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WEAK CONVERGENCE OF WEIGHTED EMPIRICAL PROCESSES WITH RANDOM WEIGHTS

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Abstract

The problem of weak convergence of certain weighted empirical processes to a well-defined Gaussian process is studied when the observations are independent but the weights are dependent random variables. Results are obtained assuming two different sets of conditions on the random weights.

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1. Introduction.

Weak convergence of empirical processes has been under extensive investigation in recent years since it has wide applications in nonparametric statistics as well as theoretical interest of its own. One application of the weak convergence of a weighted empirical process is in deriving asymptotic normality of the R, L, and M estimators of the regression parameter Δ in the model $Y_{n,j} = \Delta d_{n,j} + X_{n,j}$, $j = 1, \ldots, n$. The asymptotic normality of the above estimators rely heavily (Cf. Koul (1977)) on proving the weak convergence of the following empirical process

$$W_n(t) = \Sigma_{j=1}^{n} d_{n,j} \{I[X_{n,j} \leq t] - F_{n,j}(t)\}, \quad 0 \leq t \leq 1, \tag{1.1}$$

where $F_{n,j}(\cdot)$ is the distribution function of $X_{n,j}$, and $d_{n,j}$ are constants, $j = 1, \ldots, n$.

Situations where $d_{n,j}$ may be stochastic occur if one is sampling sequentially such that the regressors $d_{n,j}$ are randomly chosen according to $d_{n,\ell}$, $\ell < j \leq n$. In fact, the mechanism through which $d_{n,j}$ are stochastically generated need not be specified and our results below do not require any such thing but rather some mild conditions on $d_{n,j}$'s. In addition, the case of stochastic weights in (1.1) is a natural generalization of the constant regressors case. Interest in parametric regression with stochastic regressors has been recently discussed by Anderson and Taylor (1979) and Christopeit and Helmes (1979). The results reported here should prove instrumental in proving the asymptotic normality of the (nonparametric) R, L, and M estimates of Δ (for definitions of these estimates we refer the reader to Koul (1977), p. 687, p. 687, and p. 688, respectively). The independence between $X_{n,j}$ and $d_{n,j}$ in our results, to follow, is not such a severe assumption since it is fulfilled when discussing the asymptotic normality of the above mentioned estimators with stochastic regressors. Another application of our result would be to establish the asymptotic normality of linear functions of order statistics with stochastic weights (Cf. Shorack (1973)) and also study the asymptotic distributions of linear rank statistics with random score functions (Cf. Puri and Sen (1971)).

All empirical processes considered here have realizations in the space $D[0, 1]$, i.e., the space of all measurable functions on $[0,1]$ with only jump discontinuities.

The main results of this note are weak convergence of the weighted empirical process (1.1) under two sets of regularity conditions on the random weights $d_{n,j}$'s. The method of proof we employ is different from that of Billingsley (1968, p. 141) and is similar to the method of Breiman (1968, p.286), see also Shorack (1973).

It should be noted here that Lemma 2.7 below, which is needed in the proof of Theorem 2.6 below, is due to Komlos (1973) and is proved by a different and more direct way than that of Komlos and is of independent interest.

2. Main Results.

Let $X_{n,1}, \ldots, X_{n,n}$ be n independent random variables with d.f.'s $F_{n,1}, \ldots, F_{n,n}$ respectively, defined on $[0,1]$ such that

$n^{-1}\sum_{j=1}^{n} F_{n,j}(t) = t,\ 0 \le t \le 1$. Let $\{d_{n,1}, \ldots, d_{n,n}\}$ be a triangular array of random variables independent of the $X_{n,j}$'s. Define the w.e.p. by

$$W_n(t) = \sum_{j=1}^{n} d_{n,j}\{I[X_{n,j} \le t] - F_{n,j}(t)\} \tag{2.1}$$

$0 \le t \le 1$. Note that $F_{n,1}, \ldots, F_{n,n}$ need not be identical. In this section, weak convergence results for $W_n = \{W_n(t),\ 0 \le t \le 1\}$ are obtained under two different sets of assumptions on the random weights $d_{n,j}$'s.

