B) > 0$ and defi

$$W_t(m; B) = \left|P[N(t) \leq m|B] - \Phi\left(\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}}\right)\right|, \tag{2.4}$$

and set $W_t(B) = \sup_m W_t(m; B)$. Then we have,

<u>THEOREM 3</u>. <u>For any</u> $B \in \mathcal{S}$ <u>such that</u> $P(B) > 0$ <u>and all</u> $m \geq 1$,

$$W_t(m; B) \to 0, \quad \text{as} \quad t \to \infty. \tag{2.5}$$

When discussing the rate in (2.5), it has been observed by Landers and Rogge (1977) that in the mixing central limit theorems for partial sums one can not find a positive constant (independent of B) such that whenever $\gamma < \infty$, $\sup_x |P[S_m \leq x|B] - \Phi(\frac{m\mu - x}{\sigma\sqrt{m}})| \leq C \left(\frac{\gamma}{\sigma}\right)^{2+\delta} n^{-\delta/2}$, $0 < \delta \leq 1$ for any $B \in \mathcal{S}$, with $P(B) > 0$. The same is true for $N(t)$, since its rate depends on that of S_m, $m \geq 1$. In fact Landers and Rogge provided a counter example demonstrating this. It is possible, however, to establish an interesting rate of convergence in the case when $W_t(B)$ is such that $B \in \mathcal{S}_k = \sigma(X_1, \ldots, X_k)$ - the σ-field generated by the first k (k = 1, 2, ...) observations $X_1, X_2, \ldots, X_k$, where k is a fixed value. This is done in the next theorem.

<u>THEOREM 4</u>. <u>If</u> $\gamma = E^{1/(2+\delta)}|X_1 - \mu|^{2+\delta} < \infty$, $0 < \delta \leq 1$, <u>then for each</u> $2 \leq r \leq 2 + \delta$ <u>there is a positive constant</u> $C_r = C(r, \mu, \sigma, \gamma)$ <u>such that</u> <u>for all</u> $B \in \mathcal{S}_k$ <u>with</u> $P(B) > 0$,

(2.6) $$W_t(B) \le C_r\ [P(B)]^{1/r}\ [k(\tfrac{\mu}{t})]^{\delta/2}\ , \quad 0 < \delta \le 1.$$

Next, we discuss the dual results concerning the first passage (stopping) time $U(t)$. Let $\bar{W}_t(m)$, $\bar{W}_t$, $\bar{W}_t(m;\ B)$, and $\bar{W}_t(B)$ be defined as $W_t(m)$, W_t, $W_t(m;\ B)$, and $W_t(B)$, respectively with $U(t)$ instead of $N(t)$. Then we have the following dual theorems.

THEOREM 5 [Siegmund (1968)]. For any $m \ge 1$,

(2.7) $$\bar{W}_t(m) \to 0, \text{ as } t \to \infty.$$

THEOREM 6. Let $\gamma = E^{1/(2+\delta)}\ |X_1 - \mu|^{2+\delta} < \infty$, then there exists an absolute positive constant C such that for all $t \ge 0$,

(2.8) $$\bar{W}_t \le C(\tfrac{\gamma}{\sigma})^{2+\delta}\ (\tfrac{\mu}{t})^{\delta/2}, \quad 0 < \delta \le 1$$

THEOREM 7. For any $B \in S$, with $P(B) > 0$, and all $m \ge 1$

(2.9) $$\bar{W}_t(m;\ B) \to 0, \text{ as } t \to \infty.$$

THEOREM 8. If $\gamma = E^{1/(2+\delta)}\ |X_1 - \mu|^{2+\delta} < \infty$, $0 < \delta \le 1$, then for any $2 \le r \le 2 + \delta$, there is a positive constant $C_r = C(r,\ \mu,\ \sigma,\ \gamma)$ such that for any $B \in S_k$ with $P(B) > 0$,

(2.10) $$\bar{W}_t(B) \le C_r\ [P(B)]^{1/r}\ [k(\tfrac{\mu}{t})]^{\delta/2}, \quad 0 < \delta \le 1.$$

3. Proofs of the positive drift case.

Proof of Theorem 3. Let $S_\infty = \sigma(X_1,\ \ldots,\ X_n,\ \ldots)$ denote the σ-field generated by the entire sequence $\{X_n\}$. Then there are S_n-measurable r.v.'s

$\{\zeta_n\}$ $0 \leq \zeta_n \leq$ such that

$$E|P(B|\ S_\infty) - \zeta_n| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.1}$$

