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A NOTE ON WEAK CONVERGENCE OF MEAN RESIDUAL LIFE OF STATIONARY MIXING RANDOM VARIABLES

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ABSTRACT

For a sequence of strictly stationary uniform or strong mixing we estimate the mean residual time of the marginal distribution from the first n observations. Under appropriate conditions it is shown that the estimate converges weakly to a well defined Gaussian process even when the sample size is random.

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1. Introduction and statement of results.

Let T be a nonnegative random variable defined on a probability space (Ω, A, P) and with distribution function (df) $F(t)$. The mean residual life time function of T at age t is defined by (cf Barlow and Proschan (1975) for further discussion);

$$\mu(t) = E[T-t \mid T>t] = \int_t^\infty (1-F(u))du/(1-F(t)), \quad 0 \leq t < \infty. \tag{1.1}$$

Let $\bar{F}(t)=1-F(t)$ denote the survival function of T at age t for all t such that $\bar{F}(t) > 0$ then we can write $\mu(t) = \int_t^\infty \bar{F}(u)du/\bar{F}(t)$, $0 \leq t < \infty$.

Based on a random sample $T_1, \ldots, T_n$ from $F(t)$, Yang (1978) proposed an estimate of $\mu(t)$ of the form:

$$\mu_n(t) = \{\int_t^\infty \bar{F}_n(u)du/\bar{F}_n(t)\} I(T_{(n)}-t), \tag{1.2}$$

where $\bar{F}_n(t)=1-F_n(t)$ and $F_n(t)$ denote the empirical df, $T_{(n)}$ denote the largest order statistic of $T_1, \ldots, T_n$, and $I(a)=1$ or 0 according as $a > 0$ or $a \leq 0$. She proved that $\mu_n(t)$, when properly normalized, converges weakly to a Gaussian process on $[0, T_o]$ for a fixed $T_o < \infty$.

The purpose of the present note is to extend Yang's result to the case where $T_1, \ldots, T_n$ are not necessarily independent. Let $\{T_n\}_{n=1}^\infty$ be a sequence of random variables and let $F_{m,n}$ denote the σ-field generated by $\{T_m, \ldots, T_n\}$, $1 \leq m < n \leq \infty$. Further let $A \in F_{1,m}$ and $B \in F_{m+n,\infty}$ be two events. Then $\{T_n\}$ is said to be uniformly mixing if

(1.3) $$|P(AB) - P(A)P(B)| \leq \phi(n)P(A),$$

where $\phi(n)$ is a nonincreasing function of positive integers with $0 \leq \phi(n) \leq 1$ and $\lim_{n\to\infty} \phi(n) = 0$. The sequence $\{T_n\}$ is said to be strongly mixing if

(1.4) $$|P(AB) - P(A)P(B)| \leq \alpha(n),$$

where $\alpha(n)$ is a nonincreasing function of positive integers with $0 \leq \alpha(n) \leq 1$ and $\lim_{n\to\infty} \alpha(n) = 0$. Note that uniform mixing implies strong mixing, but not conversely. Assume that $\{T_n\}$ are strictly stationary with the same marginal df F defined on $[0,\infty)$, and let $T_1,\ldots,T_n$ be the first n terms in that sequence. Assume that F is continuous and define

(1.5) $$D_n(t) = \sqrt{n}(\mu_n(t) - \mu(t)), \ 0 \leq t \leq T_o < \infty$$

Let $x = F(t)$, then we can write $D_n(t)$ as $U_n(x)$, where

$$U_n(x) = \sqrt{n}(\mu_n(F^{-1}(x)) - \mu(F^{-1}(x))), \ 0 \leq x \leq b < 1,$$

where $b = F(T_o)$.

Further, let $\{N_n\}$ be a sequence of positive integer-valued random variables defined on the same probability space $(\Omega, \mathcal{A}, P)$ such that for some positive constants $c_n \to \infty$, N_n/c_n converges in probability to a strictly positive random variable N.

The following two theorems are the main results of this note.

<u>THEOREM 1.1</u>. <u>Let</u> $\{T_n\}$ <u>be a strictly stationary uniformly mixing sequence of nonnegative random variables with common continuous</u>

marginal d.f. and is such that for some $\delta > 0$, $\phi(n) = 0(n^{-2-\delta})$. *Then* $U_{N_n}(x)$ *converges weakly to a Gaussian process* $U(x)$, $0 \leq x \leq b < 1$ *as* $n \to \infty$, *where* $EU(x) = 0$ *and* $Cov(U(x), U(y)) = \Gamma(x,y)$ *is as given in* (2.20).

