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RENEWAL ANALYSIS USING BERNSTEIN DISTRIBUTION

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ABSTRACT

If the functional accuracy of a machine part is lost due to a linear non-stationary wear process, and the life of the component is terminated when the wear reaches a limiting value then the reliability of the component follows the Bernstein model. In this paper various characteristics of the renewal process are discussed when the life of the renewable units is distributed according to Bernstein law. Series solutions are developed for the renewal and renewal rate functions. The asymptotic relationships for these functions and for the variance of renewals are also presented. The results are given in both graphical and tabular form. The application of this theory is illustrated using an example of replacement of cutting tools on an automatic machining line.

1 - BERNSTEIN DISTRIBUTION

When several identical machine parts (such as cams, splines, cutting tools, machine tools, etc.,) are tested to determine the wear on their contact surfaces, it is often observed that the wear realizations are linear and time dependent. This type of linear non-stationary random wear process can be modelled by a random wear function W(t) given

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by

$$W(t) = at + b \tag{1}$$

where a and b are random variables such that

$$a = dW(t)/dt = \dot{W}(t) = \text{rate of wear}, \tag{2}$$

$$b = W(0) = \text{initial value of wear}, \tag{3}$$

and $t \in T$. T is a random variable defined as $T = \frac{W_\ell - b}{a}$ where W_ℓ (the wear limit) gives a criterion of the machine part failure if $W(t) \geq W_\ell$.

Equation (1) represents a correlated process between $W(t)$ and T (with a correlation coefficient $\rho = +1$). Gertsbakh and Kardonsky [1] have shown that with a normality assumption about the distribution of a and b, the distribution function of the life of the machine part is given by

$$F(t) = \Phi\left[\frac{t - c}{\sqrt{\alpha t^2 + \beta}}\right], \quad -\infty < t < \infty \tag{4}$$

where

$$\alpha = V(a)/E^2(a) \tag{5}$$

$$\beta = V(b)/E^2(a) \tag{6}$$

$$c = [W_\ell - E(b)]/E(a) \tag{7}$$

and the function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} z^2\right] dz$ is the Laplace function.

Equation (4) represents a three parameter model called the Bernstein distribution. When the initial value of wear is zero [i.e., $W(0) = b = 0$] then equation (4) reduces to a two parameter Bernstein distribution, which is known as the alpha-distribution [2, 3] and is written as

$$F(t) = D\Phi\left[\frac{1}{\sqrt{\alpha}} - \frac{c}{\sqrt{\alpha}\, t}\right], \quad 0 < t < \infty \tag{8}$$

where D is a multiplying factor to account for the fact that in real life testing t will always be greater than zero, c is the scale parameter and α is the shape parameter. In general

$$D = \frac{1}{\Phi(1/\sqrt{\alpha})}$$

However when $\sqrt{\alpha} < .35$, $D \cong 1$. Subsequently all analysis is for the case when $D \cong 1$ (i.e., $\sqrt{\alpha} < .35$). The probability density function $f(t) = dF(t)/dt$, of the Bernstein random variate is given by

$$f(t) = \frac{c}{\sqrt{2\pi\alpha}\ t^2} \exp\left[-\frac{1}{2}\left(\frac{1}{\sqrt{\alpha}} - \frac{c}{\sqrt{\alpha}\,t}\right)^2\right] \quad , \quad t > 0. \tag{9}$$

The probability density function f(t) is plotted for various values of $\sqrt{\alpha}$ in Figure 1. The main characteristics of Bernstein distribution are given in Table 1.

The Bernstein distribution has been successfully used in several important areas such as modelling the:

1 Life of cutting tools on automated machining lines [2, 3].

2 Fatigue life of machine parts subjected to cyclic and fluctuating loads [4].

In the following sections some characteristics of the renewal process are discussed when the life of renewable units is distributed according to Bernstein law.

2 - RENEWAL ANALYSIS USING BERNSTEIN MODEL

Consider a new piece of equipment which is installed at time $t = 0$, and suppose that at time $t = t_1 = T_1$ this equipment breaks down and is immediately replaced by a similar piece of equipment. Suppose the replacement equipment in turn breaks down and is replaced at time $t = t_2$; and so on. If the 'life times' T_1, $T_2 = t_2 - t_1$, $T_3 = t_3 - t_2$, etc., are independently and identically distributed with a distribution function F(t) , then renewal function H(t) gives the expected number of replacements E[N(t)] that are necessary upto (and including) time t where N(t) is the random variable denoting the number of replacements during the time period [0 - t] (Figure 2). It is well known from the renewal theory that H(t) = E[N(t)] satisfies the following integral equation [5]:

$$H(t) = F(t) + \int_0^t H(t - x)\, dF(x) \tag{10}$$

Moreover it can be proved that H(t) is the unique measurable solution to equation (10) which is bounded in any finite interval [6]. The renewal rate h(t) = dH(t)/dt gives the expected number of renewals per unit time and satisfies the following integral equation

$$h(t) = f(t) + \int_0^t h(t - x) f(x) dx \tag{11}$$

The variance of the number of renewals is [5, 7]:

$$V[N(t)] = 2 \int_0^t H(t-x) dH(x) + H(t) - H^2(t) \tag{12}$$

In several instances, the basic interest is in the study of the renewal process for a long period of time (i.e., when $t \to \infty$). Mathematically speaking we are concerned with the asymptotic [steady state] behavior of the renewal process. The asymptotic results in renewal theory corresponding to equations (10), (11), and (12) are [8]:

$$H_s(t) = \frac{t}{\bar{T}} + \frac{(\sigma^2 - \bar{T}^2)}{2\bar{T}^2} + 0(1) \tag{13}$$

$$h_s(t) = \frac{1}{\bar{T}} + 0(1) \tag{14}$$

$$V_s[N(t)] = \left[\frac{\sigma^2 t}{\bar{T}^3}\right] + \left[\frac{3}{4} + \frac{2\sigma^2}{\bar{T}^2} - \frac{2\mu_3}{3\bar{T}^3}\right] + 0(1) \tag{15}$$

where s identifies the steady state (asymptotic) behavior, 0(1) is a quantity of negligible magnitude, $\mu_3 = \int_0^\infty t^3 f(t) dt$, $\bar{T}$ is the mean life of the equipment and σ^2 is the variance of time to first failure.