THEOREM 2.1. *Assume that the random weights* $d_{n,j}$'s *have finite fourth moments and satisfy the following conditions:*

$$\max_{1\le j\le n} |d_{n,j}| \xrightarrow{P} 0, \quad \text{as } n \to \infty, \tag{2.2}$$

and

$$\lim_n \sup(n^2 \max_{1\le j\le n} E\, d_{n,j}^4) < \infty. \tag{2.3}$$

Assume that for all $0 \le s \le t \le 1$,

$$\sum_{j=1}^{n} d_{n,j}^2 F_{n,j}(s)(1 - F_{n,j}(t)), \tag{2.4}$$

converges in probability to a finite limit $L(s,t)$, *say. Then* W_n *converges weakly to a Gaussian process* W *such that* $P[W \in C[0,1]] = 1$ *and for all* $0 \leq s \leq t \leq 1$,

$$E\,W(t) = 0, \text{ and } \operatorname{cov}[W(s),W(t)] = L(s,t). \tag{2.5}$$

REMARK. Condition (2.3) is satisfied if $d_{n,j} = \frac{1}{\sqrt{n}}$ wp1, $j=1,\ldots,n$, $n \geq 1$.

PROOF. We need to show that (a) $[W_n(t_1), \ldots, W_n(t_m)] \xrightarrow{L} [W(t_1), \ldots, W(t_m)]$ for any finite m and $0 \leq t_1 < \ldots < t_m \leq 1$, as $n \to \infty$, and that (b) W_n is tight or relatively compact (see Billingsley (1968) for details).

(a) *Convergence of the finite dimensional distributions.* The proof hinges on the following lemma due to Dvoretzky (1972).

LEMMA 2.2. *Let* $\{\xi_{n,j}, j=1, \ldots, k_n\}$ *be a triangular array of random variables defined on the probability space* (Ω, F, P). *Let* $\{F_{n,j}, j=1, \ldots, k_n\}$ *be an array of* σ-*fields such that* $X_{n,j}$ *is measurable with respect to* $F_{n,j}$, $j=1, \ldots, k_n$, *and that for all* $n \geq 1$, $F_{n,1} \subseteq \cdots \subseteq F_{n,k_n} \subseteq F$. *Suppose that the following conditions hold:*

(i) $$\sum_{j=1}^{k_n} E[\xi_{n,j} \mid F_{n,(j-1)}] \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2.6}$$

(ii) *For some constant* $0 < \sigma^2 < \infty$

$$\sum_{j=1}^{k_n} \{E(\xi_{n,j}^2 \mid F_{n,(j-1)}) - [E(\xi_{n,j} \mid F_{n,(j-1)})]^2\} \xrightarrow{P} \sigma^2, \quad \text{as } n \to \infty. \tag{2.7}$$

(iii) *For some* $\varepsilon > 0$,

$$\sum_{j=1}^{k_n} E[\xi_{n,j}^2 \, I(|\xi_{n,j}| > \varepsilon) \mid F_{n,(j-1)}] \xrightarrow{P} 0 \quad as \quad n \to \infty. \tag{2.8}$$

Then $S_n = \sum_{j=1}^{k_n} \xi_{n,j}$ *is asymptotically normally distributed with mean zero and variance* σ^2.

PROOF. See Theorem 2.1 of Dvoretzky (1972). ||

In order to apply Lemma 2.2, Let $F_{n,j}$ denote the σ-field generated by $(d_{n,1}, \ldots, d_{n,n}, Z_{n,1}(t), \ldots, Z_{n,j}(t))$, where $Z_{n,j}(t) = I(X_{n,j} \le t) - F_{n,j}(t)$, $j = 1, \ldots, n$ and $0 \le t \le 1$. Then

$$E\,[d_{n,j}\, Z_{n,j}\,(t) \mid F_{n,(j-1)}] = d_{n,j}\, E\,[Z_{n,j}\,(t) \mid F_{n,(j-1)}] = 0. \tag{2.9}$$