THEOREM 1.2. Let $\{T_n\}$ *be a strictly stationary strongly mixing sequence of nonnegative random variables with common continuous marginal d.f. and is such that for some* $1 > \delta > 0$, $\alpha(n) = 0(n^{-2(1+\delta)/\delta})$ and $E|T_i I(F(T_i) - x)|^{2+\delta} < \infty$. *Then the conclusion of Theorem 1.1 continues to hold.*

Note that the above results may be viewed, on one hand, as generalization of the result of Wang (1978) not only to dependent variables but also to the case when the sampling was done sequentially and the final sample size is random. But, on the other hand, this work may find natural application in reliability and biometry when the data are no longer independent. This can happen in reliability if consecutive testing results in some effect between very closely tested systems and in biometry this may occur if we study the mean residual lives of persons suffering from a disease that is known that inheritance between close family members plays a significant role in inducing the disease, examples include diabetes, heart disease and maybe others.

2. Proofs.

PROOF OF THEOREM 1.1. Several preparatory Lemmas are needed.

LEMMA 2.1. *Let* $\{Z_n\}$ *be a strictly stationary uniformly mixing sequence of random variables such that for all* $n \geq 1$, $EZ_n = 0$, $|Z_n| \leq 1$, and $EZ_n^2 = \theta$ (*thus* $\|Z_n\|_r \leq \theta^{1/2r} \forall r \geq 1$). *If* $\phi(n) = O(n^{-2-\delta})$, for some $\delta > 0$, *then for any* $\varepsilon > 0$ *and sufficiently large* n, *there exists a* $\tau > 0$ *such that*

$$P[|\Sigma_{i=1}^{n} Z_i| \geq \varepsilon n^{1/2}] \leq K_1\{n^{-\tau}\theta + \theta^{1+\delta/2}\}, \tag{2.1}$$

where K_1 *is a positive constant independent of* n *and* θ.

PROOF. Follows exactly that of a lemma of Yoshihara (1978) and is omitted.□

LEMMA 2.2. *Let* $\{T_n\}$ *satisfy the conditions of Theorem 1.1. For some* $\beta \geq 0$, *define the sequence of stochastic processes*

$$V_n(x) = n^{-1/2}\Sigma_{i=1}^{n}[I(F(T_i)-x)T_i^{\beta} - E(I(F(T_i)-x)T_i^{\beta})], \tag{2.2}$$

for $x \in [0,b]$ *for some* $b<1$. *Then* $V_n(x)$ *converges weakly to a Gaussian process* $V(x)$, *as* $n\to\infty$, $0\le x\le b<1$, *where* $EV(x)=0$ *and* $EV(x)V(y)=\eta(x,y)$ *as defined in* (2.3) for all $0\le x,y\le b<1$.

PROOF. We need to show that the finite dimensional distributions of $V_n(x)$ converge to those of $V(x)$ and that $V_n(x)$ is tight. The asymptotic normality for $V_n(x)$, $0\le x\le b<1$ follows directly from Theorem 1.5 of Ibragimov (1962) since $n^{1/2}V_n(x)$ is the sum of strictly stationar uniform mixing random variables and since $\phi(n)=0(n^{-2-\delta})$, for some $\delta>0$ entails that $\Sigma_{n=1}^{\infty}\phi^{1/2}(n)<\infty$. From stationarity it is easily seen that for all $0\le x,\ y\le b<1$,

$$\text{(2.3)}\quad \mathrm{Cov}(V_n(x),V_n(y))=\eta(x,y)=\sigma_{11}^{(1,1)}(x,y)+\Sigma_{k=2}^{\infty}\sigma_{1k}^{(1,1)}(x,y)+\Sigma_{k=2}^{\infty}\sigma_{k1}^{(1,1)}(x,y),$$

where

$$\text{(2.4)}\quad \sigma_{ij}^{(\alpha,\beta)}(x,y)=\mathrm{Cov}(I(F(T_i)-x)T_i^{\alpha},\ I(F(T_j)-y)T_j^{\beta}),$$

for all $i,j=1,2,\ldots$ and all $\alpha,\beta>0$. The convergence of the finite dimensional distributions is established via the standard Cramér-Wold technique. Next, let us prove the tightness of $V_n(x)$, $0\le x\le b<1$, for let $0\le x<y\le b<1$, and note that

$$E|V_n(x)-V_n(y)|^4=n^{-2}E|\Sigma_{i=1}^{n}\{[I(F(T_i)-y)-I(F(T_i)-x)]T_i-E[I(F(T_i)-y)-I(F(T_i)-x)]T_i\}|^4$$