When F(t) is a Bernstein distribution, it is rather difficult to develop a closed form solution of the equations (10) and (11) for H(t) and h(t). For this reason instead of referring to equations (10) and (11), a slightly different approach has been used to develop expressions for H(t) and h(t). This approach starts from the basic definition of the renewal process and details are given in Appendix A. The resulting expressions for H(t) and h(t) for the Bernstein model are:

$$H(t) = \sum_{n=1}^{\infty} \Phi\left[\frac{t - nc}{\sqrt{\alpha/n}\ t}\right] \tag{16}$$

and

$$h(t) = \sum_{n=1}^{\infty} \frac{nc}{\sqrt{\alpha/n}\ \sqrt{2\pi}\ t^2} \exp\left[-\frac{1}{2}\left\{\frac{t - nc}{\sqrt{\alpha/n}\ t}\right\}^2\right] \tag{17}$$

Plots of H(t) and h(t) for various values of $\sqrt{\alpha}$ are given in Figures 3 and 4, and for a greater accuracy these functions are also tabulated in Tables 2 and 3 respectively. The variance of renewals V[N(t)] can be obtained by evaluating integral equation (12). For steady state conditions ($t \to \infty$) the following expressions can be used for Bernstein model which correspond to equations (13), (14), and (15):

$$H_s(t) \cong \frac{t}{c} - \frac{1}{2}(1 - \alpha) \tag{18}$$

$$h_s(t) \cong \frac{1}{c} \tag{19}$$

$$V_s(t) \cong \frac{\alpha t}{c} + \left[\frac{3}{4} + \frac{5}{4}\alpha^2 + 2\alpha\right] - \frac{2}{3}\frac{\mu_3}{3c^3} \tag{20}$$

3 - APPLICATIONS OF BERNSTEIN RENEWAL ANALYSIS IN AUTOMATIC MACHINING LINE

3.1 - ILLUSTRATIVE EXAMPLE

100,000 identical machine parts (circular flanges) are to be manufactured on an automatic machining line. The machining operation consist of drilling eight holes simultaneously on a multispindle drilling head consisting of eight identical drills. Each new drill can be reground five times (i.e., it can be used six times). The life characteristics of drills are as follows:

- Drill life t in number of holes is distributed according to Bernstein model
- C = 400 holes
- $\sqrt{\alpha} = .25$

If a drill is replaced upon failure it will cost C_f \$/drill and if it is replaced at a predetermined (scheduled) replacement interval, t_s, then it will cost C_s \$/drill. The replacement policy is to replace all N drills on the multispindle drilling head at an interval of t_s, and make all the failure replacements upon individual basis, whenever need arises during this interval. The general model for expected replacement costs of tool per component denoted by $E[C(t_s)]$ is:

$$E[C(t_s)] = \frac{\text{Cost of replacing N tools at time } t_s + \text{Expected cost of failure replacement in interval } (0 - t_s)}{\text{Interval length}}$$

or

$$E[C(t_s)] = \frac{N\,C_s + N\,C_f\,H(t_s)}{t_s} \tag{21}$$

or

$$\frac{E[C(t_s)]}{N\,C_f/C} = \frac{\gamma + H(t_s/C)}{t_s/C}$$

$$= [\gamma + H(\tau_s)]/\tau_s \tag{22}$$

where

$$\gamma = C_s/C_f \,, \quad \text{and} \quad \tau_s = t_s/C$$

For the given example $C_s = 5$, and $C_f = 10$, therefore $\gamma = .5$, and $N = 8$. Our objectives are :

A. Calculate the optimal scheduled replacement interval t_s^*, which will minimize the cost $E[C(t_s)]$.

B. Calculate the optimal tool replacement cost for the entire production of 100,000 components.

C. Calculate the number of drills required for the entire production.

D. Determine the optimal level of tool reliability.

<u>SOLUTION:</u>

A. For $\gamma = .5$, and $N = 8$, the left hand side of (22) is tabulated for various values of τ_s, using tables of Renewal Function (Table 2) to find the optimal value of τ_s. From the tabulated results in Table 4, $\tau_s^* = 0.7$. Therefore the optimal scheduled replacement interval is

$$t_s^* = C(\tau_s^*) = 400(.7) = 280 \text{ holes/drill}$$

which means a block replacement of 8 drills after 280 components have been machined.

TABLE 4 : SOLUTION OF EQUATION (22) TO FIND τ_s^*

$\tau_s = \frac{t_s}{C}$	$CE[C(t_s)]/8C_f$
.5	1.0006
.55	.91005
.6	.83970
.65	.79327
.7	.77605
.75	.78828
.8	.82330
.85	.87000

B. From equation (22), the optimum expected tool replacement cost per component is:

$$E[C(t_s^*)] = \frac{8.(10)}{400}\left[\frac{.5 + .04324}{.7}\right] = .1555 \text{ \$/comp.}$$

For 100,000 components the optimum replacement cost is = 15550$.