Setting $\xi_{n,j} = d_{n,j}\, Z_{n,j}(t)$, $j=1, \ldots, n$, we have

$$\sum_{j=1}^{n} E[\xi_{n,j} \mid F_{n,(j-1)}] = \sum_{j=1}^{n} d_{n,j}\, E\,[Z_{n,j}(t) \mid F_{n,(j-1)}] = 0,$$

verifying (2.6). Next note that

$$\begin{aligned}\sum_{j=1}^{n} E[\xi_{n,j}^2 \mid F_{n,(j-1)}] &= \sum_{j=1}^{n} d_{n,j}^2 E[Z_{n,j}^2(t) \mid F_{n,(j-1)}] \\ &= \{\sum_{j=1}^{n} d_{n,j}^2 F_j(t)[1-F_j(t)]\},\end{aligned}$$

which converges to $L(t,t)$ in probability as $n \to \infty$ by (2.4), verifying (2.7). Finally for any $\varepsilon > 0$

$$\sum_{j=1}^{n} E[\xi_{n,j}^2 I(|\xi_{n,j}| > \epsilon)|F_{n,(j-1)}]$$

$$= \sum_{j=1}^{n} d_{n,j}^2 \; E[Z_{n,j}^2(t) I(|Z_{n,j}(t)||d_{n,j}| > \epsilon)|F_{n,(j-1)}]$$

$$\leq \sum_{j=1}^{n} d_{n,j}^2 E[Z_{n,j}^2(t) I(|d_{n,j}|>\epsilon)|F_{n,(j-1)}]$$

$$= \sum_{j=1}^{n} E(Z_{n,j}^2(t) d_{n,j}^2 \; I(|d_{n,j}|>\epsilon)$$

$$\leq \max_{j} EZ_{n,j}^2(t) \sum_{j=1}^{n} d_{n,j}^2 I(|d_{n,j}|>\epsilon)$$

$$\leq (1/4) \sum_{j=1}^{n} d_{n,j}^2 I(|d_{n,j}|>\epsilon),$$

where the second equality follows since $Z_{n,j}(t)$ is independent of $d_{n,j}$ and $F_{n,(j-1)}$. The last bound converges to 0 in probability as $n \to \infty$, by (2.2) since $\max_{1\leq j\leq n} |d_{n,j}| \xrightarrow{p} 0$ is equivalent to $\sum_{j=1}^{n} d_{n,j}^2 I(|d_{n,j}| \geq \epsilon) \xrightarrow{p} 0$ as $n \to \infty$. Thus (2.8) is verified. Hence in view of Cramer-Wold technique, part (a) is now proved.

(b) *Tightness*. The tightness will be established through the following three lemmas.

LEMMA 2.3. *Assume that* $E\, d_{n,j}^4 < \infty$, $j=1, \ldots , n$, $n \geq 1$. *Then for any* $0 \leq s \leq t \leq 1$,

$$E|W_n(t) - W_n(s)|^4 \leq (n^2 \max_{1\leq j\leq n} Ed_{n,j}^4)[3(t - s)^2 + (t - s)/n]. \tag{2.10}$$

PROOF. Let $\eta_{n,j} = I(s < X_j \le t) - p_{n,j}$ where $p_{n,j} = F_{n,j}(t) - F_{n,j}(s)$, $j=1,\ldots,n$. In what follows we shall write $\eta_j = \eta_{n,j}$, $p_j = p_{n,j}$, and $F_j = F_{n,j}$, $j=1,\ldots,n$. Then

$$n^{-2}E|W_n(t) - W_n(s)|^4 = n^{-2}E|\sum_{j=1}^n d_{n,j}\eta_j|^4$$

$$= n^{-2}[\sum_{j=1}^n Ed_{n,j}^4 E\eta_j^4 + 3\sum\sum_{j\ne k} E(d_{n,j}d_{n,k})^2 E\eta_j^2 E\eta_k^2]$$

$$\le n^{-2}[\sum_{j=1}^n Ed_{n,j}^4\ E\eta_j^4 + 3\sum\sum_{j\ne k}(Ed_{nj}^4 Ed_{n,k}^4)^{\frac{1}{2}} E\eta_j^2 E\eta_k^2]$$