$$\text{(2.5)}\qquad =n^{-2}E|\Sigma_{i=1}^{n}h_{x,y}(T_i)|^4,\ \text{say}.$$

Now, let $Z_i=h_{x,y}(T_i)/2F^{-1}(y)$ for all $i=1,\ldots,n$, and $0\le x<y\le b<1$. Thus $EZ_i=0$, $|Z_i|\le 1$ since $|[I(F(T_i)-y)-I(F(T_i)-x)]T_i|\le |F^{-1}(y)|$, and $EZ_i^2=E[I(F(T_i)-y)-I(F(T_i)-x)]^2T_i^2=\theta(x,y)=\theta$, say $i=1,\ldots,n$. Thus an application of Lemma 2.1 leads to

$$(2.6) \qquad P[|V_n(x) - V_n(y)| \geq \varepsilon] \leq K_1\{\theta n^{-\tau} + \theta^{1+\delta/2}\},$$

where K_1 is a positive constant independent of n and θ. The balance of the proof proceeds exactly as in the empirical process case, c.f. Yoshihara (1978) for details. □

LEMMA 2.3. *Assume that the conditions of Theorem* 1.1 *hold. Define*

$$(2.7) \quad V_n(x,s) = [ns]^{1/2} V_{[ns]}(x)/n^{1/2}, \ 0 \leq s \leq 1, \text{ and } \ 0 \leq x \leq b < 1.$$

Then $V_n(x,s)$ *converges weakly to a Gaussian process* $V(x,s)$ *such that* $EV(x,s)=0$ *and* $\mathrm{Cov}(V(x,s), V(y,u))=\min(s,u)\eta(x,y)$ *for all* $0 \leq x,y \leq b$ *and* $0 \leq s,u \leq 1$, *where* $\eta(x,y)$ *is as given in* (2.3).

PROOF. Again we need to prove the convergence of the finite dimensional distributions to those of $V(x,s)$ and that $V_n(x,s)$ is tight. The former poses no problem over the corresponding part in Lemma 2.2. Thus we only need to prove the tightness of $V_n(x,s)$. First note that for all $0 \leq x < y \leq b$ and $0 \leq s < u \leq 1$,

$$(2.8) \qquad E|V_n(x,s) - V_n(y,u)|^4 \leq K_1\{\theta([nu]-[ns])^{-\tau} + \theta^{1+\delta/2}\},$$

where K_1 is a positive constant independent of θ and n. Inequality (2.8) is obtained by applying Lemma 2.1. Thus we have for all $0 \leq x < y \leq b < 1$ and $0 \leq s < u \leq 1$ that

$$(2.9) \qquad P[|V_n(x,s) - V_n(y,u)| \geq \varepsilon] \leq K_1\{\theta([nu]-[ns])^{-\tau} + \theta^{1+\delta/2})\}/\varepsilon^4$$
$$\leq K_1\{\theta(n(u-s)+1)^{-\tau} + \theta^{1+\delta/2}\}/\varepsilon^4,$$

since $[nu]-[ns] \leq n(u-s)+1$. Note that $\theta \leq |F^{-1}(y)|^2 \leq |F^{-1}(b)|^2 < \infty$. Thus $\theta^* = \theta/(F^{-1}(b))^2$ is such that $0 \leq \theta^* \leq 1$ and hence

(2.10) $$P[|V_n(x,s)-V_n(y,u)| \geq \varepsilon] \leq K_2\{\theta^*(n(u-s))^{-\tau}+\theta^{*1+\delta/2}\}/\varepsilon^4.$$

Let $\varepsilon (0 < \varepsilon < 1)$ be fixed and suppose that

$$\varepsilon \leq (n(u-s))^{\tau}\theta^{*\delta/2}(x,y).$$

Hence

(2.11) $$P[|V_n(x,s)-V_n(y,u)| \geq \varepsilon] \leq K_2\varepsilon^{-5}\theta^{*1+\delta/2}(x,y),$$

where of course θ^* is a function of x and y.

Let p and m be, respectively, a positive real number and a positive integer to be chosen later such that $mp = \gamma$, $0 < \gamma < 1$, then using the argument of Billingsley (1968) (see his (22.18)) we can see that for a fixed $0 \leq y < 1$,

(2.12) $$\sup_{\substack{0\leq s\leq 1\\ y\leq x\leq y+mp}} |V_n(x,s) - V_n(y,s)| \leq 3 \max_{\substack{0\leq i\leq p^{-1}\\ 0\leq j\leq m}} |V_n(y+jp,ip) - V_n(y,ip)| + 2pn^{1/2},$$

provided that p^{-1} is an integer and $p \geq (\frac{\varepsilon}{n})^{2/(2+\delta)}$.