Thus,

$$\begin{aligned} W_t(B;\ m)P(B) &= |P[N(t) \leq m,\ B] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})\ P(B)| \\ &\leq |P[N(t) \leq m,\ B] - P[N(t) \leq m,\ \zeta_k]| + |P[N(t) \leq m,\ \zeta_k] - \\ &\quad \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}}) E\zeta_k| + \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}}) |E\zeta_k - P(B)| \\ &\leq E|P(B|S_\infty) - \zeta_k| + E\{|\zeta_k|\,|P[N(t) \leq m|S_k] - \Phi(\frac{m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\} + \\ &\quad E|P(B|S_\infty) - \zeta_k| \\ &\leq \frac{2\epsilon}{3} + E^{\frac{1}{2}}\{|P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|^2\}, \end{aligned} \tag{3.2}$$

by choosing k large enough and using (3.1). Thus we only need to show that, as $t \to \infty$, the last terms in the above upper bound can be made arbitrarily small (less than $\epsilon/3$ say). Let $\theta \in (0, 1)$ be a real number and consider first the case $|\frac{m\mu}{t} - 1| \leq \theta$. Thus we have $\frac{t}{\mu}(1-\theta) \leq m < \frac{t}{\mu}(1+\theta)$, and as $t \to \infty$, then $m \to \infty$. Now, assuming that all the C's in what follows are not necessarily the same positive constants, look at

$$\begin{aligned} &|P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\cdot \\ &\leq |P[S_m \geq t|S_k] - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})| + |\Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}}) - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})| \\ &\leq |P[T_{k,m} \geq t - S_k] - \Phi(\frac{(m-k)\mu-t-S_k}{\sigma\sqrt{m-k}})| + |\Phi(\frac{(m-k)\mu-t-(S_k-k\mu)}{\sigma\sqrt{m-k}}) \\ &\quad - \Phi(\frac{(m-k)\mu-t}{\sigma\sqrt{m-k}})| + |\Phi(\frac{m\mu-t}{\sigma\sqrt{m-k}}) - \Phi(\frac{(m-k)\mu-t}{\sigma\sqrt{m-k}})| \\ &\quad + |\Phi(\frac{m\mu-t}{\sigma\sqrt{m-k}}) - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})| + |\Phi(\frac{m\mu-t}{\sigma\sqrt{m}}) - \Phi(\frac{(mt-\mu)\sqrt{\mu}}{\sigma\sqrt{t}})| \end{aligned} \tag{3.3}$$

$$\leq \Delta_{m-k} + C\,\frac{|S_k - k\mu|}{\sqrt{m-k}} + C\,\frac{k}{\sqrt{m-k}} + C\left|\sqrt{\frac{m}{\sqrt{m-k}}} - 1\right| + C\left|\sqrt{\frac{\mu m}{t}} - 1\right|$$

where $T_{m,k} = \sum_{i=k+1}^{m} X_i$, and in the last upper bounds we used the facts that $|\Phi(x+\varepsilon) - \Phi(x)| \leq C\,|\varepsilon|$ and $|\Phi(\varepsilon x) - \Phi(x)| \leq C\,|\varepsilon - 1|$. Now since $\left|\sqrt{\frac{m}{m-k}} - 1\right| \leq \frac{k}{m-k} \leq \sqrt{\frac{k}{m-k}}$, $\frac{1}{\sqrt{k}}\{E(S_k - k\mu)^2\}^{\frac{1}{2}} < \infty$, and using Minkowski's inequality we have that

$$(3.4) \qquad E^{\frac{1}{2}}\{|P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^2$$

$$\leq \Delta_{m-k} + C\sqrt{\frac{k}{m-k}} + C\sqrt{\frac{k}{m-k}} + C < \frac{k}{\sqrt{m-k}} + C\theta$$

$$\leq \Delta_{m-k} + C\sqrt{\frac{k}{m}} + C\,\frac{k}{\sqrt{m}} + C\theta.$$

Now, since as $m \to \infty$ (which is guaranteed in this case by $t \to \infty$) $\Delta_{m-k} \leq \frac{\varepsilon}{12}$, $C\sqrt{\frac{k}{m}} \leq \frac{\varepsilon}{12}$, and $C\,\frac{k}{\sqrt{m}} < \varepsilon/12$, and by choosing θ small enough, $\theta\,C < \varepsilon/12$, then we have for sufficiently large t (or m),

$$(3.5) \qquad E^{\frac{1}{2}}\{|P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^2 \leq \varepsilon/3.$$

From (3.2) and (3.5) we arrive at

$$(3.6) \qquad W_t(B, m)\,P(B) \leq \varepsilon, \quad \text{for } t \text{ sufficiently large.}$$

Hence the theorem is proved when $|\frac{m\mu}{t} - 1| \leq \theta$, $0 < \theta < 1$. Next let $|\frac{m\mu}{t} - 1| > \theta$ and first consider $\frac{m\mu}{t} - 1 > \theta$. Thus

$$(3.7) \qquad |P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}})|.$$

$$= |P[N(t) \geq m|S_k] - \Phi(-\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}})|$$

$$\leq P[N(t) \geq m|S_k] + \Phi(-\frac{(m\mu - t)\sqrt{\mu}}{\sigma\sqrt{t}})$$

$$\leq P[N(t) \geq m|S_k] + C\frac{\sigma^2 t}{\mu(m\mu-t)^2} \leq P[N(t) \geq m|S_k] + C\frac{\sigma^2}{\mu t},$$

where the last inequality follows since $\Phi(-x) \leq x^{-2}$ for all $x \geq 0$. But also

$$(3.8) \qquad P[N(t) \geq m|S_k] = P[S_m \leq t|S_k] = P[T_{m,k} \leq t - S_k]$$