$$= 3[n^{-1}\sum_{j=1}^n (Ed_{n,j}^4)^{\frac{1}{2}} E\eta_j^2]^2 + n^{-1}\sum_{j=1}^n Ed_{n,j}^4 [E\eta_j^4 - 3(E\eta_j^2)]$$

$$\le 3(\max_{1\le j\le n} Ed_{n,j}^4)[n^{-1}\sum_{j=1}^n E\eta_j^2]^2 + n^{-1}\sum_{j=1}^n Ed_{n,j}^4 [E\eta_j^4 - 3(E\eta_j^2)^2]$$

$$\le (\max_{1\le j\le n} Ed_{n,j}^4)\{3[n^{-1}\sum_{j=1}^n p_j(1-p_j)]^2$$

$$+ n^{-2}[\sum_{j=1}^n p_j(1-p_j)(1 - 6p_j + 6p_j^2)]\}$$

$$\le (\max_{1\le j\le n} Ed_{n,j}^4)\{3[n^{-1}\sum_{j=1}^n p_j]^2 + n^{-1}[\sum_{j=1}^n p_j]\}$$

$$= (\max_{1\le j\le n} Ed_{n,j}^4)(3\ (t-s)^2 + n^{-1}(t-s)).\ \|$$

Following Shorack (1973) we consider the partitioning $[\frac{j-1}{n}, \frac{j}{n}]$. $j = 1, \ldots, n$, of $[0,1]$ and define the stochastic process $U_n = \{U_n(t),\ 0 \le t \le 1\}$ by

$$U_n(t) = W_n(\tfrac{j-1}{n}) + [nt - (j-1)][W_n(\tfrac{j}{n}) - W_n(\tfrac{j-1}{n})], \qquad (2.11)$$

for $(j-1)/n \le t \le j/n$, $j = 1, \ldots, n$. Let $\rho(f,g) = \sup_t |f(t) - g(t)|$ be the usual Skorokhod distance function. Then we have the following lemma.

LEMMA 2.4. *Assume that* $\lim_n \sup(n^2 \max_{1\le j\le n} Ed^4_{n,j}) < \infty$. *Then*

$$\rho(U_n, W_n) \xrightarrow{p} 0 \quad \textit{as} \quad n \to \infty. \qquad (2.12)$$

PROOF. Assume that the $d_{n,j}$'s are positive random variables. Then following the argument of Shorack (1973) we have

$$\sup_{0\le t\le 1}|W_n(t) - U_n(t)| \le 2 \max_{1\le j\le n} |W_n(\tfrac{j}{n}) - W_n(\tfrac{j-1}{n})| + 4 \max_{1\le j\le n} d_{n,j}. \qquad (2.13)$$

Since $\max_{1\le j\le n} d_{n,j} \xrightarrow{p} 0$ as $n \to \infty$, it remains to show that the first term in the RHS of (2.13) converges to 0 in probability. But using Tchebychev's inequality and Lemma 2.3, we have for any $\varepsilon > 0$

$$P[\max_{1\le j\le n} |W_n(\tfrac{j}{n}) - W_n(\tfrac{j-1}{n})| > \varepsilon] \le \sum_{j=1}^{n} P[|W_n(\tfrac{j}{n}) - W_n(\tfrac{j-1}{n})| > \varepsilon]$$

$$\le \varepsilon^{-4}\sum_{j=1}^{n} E|W_n(\tfrac{j}{n}) - W_n(\tfrac{j-1}{n})|^4$$

$$\le (4/\varepsilon^4)[(n^2 \max_{1\le j\le n} Ed^4_{n,j})/n], \qquad (2.14)$$

which converges to 0 as $n \to \infty$ (since $\lim_n \sup(n^2 \max_{1 \le j \le n} Ed_{n,j}^4) < \infty$).

For general $d_{n,j}$, write $d_{n,j} = d_{n,j}^+ - d_{n,j}^-$, $j = 1, \ldots, n$, $n \ge 1$.