Hence using the extension of Bickel and Wichura (1971) of Billingsley's fluctuations inequalities we see that

(2.13) $$P[\max_{\substack{0\leq i\leq p^{-1}\\ 0\leq j\leq m}} |V_n(y+jp,ip) - V_n(y,ip)| > \varepsilon] \leq K_3\varepsilon^{-5}(\theta^*(y,y+mp))^{1+\delta/2}.$$

Thus choosing $p < \varepsilon/2\sqrt{n}$ we get

(2.14) $$P\{\sup_{\substack{0\leq s\leq p\\ y\leq x\leq y+mp}} |V_n(x,s)-V_n(y,s)| > 4\varepsilon\} \leq K_4\varepsilon^{-5}(\theta^*(y,y+mp))^{1+\delta/2} \leq K_5\varepsilon^{-5}(mp)^{1+\delta/2} = K_5\varepsilon^{-5}\gamma^{1+\delta/2},$$

since clearly $\theta^*(y,y+mp) \leq (F^{-1}(b))^2 mp$. Choose $\eta > \frac{K_5 \gamma^{\delta/2}}{\varepsilon}$, then the right-hand-side of (2.14) is less than $\eta\gamma$, provided that there exists a p such that $(\varepsilon/\eta)^{2/(2+\delta)} \leq p < \varepsilon/2\sqrt{n}$ and p^{-1} is an integer with $pm=\delta$. But this is guaranteed by choosing m such that $\frac{2\gamma\sqrt{n}}{2} \leq m \leq \frac{n^{2/(2+\delta)}\gamma}{\varepsilon^{2/(2+\delta)}}$ which is possible for sufficiently large n. Thus $V_n(x,s)$ is tight and the lemma is proved. □

Next, define the usual empirical process and a two parameter empirical process as:

(2.15) $W_n(x) = n^{-\frac{1}{2}}\Sigma_{i=1}^n [I(F(T_i)-x) - EI(F(T_i)-x)]$, $0 \leq x \leq 1$,

and $W_n(x,s) = [ns]^{\frac{1}{2}} W_n(x)/n^{\frac{1}{2}}$, $0 \leq x \leq 1$ and $0 \leq s \leq 1$.

The following two lemmas are needed to prove Theorem 1.1. Both Lemmas are due to Yoshihara (1978).

LEMMA 2.4. *Assume that the conditions of Theorem 1.1 hold. Then* $W_n(x)$ *converges weakly to a Gaussian process* $W(x)$ *with* $EW(x) = 0$ *and* $Cov(W(x),W(y)) = \zeta(x,y)$, *where*

(2.16) $$\zeta(x,y)=\sigma_{11}^{(0,0)}(x,y)+\Sigma_{k=2}^{\infty}\sigma_{1k}^{(0,0)}(x,y)+\Sigma_{k=2}^{\infty}\sigma_{k1}^{(0,0)}(x,y),$$

for all $0 \leq x, y \leq 1$ *and* $\sigma_{ij}^{(\alpha,\beta)}(x,y)$ *is as given in* (2.4).

LEMMA 2.5. *Assume that the conditions of Theorem* 1.1 *hold. Then* $W_n(x,s)$ *converges weakly to a Gaussian process* $W(x,s)$ *with* $EW(x,s)=0$ *and* $Cov(W(x,s),W(y,u)) = \min(s,u)\ \zeta(x,y)$ *for all* $0 \leq x,y,s,u \leq 1$.

We are now in a position to prove Theorem 1.1. This accomplished in three steps.

<u>STEP ONE</u>. Let us prove that $U_n(x)$ converges weakly to a Gaussian process $U(x)$ with $EU(x)=0$ and $\mathrm{Cov}\,(U(x),U(y))=\Gamma(x,y)$. To this end we note that $U_n(x)$ has the same limiting distribution as the process $U_n^*(x)$ defined by:

$$(2.17)\qquad U_n^*(x) = (1-x)V_n(x) - [F^{-1}(x)(1-x) + \int_x^1 (1-y)\,dF^{-1}(y)]W_n(x)$$

$$= C_1(x)V_n(x) + C_2(x)W_n(x),\ \text{say.}$$

Now the tightness of $U_n^*(x)$ is guaranteed by Lemmas 2.2 and 2.4. Thus we need only to establish the convergence of the finite dimensional distributions of $U_n^*(x)$. But this poses no new problems and we need only to establish the covariance function. Let $0 \le x,y \le b < 1$, then

$$(2.18)\qquad \mathrm{Cov}(U_n^*(x),U_n^*(y)) = C_1(x)C_1(y)\mathrm{Cov}(V_n(x),V_n(y)) + C_2(x)C_2(y)\mathrm{Cov}(W_n(x),W_n(y))$$