C. Total number of drills required for machining 100,000 components:

$$= 8\left[\frac{100,000}{t_s^*} + \frac{100,000\ H(t^*)}{t_s^*}\right]$$

$$= 8[358 + 358(.04324)] = 2988 \text{ drills}$$

Since each new drill can be used 6 times therefore total number of new drills required for entire production lot is

$$= \frac{2988}{6} \approx 498.$$

D. The optimal level of tool reliability is:

$$R(t_s^*) = \Phi\left[\frac{1}{\sqrt{\alpha}}\left(\frac{C}{t_s^*} - 1\right)\right] = \Phi(1.714)$$

$$= .9569$$

3.2 - GENERAL SOLUTION OF SCHEDULED REPLACEMENT MODEL

To develop a general solution for $\tau_s^* = t_s^*/C$, differentiate equation (22) with respect to t_s and equate the resulting expression to zero. As a result the following equation is obtained:

$$\gamma = \tau_s^*\ h(\tau_s^*) - H(\tau_s^*) \qquad (23)$$

For any value of γ and $\sqrt{\alpha}$, the value of τ_s^* can be obtained using the tables of $H(\tau)$ and $h(\tau)$ given in this paper and the results can be plotted for general use in non-dimensional form, as γ versus τ_s^* is plotted for different values of $\sqrt{\alpha}$.

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APPENDIX A

If X has a Bernstein distribution

$$F(c,\ \alpha,\ x) = \Phi\left[\frac{1}{\sqrt{\alpha}}\left(1 - \frac{c}{x}\right)\right] = B(c,\ \alpha),$$

then the random variable $\frac{1}{X}$ is normally distributed with mean $E\left(\frac{1}{X}\right) = \frac{1}{c}$ and variance $V\left(\frac{1}{X}\right) = \frac{\alpha}{c^2}$. Using this fact the following theorem can easily be proved (see Ahmad and Sheikh [9]):

THEOREM:

If T_i, $i = 1, 2, \ldots, n$ are identical and independent random variables each having a Bernstein distribution, then the random variable

$$T = \frac{n^2}{\frac{1}{T_1} + \frac{1}{T_2} + \ldots + \frac{1}{T_n}} \qquad \text{(A.1)}$$

is also Bernstein distributed and is given by

$$B\left(nc,\ \frac{\alpha}{n}\right) = F\left(nc,\ \frac{\alpha}{n},\ t\right)$$
$$= \Phi\left\{\frac{1}{\sqrt{\alpha/n}}\left(1 - \frac{nc}{t}\right)\right\}. \qquad \text{(A.2)}$$

RENEWAL FUNCTION OF BERNSTEIN MODEL

Referring to Figure 2, the basic definition of Renewal Function is [8]:

$$H(t) = \sum_{n=1}^{\infty} F_n(t) \qquad \text{(A.3)}$$

where

$$F_n(t) = P[T_1 + T_2 + \ldots + T_n < t]$$
$$= P\left[n \cdot \frac{1}{n}\ (T_1 + T_2 + \ldots + T_n) < t\right]. \qquad \text{(A.4)}$$

It can be shown that when T_i, s are independent and identically distributed random variables then the random variable

$$\varepsilon = \frac{T_1 + T_2 + \ldots + T_n}{n} = \frac{T}{n}$$

can be replaced by an approximately equivalent random variable defined as

$$\tilde{\varepsilon} = \frac{n}{\frac{1}{T_1} + \frac{1}{T_2} + \ldots + \frac{1}{T_n}} \; .$$

With this substitution in equation (A.4) and using the results of the above-mentioned theorem we obtain

$$F_n(t) = \Phi\left[\frac{1}{\sqrt{\alpha/n}}\left(1 - \frac{nc}{t}\right)\right] \tag{A.5}$$

and from equation (A.3)

$$H(t) = \sum_{n=1}^{\infty} \Phi\left[\frac{1}{\sqrt{\alpha/n}}\left(1 - \frac{nc}{t}\right)\right] \tag{A.6}$$

By differentiating equation (A.6) with respect to t, we obtain the renewal rate function

$$h(t) = \dot{H}(t) = \sum_{n=1}^{\infty} \frac{nc}{\sqrt{\alpha/n}\ \sqrt{2\pi}\ t^2}\ \exp\left[-\frac{1}{2}\left(\frac{t - nc}{\sqrt{\alpha/n}\ t}\right)^2\right] \tag{A.7}$$