Then $W_n(t) = W_n^+(t) - W_n^-(t)$, where $W_n^+(t) = \sum_{j=1}^n d_{n,j}^+\{I(X_j \le t) - F_j(t)\}$,

and $W_n^-(t) = \sum_{j=1}^n d_{n,j}^-\{I(X_j \le t) - F_j(t)\}$. Thus we have shown that

$\rho(W_n^+, U_n^+) \xrightarrow{P} 0$, as $n \to \infty$. Similarly we can show that $\rho(W_n^-, U_n^-) \xrightarrow{P} 0$ as $n \to \infty$, where U_n^+ and U_n^- are as given in (2.11) with W_n replaced by W_n^+ and W_n^-, respectively. ||

<u>LEMMA 2.5.</u> *Let* $\{U_n = U_n(t),\ 0 \le t \le 1\}$ *be as defined in* (2.11). *Assume that* $Ed_{n,j}^4 < \infty$, $j = 1, \ldots, n$, $n \ge 1$. *Then, for any* $0 \le s \le t \le 1$, *and some positive constant* K,

$$E|U_n(t) - U_n(s)|^4 = K(n^2 \max_{1 \le j \le n} Ed_{n,j}^4)(t-s)^2. \tag{2.15}$$

<u>PROOF.</u> Shorack (1973, p.149) gives a proof of this lemma assuming that $d_{n,j} = \frac{1}{\sqrt{n}}$, wp1, $j = 1, \ldots, n, n \ge 1$. In view of Lemma 2.3, with obvious modifications on Shorack's proof, (2.15) follows immediately. ||

Note that (2.15) is a sufficient condition for the tightness of U_n. (See Billingsley (1968, p.95)). The tightness of W_n is now established by the facts that U_n is tight and $\rho(U_n, W_n) \xrightarrow{P} 0$ as $n \to \infty$. This completes the proof of part (b) and the theorem. ||

Note that Condition (2.2) is a consequence of Condition (2.3) but is stated explicitly since it is the only condition needed to prove the convergence of the finite dimensional distributions while (2.3) is needed to prove the tightness.

In the next theorem we shall establish the weak convergence of W_n under a different set of conditions imposed on the behavior of the random weights $d_{n,j}$'s.

THEOREM 2.6. *Assume that the random weights* $d_{n,j}$'s *have finite fourth moments, that for all* $0 \le s \le t \le 1$, $\sum_{j=1}^{n} Ed_{n,j}^2 F_{n,j}(s)[1-F_{n,j}(t)]$ *has a finite limit* $L(s,t)$, *say, as* $n \to \infty$, *and that Condition* (2.3) *holds. Assume further that the following conditions are satisfied:*

$$E(d_{n,j_1}^2 d_{n,j_2}^2) = Ed_{n,j_1}^2 Ed_{n,j_2}^2, \quad 1 \le j_1 < j_2 \le n, \; n \ge 1, \tag{2.16}$$

$$E(d_{n,j_1}^2 \cdots d_{n,j_\ell}^2) \le C\, E\, d_{n,j_1}^2 \cdots E\, d_{n,j_\ell}^2, \quad 1 \le j_1 \cdots j_\ell \le n, \; n \ge 1, \tag{2.17}$$

for some positive constant C, *and for any* $\varepsilon > 0$

$$\sum_{j=1}^{n} E\, d_{n,j}^2\, I\,(|d_{n,j}| > \varepsilon) \to 0 \quad as \quad n \to \infty. \tag{2.18}$$

Then W_n *converges weakly to* W.

PROOF. Again we will establish that the finite dimentional distributions of W_n converge in law to those of W, and that W_n is tight. For the former we need the following lemma.