$$= \int_{-\infty}^{t} P[T_{m,k} \leq t-u]\, dP[S_k \leq u] \leq P[T_{m,k} \leq t]\, P[S_k \leq t]$$

$$\leq P[S_{m-k} \leq t] = P[N(t) \geq m-k]$$

$$\leq \frac{\sigma^2 t}{\mu[(m-k)\mu-t]^2} \leq \frac{\sigma^2}{\mu t}(\theta - \frac{k\mu}{t})^2,$$

where in the third inequality we used Tchebychev's inequality, and in the final one we used the fact that $\frac{m\mu}{t} - 1 > \theta$. Hence the theorem is proved in this case also. Finally let $\frac{m\mu}{t} - 1 < -\theta$. Then

$$(3.9) \qquad |P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|$$

$$\leq P[S_m \geq t|S_k] + \Phi(-\frac{\theta\sqrt{\mu t}}{\sigma}) \leq P[S_m \geq t|S_k] + \frac{\sigma^2}{\theta^2\mu t}$$

$$\leq \frac{E[(S_m-m\mu)^2|S_k]}{(t-m\mu)^2} + \frac{\sigma^2}{\theta^2\mu t} \leq \frac{(S_k-k\mu)^2 + (m-k)\sigma^2}{\theta^2 t^2} + \frac{\sigma^2}{\theta^2\mu t}$$

$$\leq \frac{(S_k-k\mu)^2 + (m-k)\sigma^2}{\theta^2 t^2} + \frac{\sigma^2}{\theta^2\mu t} \leq \frac{(S_k-k\mu)^2}{\theta^2 t^2} + \frac{\sigma^2(1-\theta)}{\theta^2\mu t} + \frac{\sigma^2}{\sigma\mu t}.$$

Thus using Minkowski's inequality we get that in this case

$$(3.10) \qquad E^{\frac{1}{2}}\{|P[N(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^2 \leq \frac{k}{\theta^2 t^2}[\frac{1}{k}E(S_k-k\mu)^2] + \frac{\sigma^2(1-\theta)}{\theta^2\mu t} + \frac{\sigma^2}{\theta\mu t},$$

which approaches 0 as $t \to \infty$. Thus theorem 1 is completely proved. QED.

Proof of Theorem 4. Using Hölder's inequality with $2 \le r \le 2 + \delta$, $0 < \delta \le 1$ we have

$$(3.11) \quad W_t(B)P(B) \le [P(B)]^{(1-1/r)} E^{1/r} \{\sup_m |P[N(t) \le m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^r .$$

First, assume that $|\frac{m\mu}{t} - 1| \le \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2} t} (\frac{t}{\mu})^{(2+\delta)/4}$. Thus using (3.3), Theorem 6 p. 115 of Petrov (1975) and Minkowski's inequality, we get that

$$(3.12) \quad E^{1/r} \{\sup_m |P[N(t) \le m|S_k] - \Phi(\frac{(M\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^r$$

$$\le C(\frac{\gamma}{\sigma})^{2+\delta}(m-k)^{-\delta/2} + C(\frac{1}{k})^{r/2} E^{1/r}|S_k - k\mu| /\sigma\sqrt{m-k} + C \frac{k}{\sqrt{m-k}} + \frac{k}{m-k}$$

$$+ \sup_m |\Phi(\frac{m\mu-t}{\sigma\sqrt{m}}) - (\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})| .$$

But as in Englund (1980) for m, t such that $|\frac{m\mu}{t} - 1| \le \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2} t} (\frac{t}{\mu})^{(2+\delta)/4}$ we have $1 - \theta \le \frac{m\mu}{t} \le 1 + \theta$ for some $0 < \theta < 1$ and also that (see his Lemma 2.3) $\sup_m |\Phi(\frac{m\mu-t}{\sigma\sqrt{m}}) - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})| \le C\, (\frac{\gamma}{\sigma})^{2+\delta} (\frac{t}{\mu})^{-\delta/2}$. Now, if we choose, without loss of generality, $k \le m/2$, and use the above remark and apply the above results we find that the right-hand side of (3.12) is less than or equal to $C(\mu, \sigma, \gamma)\ [k(\frac{\mu}{t})^{\delta/2}]$, $0 < \delta \le 1$. Next, assume that $|\frac{m\mu}{t} - 1| \ge \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2} t} (\frac{t}{\mu})^{(2+\delta)/4}$. Consider first $\frac{m\mu}{t} - 1 \ge \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2} t} (\frac{t}{\mu})^{(2+\delta)/4}$. Then using argument similar to those in (3.7) and (3.8) we have

$$(3.13) \quad |P[N(t) \le m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|$$

$$\le P[N(t) \ge m - k] + \Phi(-\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})$$

$$\le \frac{\sigma^2 t}{\mu[(m-k)\mu-t]^2} + \frac{\sigma^2 t}{\mu(m\mu-t)} = \frac{\sigma^2}{\mu t[\frac{(m-k)\mu}{t}-1]^2} + \frac{\sigma^2}{\mu t(\frac{m\mu}{t}-1)^2}$$