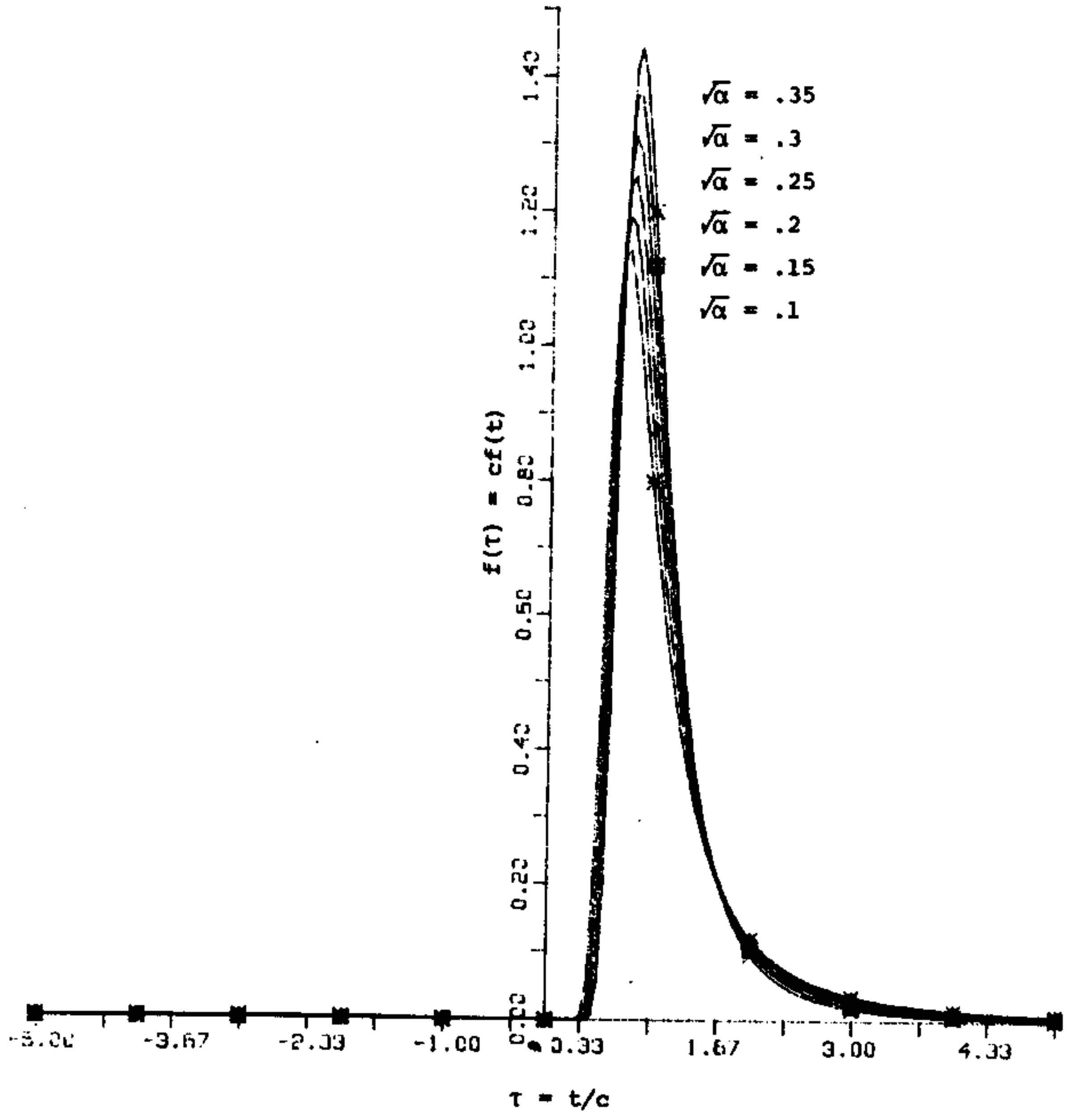

FIGURE 1. Probability Density Function of Bernstein Model for $\sqrt{\alpha} < .35$.

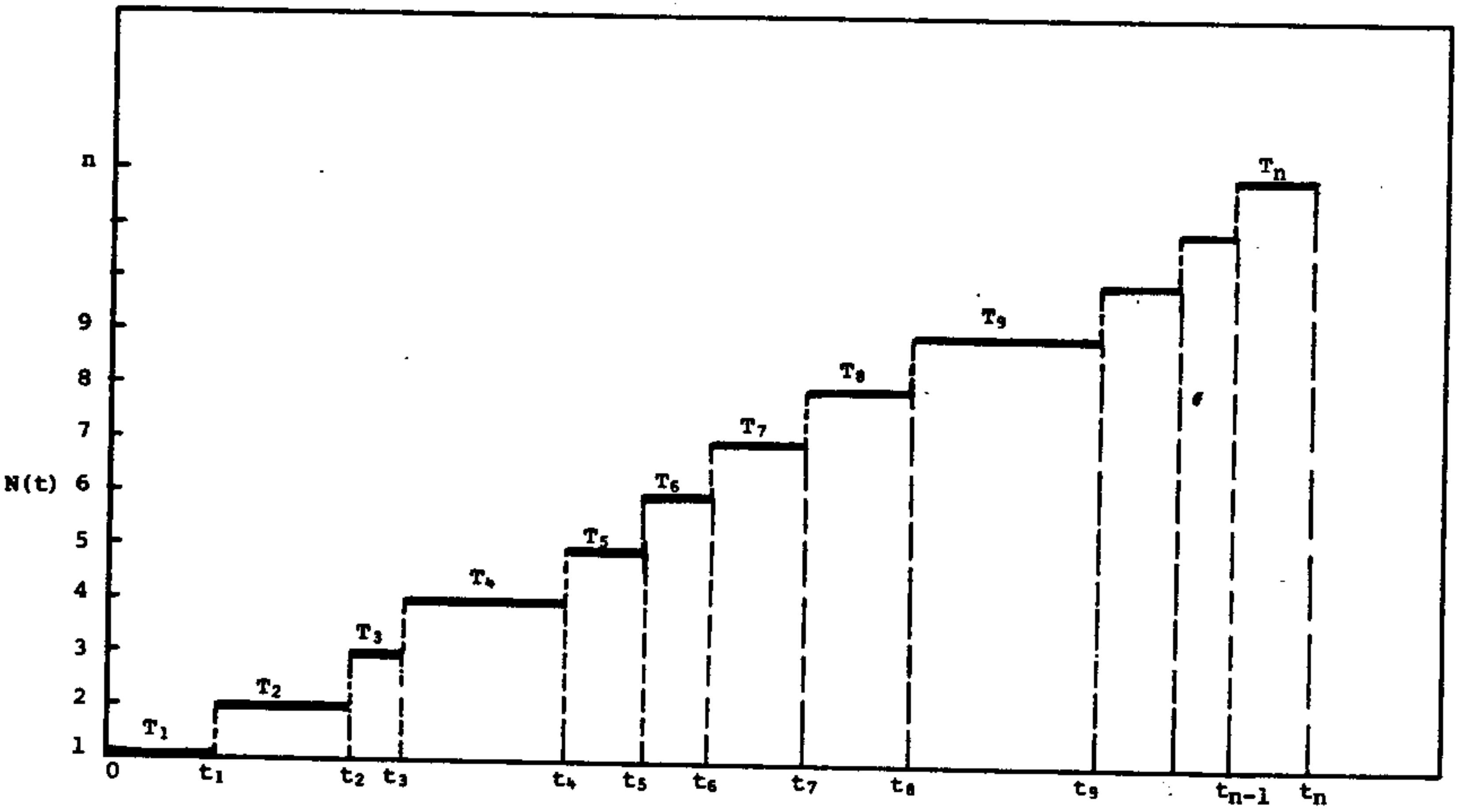

FIGURE 2. A Single Realization of the Renewal Process. [N(t) is the Number of Failures in time interval $t_{n-1} - t_n$.]