LEMMA 2.7. *Let* $\{\xi_{n,j},\ j = 1, \ldots, k_n\}$ *be an array of random variables satisfying Conditions* (2.16) - (2.18) *Assume further that*

$$E(\xi_{n,j_1} \cdots \xi_{n,j_\ell}) = 0, \quad 1 \le j_1 < \cdots < j_\ell \le n, \; n \ge 1, \tag{2.19}$$

and that for some $0 < \sigma^2 < \infty$,

$$\sum_{j=1}^{k_n} E\xi_{n,j}^2 \to \sigma^2 \quad as \quad n \to \infty. \tag{2.20}$$

Then $S_n = \sum_{j=1}^{k_n} \xi_{n,j}$ *is asymptotically normally distributed with mean* 0 *and variance* σ^2.

<u>PROOF</u>. A slightly different result is given by Komlós (1973). This lemma is of interest in its own right. We shall present a proof based on Theorem 2.1 of McLeish (1974) which is different from and more direct than that of Komlós. Let $T_n = \prod_{j=1}^{k_n} (1 + it\xi_{n,j})$. We need to show that $ET_n \to 1$ as $n \to \infty$, $\{T_n\}$ is uniformly integrable, $\max_{1 \le j \le k_n} |\xi_{n,j}| \xrightarrow{P} 0$, and that $\sum_{j=1}^{k_n} \xi_{n,j}^2 \xrightarrow{P} \sigma^2$ as $n \to \infty$. In our case it is clear that $ET_n = 1$, from Condition (2.16). Next, note that

$$E|T_n|^2 = E \prod_{j=1}^{k_n} (1 + t^2\xi_{n,j}^2) \le C^* \prod_{j=1}^{k_n} (1 + t^2E\xi_{n,j}^2)$$

$$\le C^* \exp(t^2\sum_{j=1}^{k_n} E\xi_{n,j}^2). \tag{2.21}$$

The first inequality is obtained from Condition (2.18) with some positive constant C^*, the second from the fact that $1 + y \le e^y$ for all $y \ge 0$. The last upper bound converges to $\exp(t^2\sigma^2)$ by Condition (2.20). Hence $\{T_n\}$ is uniformly integrable. Now for any $\delta > 0$

$$P[\max_{1 \le j \le k_n} |\xi_{n,j}| > \delta^{\frac{1}{2}}] \le \delta^{-1}E(\max_{1 \le j \le k_n} {}_n\xi_{n,j}^2)$$

$$\le \delta^{-1}[\delta^2 + \sum_{j=1}^{k_n} E\xi_{n,j}^2 I(|\xi_{n,j}| > \delta)]. \tag{2.22}$$

The last expression follows from the fact that for any $\delta > 0$, $\max_{1\le j\le k_n} \xi_{n,j}^2 \le \delta^2 + \sum_{j=1}^{k_n} \xi_{n,j}^2 \, I(|\xi_{n,j}| > \delta)$. Thus the RHS for (2.22) converges to 0 as $n \to \infty$ and $\delta \to 0$. Hence $\max_{1\le j\le k_n} |\xi_{n,j}| \xrightarrow{P} 0$ as $n \to \infty$. Finally we show that $\sum_{j=1}^{k_n} \xi_{n,j}^2 \xrightarrow{P} \sigma^2$ as $n \to \infty$. Let $\varepsilon > 0$ be given and let $\xi_{n,j}^* = \xi_{n,j} I(|\xi_{n,j}| \le \varepsilon)$. Set $S_n^* = \sum_{j=1}^{k_n} \xi_{n,j}^*$. Then it follows from (2.19) that $P[S_n \ne S_n^*] \to 0$ as $n \to \infty$. Therefore it suffices to show that $\sum_{j=1}^{k_n} \xi_{n,j}^{*2} \xrightarrow{P} \sigma^2$ as $n \to \infty$. Since $\sum_{j=1}^{k_n} E\xi_{n,j}^{*2} \to \sigma^2$ as $n \to \infty$, it suffices to show that $\mathrm{Var}(\sum_{j=1}^{k_n} \xi_{n,j}^{*2}) \to 0$ as $n \to \infty$. But

$$\mathrm{Var}\,(\sum_{j=1}^{k_n} \xi_{n,j}^{*2}) = E(\sum_{j=1}^{k_n} \xi_{n,j}^{*2})^2 - (\sum_{j=1}^{k_n} E\xi_{n,j}^{*2})^2. \qquad (2.23)$$