$$\le C\,(\tfrac{\gamma}{\sigma})^{2+\delta}\,(\tfrac{t}{\mu})^{-\delta/2}\,[(1-\mu k/(\frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}}\,(\tfrac{t}{\mu})^{(2+\delta)/4})^{-2}+1] \le C(\tfrac{\gamma}{\sigma})^{2+\delta}(\tfrac{t}{\mu})^{-\delta/2},$$

since as in Englund (1980), $(\frac{t}{\mu}) \ge (C(\frac{\gamma}{\sigma})^{2+\delta})^{1/\delta}$. Finally, assume that $\frac{m\mu}{t} - 1 \le -\frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}t}(\frac{t}{\mu})^{(2+\delta)/4}$. Then the left hand side in (3.13) is less than or equal to

$$(3.14)\quad P[S_m \ge t|S_k] + \phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})$$

$$\le \frac{(S_k-k\mu)^2}{(t-m\mu)^2} + \frac{(m-k)\sigma^2}{(t-m\mu)^2} + \phi(-\frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}}\,(\tfrac{t}{\mu})^{(2+\delta)/4})$$

$$\le (S_k-k\mu)^2\,(\tfrac{\gamma}{\sigma})^{2+\delta}/\sigma^2(\tfrac{t}{\mu})^{(2+\delta)/2} + C(\tfrac{\gamma}{\sigma})^{2+\delta}/(\tfrac{t}{\mu})^{\delta/2} + C(\tfrac{\gamma}{\sigma})^{2+\delta}/(\tfrac{t}{\mu})^{\delta/2},$$

since for this choice $\frac{m\mu}{t} \le 1 - \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}}(\frac{t}{\mu})^{(2+\delta)/4} \le 1 - \theta$, a constant less than 1 (cf. Englund (1980)). Hence whenever $|\frac{m\mu}{t} - 1| \ge \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}t}(\frac{t}{\mu})^{(2+\delta)/4}$ we have, in view of Minkowski's inequality and the fact that $\frac{1}{\sqrt{k}} E^{1/r}|S_k-k\mu|^r < \infty$,

$$(3.15)\quad E^{1/r}\{\sup_m |P[N(t) \le m|S_k] - \phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^r \le C(\mu,\sigma,\gamma)\quad [k(\tfrac{\mu}{t})^{\delta/2}].\ \text{QED.}$$

<u>Proof of Theorem 6</u>. Proceeds exactly as in Englund (1980), using $\bar{S}_n$ in all places of S_n, and using Theorem 1 of Ahmad (1979) (after easy refining to obtain the upper bound explicitly) in place of the Berry-Esseen inequality in his (2.22), while we use Kolmogorov's inequality instead of Tchebychev's inequality in his (2.13). The rest of the proof remains unaffected. QED.

<u>Proof of Theorem 7</u>. We shall only mention the alterations in the proof of Theorem 3. In (3.2) we get for any $\varepsilon > 0$,

$$(3.16)\quad \bar{W}_t(B\ ;\ m)\quad P(B) \leq \frac{2\varepsilon}{3} + E^{\frac{1}{2}}\{|P[U(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|^2\}.$$

Also in place of (3.3) we get for all $|\frac{m\mu}{t} - 1| \leq \theta$ for some $\theta \in (0, 1)$.

$$(3.17)\quad |P[U(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|$$

$$\leq |P[\bar{S}_m \geq t|S_k] - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})| + C\ \frac{|S_k-k\mu|}{\sigma\sqrt{m-k}} + C\ \frac{k}{\sqrt{m-k}} + C\ \frac{k}{m-k}$$

$$+ C\ |\frac{m\mu}{t} - 1|$$

$$\leq |P[\bar{S}_m \geq t|S_k] - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})| + \frac{\varepsilon}{6},$$

again by choosing θ small enough that $C\,\theta < \varepsilon/18$.
But it follows from Theorem 3.1 of Ahmad (1981b), that for $k \leq m/2$

$$(3.18)\quad P[\bar{S}_m \geq t\,|S_k] - \Phi(\frac{m\mu-t}{\sigma\sqrt{m}})|$$

$$\leq \sup_t|\ P[\bar{S}_{m-k} \leq t] - \Phi(\frac{t-(m-k)\mu}{\sigma\sqrt{m-k}})| + C\ \frac{|S_k-k\mu|}{\sqrt{m-k}} + C\ \frac{k}{m-k}$$

$$+ P[\bar{S}_k > t\sigma\sqrt{m} + m\mu|\ S_k]\ \Phi(t)$$

$$\leq \bar{\Delta}_{m-k} + C\ \frac{|S_k-k\mu|}{\sqrt{m}} + C\ \frac{k}{m} + P[\bar{S}_k \geq t\ \sigma\sqrt{m} + m\mu|S_k]\ \Phi(t).$$

where $\bar{\Delta}_n = \sup_x|\ P[\bar{S}_n \leq x] - \Phi(\frac{x-n\mu}{\sigma\sqrt{n}})|$. Thus using Minkowski's inequality we find cf., (Corollary 3.2 of Ahmad (1981b)) that the second term in the right-hand-side of (3.16) is less than or equal to:

$$(3.19)\quad \bar{\Delta}_{m-k} + C\sqrt{\frac{k}{m}} + C\ \frac{k}{m} + \{P[\bar{S}_k \geq \sigma t\ \sqrt{m} + m\mu]\}^{\frac{1}{2}}\ \Phi(t).$$