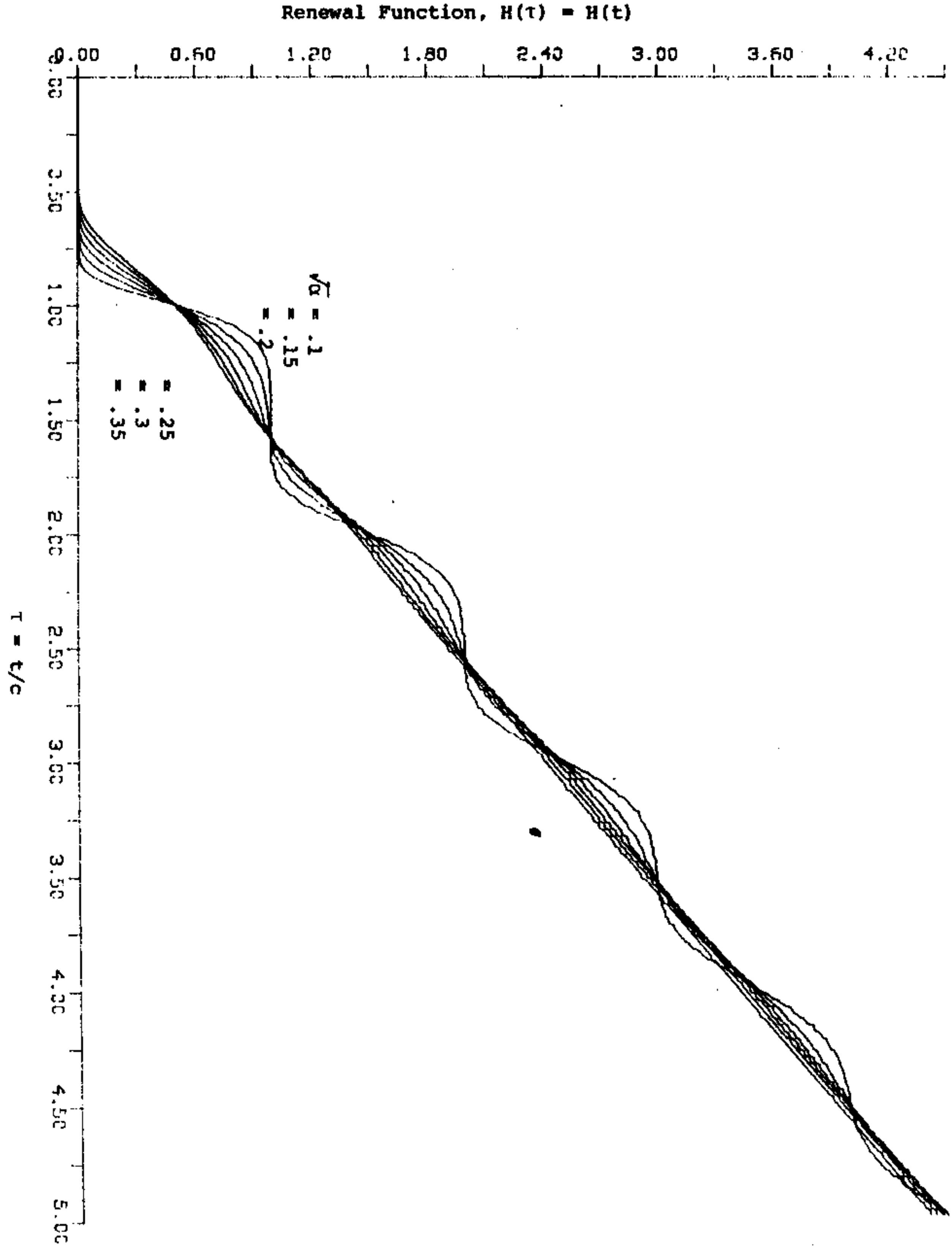

FIGURE 3. Renewal Function of Bernstein Model

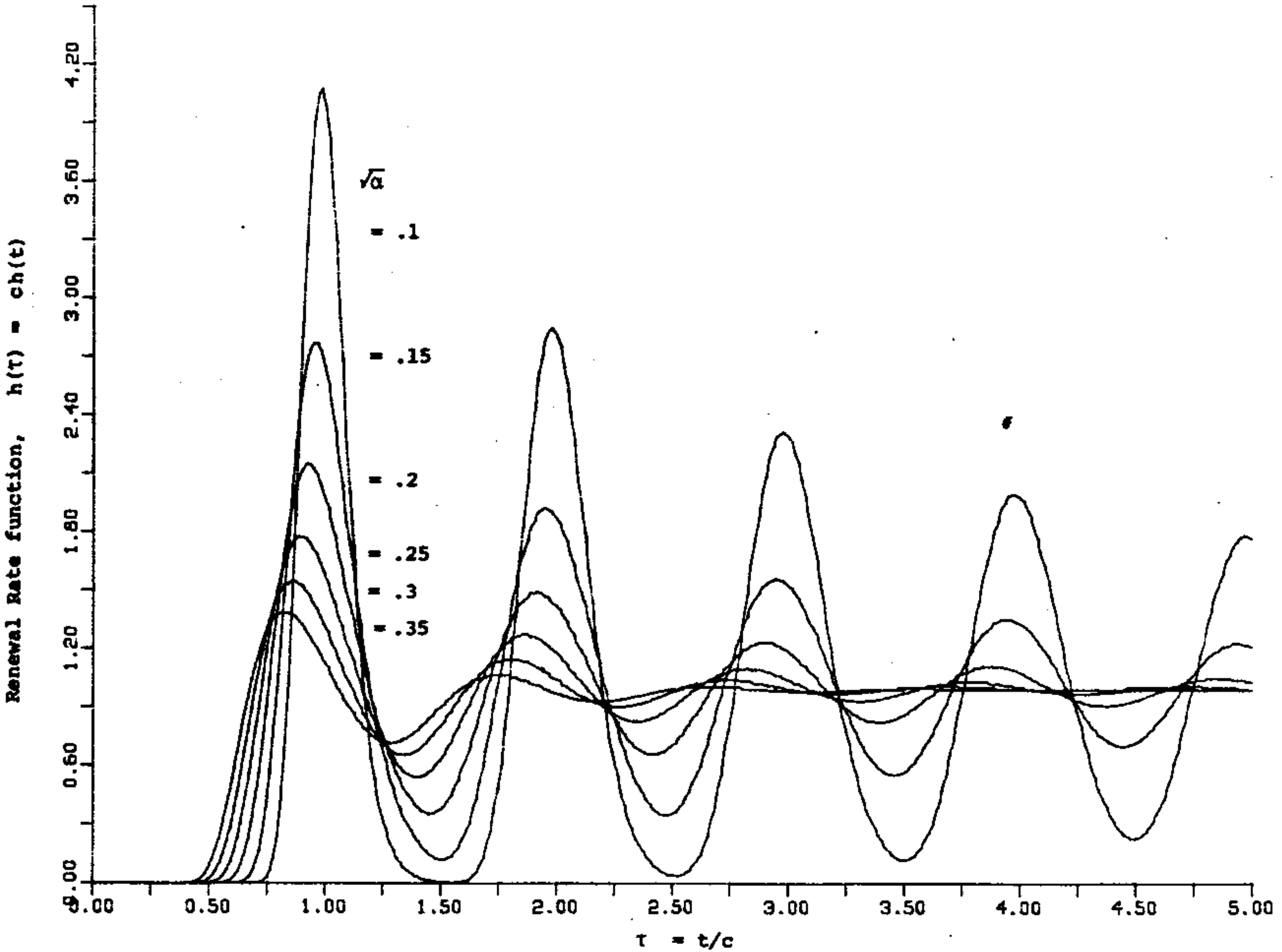

Figure 4. Renewal Rate Function (Renewal Density) of Bernstein Model

TABLE 1 : CHARACTERISTICS OF BERNSTEIN'S DISTRIBUTION

1. **Probability Density Function:**

$$f(t) = \frac{C}{\sqrt{2\pi\alpha}\, t^2} \exp\left\{-\frac{1}{2}\left[\frac{t-C}{\sqrt{\alpha}\, t}\right]^2\right\}$$

$-\infty < t < \infty, \quad C, \alpha > 0$

for $\sqrt{\alpha} < .3 \qquad 0 < t < -\infty$

2. **Cumulative Distribution Function:**†

$$F(t) = \Phi\left\{\frac{1}{\sqrt{\alpha}}\left(1 - \frac{C}{t}\right)\right\}$$

3. **Reliability Function:**

$$R(t) = 1 - \Phi\left\{\frac{1}{\sqrt{\alpha}}\left(1 - \frac{C}{t}\right)\right\} = \Phi\left\{\frac{1}{\sqrt{\alpha}}\left(\frac{C}{t} - 1\right)\right\}$$

4. **Hazard Rate:**

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{C \exp\left[-\frac{1}{2}\left(\frac{t-C}{\sqrt{\alpha}\, t}\right)^2\right]}{\sqrt{2\pi\alpha}\, t^2\, \Phi\left\{\frac{1}{\sqrt{\alpha}}\left(\frac{C}{t} - 1\right)\right\}}$$

5. **Median:** $\bar{T}_M = C$

6. **Harmonic Mean:** $\bar{\bar{T}}_h = \bar{T}_M = C$

7. **Mean Value:** $\bar{T} \cong C, \quad$ for $\sqrt{\alpha} < .35$

8. **Variance:** $V(T) = \sigma^2 \simeq \sqrt{\alpha}\, C^2, \quad$ for $\sqrt{\alpha} < .35$

9. **Coefficient of Variation:** $K = \sqrt{\alpha}, \quad$ for $\sqrt{\alpha} < .35$

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† $\Phi(Z) = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} Z^2\right) dZ$, and $\Phi(-Z) = 1 - \Phi(Z)$