Now

$$E(\sum_{j=1}^{k_n} \xi_{n,j}^{*2})^2 = \sum_{j=1}^{k_n} E\xi_{n,j}^{*4} + \sum\sum_{j\ne k} E\xi_{n,j}^{*2}\xi_{n,k}^{*2}$$

$$\le \sum_{j=1}^{k_n} E\xi_{n,j}^{*4} + \sum_{j=1}^{k_n} \sum_{k=1}^{k_n} E\xi_{n,j}^2 \xi_{n,k}^2. \qquad (2.24)$$

Hence

$$\mathrm{Var}(\sum_{j=1}^{k_n} \xi_{n,j}^{*2}) \le \sum_{j=1}^{k_n} E\xi_{n,j}^{*4} + (\sum_{j=1}^{k_n} E\xi_{n,j}^2 - \sum_{j=1}^{k_n} E\xi_{n,j}^{*2})$$

$$\times (\sum_{j=1}^{k_n} E\xi_{n,j}^2 + \sum_{j=1}^{k} E\xi_{n,j}^{*2}). \qquad (2.25)$$

The second term in the last expression is readily seen to converge to 0 as $n \to \infty$, while the first term is

$$\Delta_n = \sum_{j=1}^{k_n} E\xi_{n,j}^{*4} = \sum_{j=1}^{k_n} E\xi_{n,j}^4 I(|\xi_{n,j}| \le \varepsilon). \tag{2.26}$$

Now for all $0 < \varepsilon_1 < \varepsilon$ we have

$$\Delta_n \le \sum_{j=1}^{k_n} E\xi_{n,j}^4 I(|\xi_{n,j}| \le \varepsilon_1) + \sum_{j=1}^{k_n} E\xi_{n,j}^4 I(\varepsilon_1 < |\xi_{n,j}| \le \varepsilon)$$

$$\le \varepsilon_1^2 \sum_{j=1}^{k_n} E\xi_{n,j}^2 + \varepsilon^2 \sum_{j=1}^{k_n} E\xi_{n,j}^2 I(|\xi_{n,j}| > \varepsilon_1), \tag{2.27}$$

the last upper bound converges to 0 by first letting $n \to \infty$ and then letting $\varepsilon_1 \to 0$. The lemma is now proved.‖

The above lemma will be applied to show that for fixed t, $W_n(t)$ is asymptotically normally distributed with mean 0 and variance $L(t, t)$. Let $\xi_{n,j} = d_{n,j} Z_{n,j}(t)$, where $Z_{n,j}(t) = I(X_{n,j} \le t) - F_{n,j}(t)$, $j = 1, \dots, n$. Clearly Conditions (2.16) - (2.18) are satisfied for $\xi_{n,j}$'s. Also $\sum_{j=1}^n E\xi_{n,j}^2 \to L(t,t)$. Thus it remains only to check (2.19) for $\xi_{n,j}$'s. Note that for any $\varepsilon > 0$,

$\sum_{j=1}^n E\xi_{n,j}^2 I(|\xi_{n,j}| > \varepsilon)$ can be evaluated as follows:

$$\sum_{j=1}^{n} E\xi_{n,j}^{2} I(|\xi_{n,j}| > \varepsilon)$$

$$\leq \sum_{j=1}^{n} Ed_{n,j}^{2} EZ_{n,j}^{2}(t) I(|Z_{n,j}(t)||d_{n,j}| > \varepsilon)$$

$$\leq \max_{1 \leq j \leq n} EZ_{n,j}^{2}(t) \sum_{j=1}^{n} Ed_{n,j}^{2} I(|d_{n,j}| > \varepsilon), \qquad (2.8)$$

which converges to 0 as $n \to \infty$ since $\max_{1 \leq j \leq n} EZ_{n,j}^{2}(t) \leq 1/4$ and using condition (2.19). Thus in view of the Cramer-Wold technique the finite dimensional distributions of W_n converge in law to those of W. The tightness of W_n can be proved in a fashion similar to that of the tightness of Theorem 2.1. Details are omitted. ‖

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