But since $t \geq 0$ and from Kolmogorov's inequality, $P[\bar{S}_k \geq \sigma t\sqrt{m} + m\mu] \leq P[\bar{S}_k \geq m\mu] \leq C(\frac{k}{m})$. Thus since in this case $t \to \infty \Rightarrow m \to \infty$ the conclusion follows from the well-known fact that $\bar{\Delta}_{m-k} \to 0$ as $m \to \infty$. The two cases $|\frac{m\mu}{t} - 1| \geq \theta$ proceed as in Theorem 3 except that Kolmogorov's inequality for $\bar{S}_{m-k}$ is used in place of Tchebychev's inequality for S_{m-k}. Details are omitted. QED.

Proof of Theorem 8. Again we split the region into $|\frac{m\mu}{t} - 1| \leq \frac{\sigma^{2+\delta/2}}{\gamma^{1+\delta/2}t}(\frac{t}{\mu})^{(2+\delta)/4}$ and its complement. Using Theorem 3.1 of Ahmad (1981b) we get here that

$$(3.20)\quad E^{1/r}\{\sup_m |[P\, U(t) \leq m|S_k] - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|\}^r$$

$$\leq \bar{\Delta}_{m-k} + C\sqrt{\frac{k}{m-k}} + C\frac{k}{\sqrt{m-k}} + C\frac{k}{m-k} + \sup_m |\{P[S_k \geq \sigma\, t\, \sqrt{m} + m\mu].$$

$$\Phi(t)\} + \sup_m |\Phi(\frac{m\mu-t}{\sigma\sqrt{m}}) - \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}})|$$

$$\leq \bar{\Delta}_{m-k} + C\sqrt{\frac{k}{m-k}} + C\frac{k}{\sqrt{m-k}} + C\frac{k}{m-k} + C\frac{k}{m} + \sup_m |\Phi(\frac{m\mu-t}{\sigma\sqrt{m}})$$

$$- \Phi(\frac{(m\mu-t)\sqrt{\mu}}{\sigma\sqrt{t}}) .$$

Thus we obtain the same rate as in (3.12) upon using Theorem 1 of Ahmad (1979) regarding $\bar{\Delta}_{m-k}$. The alterations for the complement set simply involve using Kolmogorov's inequality in place of Tchebychev's inequality. The procedure is analogous. QED.

4. The zero drift case.

Again we start by dealing with the renewal variable $N(t)$. Define

(4.1) $$V_t(m) = |P[N(t) \le m] - (1 - \Phi(t/\sigma\sqrt{m}))|, \quad t \ge 0,$$

and set $V_t = \sup_m V_t(m)$.

THEOREM 9 [Gut (1974)]. For any $m \ge 1$,

(4.2) $$V_t(m) \to 0, \quad \text{as} \quad t \to \infty.$$

Next, we discuss the rate of convergence thus complementing Theorem 2 of Englund (1980).

THEOREM 10. If $\gamma = E^{1/(2+\delta)}|X_1|^{2+\delta} < \infty$, $0 < \delta \le 1$, then there is a positive constant C such that for all $t > 0$,

(4.3) $$V_t \le C\left(\frac{\gamma}{\sigma}\right)^{2+\delta}\left(\frac{\mu}{t}\right)^{\delta}, \quad 0 < \delta \le 1.$$

Next, we discuss the mixing central limit theorem for V_t. Theorems 11 and 12 below are the zero drift analogues of Theorem 3 and 4 of Section 2. We define $V_t(m; B)$ and $V_t(m)$ analogously.

THEOREM 11. For any $B \in S$ with $P(B) > 0$, and all $m \ge 1$

(4.4) $$V_t(m; B) \to 0, \quad \text{as} \quad t \to \infty.$$

THEOREM 12. If $\gamma = E^{1/(2+\delta)}|X_1|^{2+\delta} < \infty$, $0 < \delta \le 1$, then for each $2 \le r \le 2 + \delta$ there is a positive constant $C_r = C_r(r, \sigma, \gamma)$ such that for all $B \in S_k$ with $P(B) > 0$,

$$V_t(B) \le C_r\ [P(B)]^{1/\gamma}[k(\tfrac{\sigma}{t})]^{\delta},\quad 0 < \delta \le 1,\quad t > 0.$$

Next, we deal with the stopping time $U(t)$. Define

(4.5) $\bar{V}_t(t) = |P[U(t) \le m] - 2(1 - \Phi(t/\sigma\sqrt{m}))|,$

and set $\bar{V}_t = \sup_m \bar{V}_t(m)$. Then we have,

<u>THEOREM 13</u> [Teicher (1973)]. <u>For any</u> $m \ge 1$,

(4.6) $\bar{V}_t(m) \to 0,$ <u>as</u> $t \to \infty$.