TABLE 2. Renewal Function H(T) = H(t), for Bernstein Model

τ = t/c	$\sqrt{\alpha}$ = .1	$\sqrt{\alpha}$ = .15	$\sqrt{\alpha}$ = .2	$\sqrt{\alpha}$ = .25	$\sqrt{\alpha}$ = .3	$\sqrt{\alpha}$ = .35
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.05	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.15	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.30	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.35	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.40	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
0.45	0.00000	0.00000	0.00000	0.00000	0.00002	0.00024
0.50	0.00000	0.00000	0.00000	0.00003	0.00043	0.00214
0.55	0.00000	0.00000	0.00002	0.00053	0.00319	0.00970
0.60	0.00000	0.00000	0.00043	0.00383	0.01313	0.02841
0.65	0.00000	0.00017	0.00355	0.01563	0.03634	0.06197
0.70	0.00001	0.00214	0.01606	0.04324	0.07656	0.11038
0.75	0.00043	0.01313	0.04779	0.09121	0.13326	0.17045
0.80	0.00621	0.04779	0.10565	0.15865	0.20233	0.23752
0.85	0.03881	0.11970	0.18879	0.24013	0.27819	0.30706
0.90	0.13326	0.22942	0.28926	0.32836	0.35555	0.37545
0.95	0.29933	0.36284	0.39621	0.41663	0.43037	0.44024
1.00	0.50000	0.50000	0.50000	0.50000	0.50000	0.50003
1.05	0.68302	0.62455	0.59409	0.57553	0.56307	0.55424
1.10	0.81834	0.72776	0.67528	0.64193	0.61912	0.60294
1.15	0.90394	0.80773	0.74285	0.69908	0.66838	0.64671
1.20	0.95221	0.86674	0.79767	0.74758	0.71158	0.68656
1.25	0.97725	0.90879	0.84135	0.78849	0.74984	0.72381
1.30	0.98949	0.93803	0.87578	0.82317	0.78469	0.75995
1.35	0.99524	0.95804	0.90289	0.85336	0.81787	0.79643
1.40	0.99786	0.97162	0.92465	0.88111	0.85122	0.83450
1.45	0.99904	0.98090	0.94329	0.90871	0.88642	0.87505
1.50	0.99957	0.98770	0.96142	0.93846	0.92478	0.91855
1.55	0.99983	0.99410	0.98202	0.97236	0.96711	0.96504
1.60	1.00011	1.00300	1.00815	1.01183	1.01364	1.01422
1.65	1.00131	1.01843	1.04237	1.05753	1.06409	1.06553
1.70	1.00626	1.04504	1.08628	1.10929	1.11778	1.11827
1.75	1.02166	1.08686	1.14013	1.16626	1.17377	1.17170
1.80	1.05803	1.14588	1.20288	1.22709	1.23102	1.22513
1.85	1.12573	1.22118	1.27239	1.29018	1.28848	1.27800
1.90	1.22830	1.30905	1.34593	1.35390	1.34525	1.32990
1.95	1.35840	1.40388	1.42061	1.41675	1.40062	1.38061
2.00	1.49995	1.49954	1.49377	1.47750	1.45414	1.43009
2.05	1.63488	1.59059	1.56325	1.53527	1.50560	1.47847
2.10	1.74963	1.67300	1.62751	1.58958	1.55506	1.52599
2.15	1.83807	1.74445	1.68565	1.64032	1.60279	1.57295
2.20	1.90070	1.80416	1.73744	1.68776	1.64923	1.61970
2.25	1.94194	1.85252	1.78316	1.73248	1.69489	1.66655
2.30	1.96745	1.89073	1.82364	1.77529	1.74033	1.71376
2.35	1.98240	1.92049	1.86011	1.81711	1.78606	1.76152
2.40	1.99079	1.94383	1.89411	1.85891	1.83249	1.80993
2.45	1.99535	1.96306	1.92737	1.90156	1.87991	1.85900
2.50	1.99792	1.98075	1.96161	1.94576	1.92847	1.90868

TABLE 2 (continued)

τ = t/c	$\sqrt{\alpha}$ = .1	$\sqrt{\alpha}$ = .15	$\sqrt{\alpha}$ = .2	$\sqrt{\alpha}$ = .25	$\sqrt{\alpha}$ = .3	$\sqrt{\alpha}$ = .35
2.50	1.99792	1.98075	1.96161	1.94576	1.92847	1.90868
2.55	1.99997	1.99975	1.99840	1.99198	1.97818	1.95884
2.60	2.00330	2.02301	2.03894	2.04043	2.02890	2.00934
2.65	2.01081	2.05321	2.08398	2.09106	2.08043	2.06001
2.70	2.02701	2.09245	2.13373	2.14360	2.13249	2.11071
2.75	2.05759	2.14181	2.18789	2.19761	2.18480	2.16131
2.80	2.10795	2.20117	2.24574	2.25253	2.23706	2.21172
2.85	2.18093	2.26917	2.30619	2.30778	2.28904	2.26188
2.90	2.27510	2.34348	2.36802	2.36279	2.34057	2.31179
2.95	2.38447	2.42117	2.42993	2.41708	2.39153	2.36147
3.00	2.49993	2.49911	2.49075	2.47029	2.44192	2.41097
3.05	2.61170	2.57445	2.54948	2.52220	2.49175	2.46035
3.10	2.71176	2.64483	2.60542	2.57276	2.54114	2.50970
3.15	2.79520	2.70862	2.65818	2.62205	2.59023	2.55908
3.20	2.86045	2.76494	2.70771	2.67029	2.63915	2.60854
3.25	2.90860	2.81366	2.75429	2.71776	2.68807	2.65812
3.30	2.94232	2.85529	2.79845	2.76484	2.73711	2.70785
3.35	2.96486	2.89091	2.84094	2.81187	2.78639	2.75774
3.40	2.97941	2.92208	2.88263	2.85918	2.83596	2.80776
3.45	2.98877	2.95069	2.92442	2.90703	2.88585	2.85790
3.50	2.99546	2.97884	2.96712	2.95562	2.93605	2.90811
3.55	3.00197	3.00864	3.01142	3.00502	2.98652	2.95837
3.60	3.01118	3.04205	3.05778	3.05523	3.03718	3.00865
3.65	3.02653	3.08061	3.10643	3.10617	3.08796	3.05890
3.70	3.05189	3.12532	3.15736	3.15767	3.13877	3.10912
3.75	3.09090	3.17652	3.21030	3.20954	3.18953	3.15929
3.80	3.14607	3.23383	3.26484	3.26155	3.24020	3.20941
3.85	3.21779	3.29626	3.32040	3.31349	3.29072	3.25949
3.90	3.30392	3.36232	3.37635	3.36518	3.34107	3.30952
3.95	3.39996	3.43021	3.43209	3.41648	3.39126	3.35954
4.00	3.49990	3.49808	3.48706	3.46730	3.44130	3.40955
4.05	3.59742	3.56417	3.54081	3.51762	3.49123	3.45956
4.10	3.68707	3.62698	3.59305	3.56745	3.54108	3.50960
4.15	3.76506	3.68546	3.64368	3.61688	3.59091	3.55966
4.20	3.82950	3.73898	3.69273	3.66601	3.64075	3.60976
4.25	3.88029	3.78744	3.74042	3.71497	3.69063	3.65989
4.30	3.91864	3.83121	3.78708	3.76390	3.74058	3.71006
4.35	3.94660	3.87107	3.83310	3.81292	3.79062	3.76025
4.40	3.96661	3.90814	3.87894	3.86213	3.84075	3.81046
4.45	3.98128	3.94380	3.92503	3.91161	3.89096	3.86068
4.50	3.99334	3.97949	3.97174	3.96139	3.94123	3.91090
4.55	4.00567	4.01660	4.01936	4.01147	3.99155	3.96112
4.60	4.02136	4.05636	4.06808	4.06182	4.04189	4.01133
4.65	4.04356	4.09967	4.11794	4.11239	4.09223	4.06153
4.70	4.07526	4.14707	4.16889	4.16310	4.14254	4.11172
4.75	4.11877	4.19865	4.22075	4.21387	4.19282	4.16191
4.80	4.17524	4.25407	4.27327	4.26463	4.24305	4.21208
4.85	4.24429	4.31263	4.32616	4.31530	4.29324	4.26225
4.90	4.32385	4.37336	4.37911	4.36583	4.34337	4.31241
4.95	4.41048	4.43509	4.43182	4.41618	4.39347	4.36258
5.00	4.49986	4.49664	4.48403	4.46635	4.44354	4.41275