$$+ C_1(x)C_2(y)\mathrm{Cov}(V_n(x),W_n(y)) + C_2(x)C_1(y)\mathrm{Cov}(W_n(x),V_n(y))$$

$$= I + II + III + IV$$

$$= C_1(x)C_2(y)\eta(x,y) + C_2(x)C_2(y)\zeta(x,y) + III + IV.$$

But easy calculations lead to:

$$(2.19)\qquad \mathrm{Cov}(V_n(x),W_n(y)) = \sigma_{11}^{(1,0)}(x,y) + \Sigma_{k=2}^{\infty}\sigma_{1k}^{(1,0)}(x,y) + \Sigma_{k=2}^{\infty}\sigma_{k1}^{(1,0)}(x,y)$$

$$= \xi(x,y),\ \text{say.}$$

and

$$(2.20)\qquad \mathrm{Cov}(W_n(x),V_n(y)) = \sigma_{11}^{(0,1)} + \Sigma_{k=2}^{\infty}\sigma_{1k}^{(0,1)}(x,y) + \Sigma_{k=2}^{\infty}\sigma_{k1}^{(0,1)}(x,y) = \tau(x,y).$$

Hence we arrive at

(2.20) $$\mathrm{Cov}(U_n^*(x),U_n^*(y)) = \Gamma(x,y)$$

$$= C_1(x)C_1(y)\eta(x,y) + C_2(x)C_2(y)\zeta(x,y) + C_1(x)C_2(y)\xi(x,y) + C_2(x)C_1(y)\tau(x,y).$$

STEP TWO. Define the two parameter stochastic processes

(2.21) $$U_n(x,s) = [ns]^{1/2}U_n(x)/n^{1/2}, \quad 0 \le x \le b < 1 \quad \text{and} \quad 0 \le s \le 1.$$

Thus using Lemmas 2.3 and 2.5 we can see that $U_n(x,s)$ is tight and it follows from the argument of Step One that

(2.22) $$\mathrm{Cov}(U_n(x,s),U_n(y,u)) = \min(s,y)\Gamma(x,y) \quad \text{for all} \quad 0 \le x,y \le b < 1 \quad \text{and} \quad 0 \le s,u \le 1.$$

Hence $U_n(x,s)$ converges weakly to a Gaussian process $U(x,s)$ such that $EU(x,s)=0$ and $\mathrm{Cov}(U(x,s),U(y,u))=\min(s,u)\Gamma(x,y)$.

STEP THREE. By straight forward adaptation of the arguments of Theorem 17.2 of Billingsley (1968) it is easily seen that Step 2 implies that $U_{N_n}(x)$ converges weakly to $U(x)$, $0 \le x \le b < 1$. This concludes the proof of Theorem 1.1.

PROOF OF THEOREM 1.2. We start with the following lemma.

LEMMA 2.6. *Let $\{Z_n\}$ be a strictly stationary strong mixing sequence of random variables such that for all $n \ge 1$, $|Z_n| \le 1$, $EZ_n^2 = \theta$. If $\alpha(n) = O(n^{-5/2-\delta})$ for some $\delta > 0$, then for any $\varepsilon > 0$ and sufficiently large n, there exists a $\tau > 0$ such that*

$$P[|\Sigma_{i=1}^n Z_i| \ge \varepsilon n^{1/2}] \le K_1^*\{n^{-\rho}\tau^{1-\gamma} + \tau^{6/5}\},$$

where γ is a positive number such that $2/(2+\delta) < \gamma < 1$ and $\rho = (5/2+\delta)\gamma - 2$.

PROOF. Follows exactly as that of Lemma 2 of Yoshihara (1975).

With the aid of the above lemma, four lemmas analogous to Lemmas 2.2-2.5 can be established except that here in place of Theorem 1.5 of Ibragimov we say Theorem 1.7 which is valid since $E|T_i I(F(T_i)-x)|^{2+\delta} < \infty$ for some $\delta > 0$ and that $\alpha(n) = 0(n^{-2(1+\delta)/\delta})$ implies that $\sum_{k=1}^{\infty} (\alpha(n))^{\delta/(2+\delta)} < \infty$. The rest of the proof is unaffected since $\alpha(n) = 0(n^{-2(1+\delta)/\delta})$ implies $\alpha(n) = 0(n^{-5/2-\delta})$ for any $\delta < 1$, and then we proceed in three steps to prove Theorem 1.2 in exactly the same fashion as Theorem 1.1. For the sake of brevity we omit the details.

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