<u>THEOREM 14</u>. <u>If</u> $\gamma = E^{1/(2+\delta)}|X_1|^{2+\delta} \le 1$, $0 < \delta \le 1$, <u>then there is a positive constant</u> C <u>such that for all</u> $t \ge 0$,

(4.7) $\bar{V}_t \le C(\frac{\gamma}{\sigma})^{2+\delta}\ (\frac{\sigma}{t})^{\delta/2},\ 0 < \delta \le 1,\quad t > 0.$

Next, we deal with the mixing versions of theorems 13 and 14. Define

(4.8) $\bar{V}_t(m;\ B) = |P[U(t) \le m\ |B]\ - 2(1 - \Phi(t/\sigma\sqrt{m}))|,$

and set $\bar{V}_t(B) = \sup_m \bar{V}_t(m;\ B)$. Then we can state,

<u>THEOREM 15</u>. <u>If</u> $B \in S$ <u>with</u> $P(B) > 0$, <u>then for any</u> $m \ge 1$,

(4.9) $\bar{V}_t(m;\ B) \to 0,$ <u>as</u> $t \to \infty$.

<u>THEOREM 16</u>. <u>If</u> $B \in S_k$ <u>with</u> $P(B) > 0$ <u>and if</u> $\gamma = E^{1/(2+\delta)}|X_1|^{2+\delta} < \infty$, $0 < \delta \le 1$, <u>then for each</u> $2 \le r \le 2 + \delta$ <u>there is a positive constant</u> $C_r = C(r, \sigma, \gamma)$ <u>such that</u>

$$\bar{V}_t(B) \le C_r[P(B)]^{1/r}[k(\tfrac{\sigma}{t})]^{\delta},\quad 0 < \delta \le 1,\quad t > 0.$$

5. Proofs of the zero drift case.

Proof of Theorem 10. Since $P[N(t) \le m] = P[S_m \ge t]$ we have when $\frac{\sigma\sqrt{m}}{t} \le 1$ and by using Lemma 1 of Ahmad (1979) with $g(x) = x^\delta$, $0 < \delta \le 1$,

$$(5.1) \quad V_t(m) = |P[S_m \le t] - \Phi(t/\sigma\sqrt{m})|$$

$$\le C(\tfrac{\gamma}{\sigma})^{2+\delta}\, m^{-\delta/2}\, (\tfrac{\sigma\sqrt{m}}{t})^{2+\delta} \le C(\tfrac{\gamma}{\sigma})^{2+\delta}\, (\tfrac{\sigma}{t})^{\delta}, \quad 0 < \delta \le 1.$$

Next, assume that $\frac{\sigma\sqrt{m}}{t} > 1$. Then by Theorem 6 of Petrov p. 115 (1975) we have for any $m \ge 1$

$$(5.2) \quad V_t(m) \le C(\tfrac{\gamma}{\sigma})^{2+\delta}\, m^{-\delta/2} \le C(\tfrac{\gamma}{\sigma})^{2+\delta}\, (\tfrac{\sigma}{t})^{\delta}, \quad 0 < \delta \le 1. \quad \text{QED}$$

Proof of Theorem 11. Proceeding as in Theorem 3 above it suffices to show that for any $m \ge 1$,

$$(5.3) \quad E^{\frac{1}{2}}\{|P[N(t) \le m|S_k] - (1 - \Phi(t/\sigma\sqrt{m}))|^2\} \to 0 \quad \text{as} \quad t \to \infty.$$

Assume first that $\frac{\sigma\sqrt{m}}{t} > 1$. Note that in this case as $t \to \infty$, then $m \to \infty$ and (5.3) is equivalent to

$$(5.4) \quad E^{\frac{1}{2}}\{|P[S_m \le t|S_k] - \Phi(t/\sigma\sqrt{m})|^2\}$$

But for all t, using Minkowski's inequality,

$$(5.5)\ E^{\frac{1}{2}}\{|P[S_m \le t|S_k] - \Phi(t/\sigma\sqrt{m})|^2\} \le \Delta_{m-k} + \frac{E^{\frac{1}{2}}|S_k|^2}{\sigma\sqrt{m-k}} + C\,\frac{k}{\sqrt{m-k}},$$

which converges to 0 as $m \to \infty$ (which happens if $t \to \infty$ in this case). Next, let $\frac{\sigma\sqrt{m}}{t} \le 1$, then

$$(5.6) \quad |P[S_m \le t|S_k] - \Phi(t/\sigma\sqrt{m})| \le P[S_m \ge t|S_k] + \Phi(-\tfrac{t}{\sigma\sqrt{m}})$$

$$\le \frac{E(S_m^2|S_k)}{t^2} + \frac{\sigma^2 m}{t^2} = \frac{S_k^2 + (m-k)\sigma^2 + m\sigma^2}{t^2}$$

$$= \frac{S_k^2 + 2m\sigma^2 - k\sigma^2}{t^2},$$

which, upon using Minkowski's inequality, converges to 0 as $t \to \infty$. QED.

Proof of Theorem 12. Again using Hölder's inequality with $2 \le r \le 2 + \delta$; $0 < \delta \le 1$ we have

$$(5.7) \quad V_t(B)P(B) \le [P(B)]^{1-1/r} E^{1/r} \{\sup_m |P[N(t) \le m|S_k] - \Phi(t/\sigma\sqrt{m})|\}^r.$$

Proceeding as in Theorem 4 above with the breaking used in Theorem 11 but using the bound $\Delta_{m-k} \le C/(m-k)^{\delta/2}$ in the right hand side of (5.5) we have the bound $C_r(P(B))^{1/r} \; [k(\frac{\sigma}{t})^\delta]$ since $\frac{\sigma\sqrt{m}}{t} > 1$. The other case follows by taking the expectation on the right hand side of (5.6). QED.

Proof of Theorem 14. Since $P[U(t) \le m] = P[\bar{S}_m \ge t]$ we easily see that $V_t(m) = |P[\bar{S}_m < t] - G(\frac{t}{\sigma\sqrt{m}})|$, where $G(x) = 2\phi(x) - 1$, $x > 0$. Thus we have for $\frac{\sigma\sqrt{m}}{t} \le 1$ and using Proposition 1 of the appendix with $g(y) = y^\delta$, $0 < \delta \le 1$ that

$$(5.8) \quad \bar{V}_t(m) \le C(\tfrac{\gamma}{\sigma})^{2+\delta} m^{-\delta/2} (\tfrac{\sigma\sqrt{m}}{t})^{2+\delta} \le C(\tfrac{\gamma}{\sigma})^{2+\delta} (\tfrac{\sigma}{t})^\delta, \quad 0 < \delta \le 1.$$

Again if $\frac{\sigma\sqrt{m}}{t} > 1$, then using the same proposition we arrive at

$$(5.9) \quad \bar{V}_t(m) \le C(\tfrac{\gamma}{\sigma})^{2+\delta} m^{-\delta/2} \le C(\tfrac{\gamma}{\sigma}) (\tfrac{\sigma}{t})^\delta, \quad 0 < \delta \le 1. \text{ QED.}$$

Proof of Theorem 15. Proceeding as in Theorem 7 above we need only to show that for any $m \ge 1$, (5.10) $E^{\frac{1}{2}}\{|P[U(t) \le m|S_k] - 2(1 - \Phi(t/\sigma\sqrt{m}))|\}^2 \to 0$ as $t \to \infty$.

But for all t,

(5.11) $$|P[U(t) \le m|S_k] - 2\ (1 - \Phi(t/\sigma\sqrt{m}))|$$

$$\le \bar{\Delta}_{m-k} + C\frac{|S_k|}{\sigma\sqrt{m-k}} + C\frac{k}{\sqrt{m-k}} + P[\bar{S}_k \ge t\ \sigma\sqrt{m}|\ _k]\ \Phi(t),$$

which can be used together with Minkowski's inequality and the argument of Theorem 7, to show that (5.10) converges to 0 as $t \to \infty$. QED.

Proof of Theorem 16. Again we proceed as in Theorem 8 but here we use the bound given in proposition 1 of the appendix. We shall not dwell on the details. QED.

6. Appendix.

In this section we shall give a nonuniform bound on the rate of convergence of

(6.1) $$\bar{\Delta}_n(x) = |P[\bar{S}_n \le x\sigma\sqrt{n}] - (2\Phi(x)-1)|$$

Let

(6.2) $$\theta_n(x) = \int_{|w| < \sigma\sqrt{n}(1+|x|)} |w|^3 dF(w) + \sigma\sqrt{n}(1+|x|)\int_{|w|>\sigma\sqrt{n}(1+|x|)} w^2 dF(w)$$

The method of proof of the following proposition relies on Lemma 1 of Ahmad (1979) and a theorem of Ahmad and Lin (1977).

Proposition 1. For all $x \ge 0$, and all $n \ge 1$, (6.3) $\bar{\Delta}_n(x) \leq C\theta_n(x) \,/\, \sigma^2\sqrt{n}\,(1 + |x|^3)$.

<u>If we let</u> G <u>denote the class of all functions</u> g <u>such that:</u> g <u>is non-negative,</u> $g(x)$ <u>and</u> $x/g(x)$ <u>tend to</u> ∞ <u>as</u> $x \to \infty$, <u>then for all</u> $x \geq 0$, $n \geq 1$, <u>and a positive constant</u> C.

$$(6.4)\qquad \bar{\Delta}_n(x) \leq C\,\frac{EX_1^2 g(|X_1|)}{\sigma^3(1+x^2)\;g(\sigma\sqrt{n}(1+|x|))}\,.$$

<u>Proof</u>. For any x, define $X_{n,i} = X_i$ if $|X_i| < \sigma\sqrt{n}(1+|x|)$ and $X_{n,i} = 0$ otherwise. Let $\mu_n = EX_{n,i}$, $\sigma_n^2 = \mathrm{Var}\,X_{n,i}$, $X^*_{n,i} = X_{n,i} - \mu_n$ and set $\bar{\xi}_n = \max(X_{n,1}, X_{n,1} + X_{n,2}, \ldots, X_{n,1}+X_{n,2}+\ldots+X_{n,n})$ and $\bar{\eta}_n = \max(X^*_{n,1}, X^*_{n,1}+X^*_{n,2}, \ldots, X^*_{n,1} + \ldots + X^*_{n,n})$. Assume first that $\theta_n(x) \leq \sigma^2\sqrt{n}/4$ in which case one can see easily that

$$(6.5)\qquad 0 \leq \sigma^2 - \sigma_n^2 \leq 2\,\frac{\theta_n(x)}{\sigma\sqrt{n}(1+|x|)} \leq \sigma^2/2.$$

Note on the other hand that

$$(6.6)\qquad \bar{\Delta}_n(x) \leq |P[\bar{S}_n \leq \sigma x\sqrt{n}] - P[\bar{\xi}_n \leq \sigma x\sqrt{n}]| + |P[\bar{\xi}_n \leq \sigma x\sqrt{n}] - P[\bar{\xi}_n \leq \sigma_n x\sqrt{n}]|$$

$$+ |P[\bar{\xi}_n \leq \sigma_n x\sqrt{n}] - P[\bar{\eta}_n \leq \sigma_n x\sqrt{n}]| + |P[\bar{\eta}_n \leq \sigma_n x\sqrt{n}] - (2\Phi(x)-1)|$$

$$= \delta_n^1(x) + \delta_n^2(x) + \delta_n^3(x) + \delta_n^4(x), \text{ say.}$$

Now, for any $x \geq 0$ and all $n \geq 1$,

$$(6.7)\qquad \delta_n^1(x) \leq n\,P[|X_1| > \sigma\sqrt{n}\,(1 + |x|)] \leq \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x^3)}\,.$$

Next, by theorem 1 of Arak (1974), since $E|X_{n,1}|^3 \leq 3\,\theta_n(x)$, (6.8) $$\delta_n^4(x) \leq C \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x^3)} .$$

Thus it remains to evaluate $\delta_n^2(x)$ and $\delta_n^3(x)$. Note that

(6.9) $$\delta_n^2(x) \leq P[\bar{\eta}_n \leq \sigma_n\sqrt{n}\ (x \frac{\sigma}{\sigma_n} + \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x)^2})] - P[\bar{\eta}_n \leq \sigma_n \sqrt{n}\ (x - \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x)^2})]$$

$$\leq C \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x^3)} ,$$

where the first inequality of (6.9) follows from (6.8), (6.5) and the following easily verifiable inequality

(6.10) $$P[\bar{\eta}_n \leq x\sigma_n\sqrt{n} - n|\mu_n|] \leq P[\bar{\xi}_n \leq \sigma_n\ x\ \sqrt{n}] \leq P[\bar{\eta}_n \leq x\sigma_n\sqrt{n} + n|\mu_n|] .$$

Finally, assuming that $x > 1$ (the case $x \leq 1$ is easier and simpler to do) we have that $|\mu_n| \leq \theta_n(x) / n\sigma^2(1 + x^2)$ and thus with $[\delta_n^3(x)]^+$ denoting the positive part of $\delta_n^3(x)$,

(6.11) $$[\delta_n^3(x)]^+ = P[\bar{\xi}_n \leq x\ \sigma_n \sqrt{n}] - P[\bar{\eta}_n \leq x\ \sigma_n \sqrt{n}]$$

$$\geq P[\bar{\xi}_n \leq x\ \sigma_n \sqrt{n} - n|\mu_n|] - P[\bar{\eta}_n < x\ \sigma_n \sqrt{n}]$$

$$\geq 2[\Phi(\ x - \frac{2\theta_n(x)}{\sigma^3\sqrt{n}(1+x^3)}) - \Phi(x)] - C \frac{\theta_n(x)}{\sigma_n^3\sqrt{n}(1+x^3)}$$

$$\geq - C \frac{\theta_n(x)}{\sigma_n^3\sqrt{n}(1+x^3)} .$$

Similarly with $[\delta_n^3(x)]^-$ the negative part of $\delta_n^3(x)$, we can show that

(6.12) $$[\delta_n^3(x)]^- = P[\bar{\eta}_n \leq x\ \sigma_n \sqrt{n}] - P[\bar{\xi}_n \leq x\ \sigma_n \sqrt{n}] \leq C \frac{\theta_n(x)}{\sigma\sqrt{n}(1+x^3)} .$$

Hence from (6.11) and (6.12) we have shown that

$$\delta_n^3(x) \leq C \frac{\theta_n(x)}{\sigma^3\sqrt{n}(1+x^3)} .$$

Hence (6.3) is now proved whenever $\theta_n(x) \leq \sigma^3\sqrt{n}/4$. The complement case is trivial since the bound itself is trivial in this case. Thus (6.3) is now proved.

In order to prove (6.4) note that the following is true:

$$\text{(6.13)} \quad \theta_n(x) \leq \frac{E\, X_1^2 g(|X_1|)}{g(\sigma\sqrt{n}(1+|x|))} \; \sigma\sqrt{n}\; (1 + |x|).$$

Hence the proposition is now proved. QED.

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