TABLE 3. Renewal Rate (Renewal Density) Function for Bernstein Model, $h(\tau) = ch(t)$

$\tau = t/c$	$\sqrt{\alpha} = .1$	$\sqrt{\alpha} = .15$	$\sqrt{\alpha} = .2$	$\sqrt{\alpha} = .25$	$\sqrt{\alpha} = .3$	$\sqrt{\alpha} = .35$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.05	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.15	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.25	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.30	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.35	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.40	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
0.45	0.00000	0.00000	0.00000	0.00000	0.00003	0.00073
0.50	0.00000	0.00000	0.00000	0.00005	0.00163	0.01266
0.55	0.00000	0.00000	0.00003	0.00214	0.02056	0.07696
0.60	0.00000	0.00000	0.00153	0.02491	0.10664	0.24517
0.65	0.00000	0.00038	0.02142	0.12662	0.31272	0.51606
0.70	0.00000	0.01002	0.12591	0.37135	0.62864	0.82614
0.75	0.00084	0.09162	0.40980	0.74924	0.97821	1.09916
0.80	0.02742	0.40027	0.88423	1.16629	1.27521	1.28754
0.85	0.27387	1.03621	1.42694	1.51231	1.46829	1.37998
0.90	1.16367	1.84259	1.87060	1.72161	1.54815	1.38932
0.95	2.65668	2.49565	2.11044	1.78481	1.53291	1.33807
1.00	3.84866	2.77101	2.13499	1.72941	1.45098	1.24894
1.05	3.98943	2.65962	1.99472	1.59577	1.32987	1.14076
1.10	3.23070	2.29382	1.75871	1.42140	1.19146	1.02801
1.15	2.18109	1.82927	1.48674	1.23454	1.05153	0.92205
1.20	1.28853	1.37796	1.21936	1.05364	0.92142	0.83226
1.25	0.69085	0.99629	0.97894	0.88993	0.81015	0.76619
1.30	0.34557	0.69980	0.77477	0.75072	0.72559	0.72882
1.35	0.16469	0.48195	0.60898	0.64251	0.67413	0.72158
1.40	0.07598	0.32784	0.48184	0.57209	0.65932	0.74195
1.45	0.03436	0.22228	0.39601	0.54562	0.68041	0.78400
1.50	0.01538	0.15477	0.35818	0.56603	0.73207	0.83968
1.55	0.00693	0.12403	0.37696	0.63060	0.80518	0.90037
1.60	0.00409	0.14137	0.45756	0.73027	0.88867	0.95821
1.65	0.00988	0.22833	0.59627	0.85095	0.97149	1.00707
1.70	0.04667	0.40503	0.77812	0.97636	1.04426	1.04297
1.75	0.17365	0.67337	0.97906	1.09111	1.10031	1.06414
1.80	0.47873	1.00625	1.17156	1.18311	1.13596	1.07077
1.85	1.01321	1.35139	1.33099	1.24483	1.15035	1.06466
1.90	1.70827	1.64813	1.44025	1.27347	1.14505	1.04872
1.95	2.36928	1.84741	1.49173	1.27025	1.12341	1.02653
2.00	2.77858	1.92492	1.48660	1.23950	1.09000	1.00182
2.05	2.82100	1.88323	1.43263	1.18753	1.05001	0.97805
2.10	2.53009	1.74523	1.34138	1.12181	1.00867	0.95802
2.15	2.03975	1.54370	1.22593	1.05022	0.97073	0.94368
2.20	1.50052	1.31197	1.09919	0.98043	0.94003	0.93599
2.25	1.02037	1.07771	0.97317	0.91927	0.91914	0.93498
2.30	0.64865	0.86072	0.85860	0.87226	0.90924	0.93986
2.35	0.38925	0.67353	0.76477	0.84304	0.91013	0.94928
2.40	0.22241	0.52387	0.69913	0.83312	0.92043	0.96156
2.45	0.12214	0.41747	0.66673	0.84184	0.93785	0.97495
2.50	0.06623	0.36000	0.66946	0.86655	0.95962	0.98781
	0.04129	0.35719	0.70564	0.90311	0.98284	0.99884

TABLE 3 (continued)

$\tau = t/c$	$\sqrt{\alpha} = .1$	$\sqrt{\alpha} = .15$	$\sqrt{\alpha} = .2$	$\sqrt{\alpha} = .25$	$\sqrt{\alpha} = .3$	$\sqrt{\alpha} = .35$
2.50	0.04129	0.35719	0.70564	0.90311	0.98284	0.99884
2.55	0.04639	0.41295	0.76997	0.94647	1.00483	1.00715
2.60	0.09614	0.52633	0.85415	0.99138	1.02337	1.01228
2.65	0.21947	0.68898	0.94804	1.03296	1.03692	1.01423
2.70	0.44800	0.88437	1.04098	1.06719	1.04468	1.01336
2.75	0.79415	1.08971	1.12321	1.09126	1.04658	1.01028
2.80	1.23016	1.27991	1.18695	1.10370	1.04319	1.00579
2.85	1.68435	1.43225	1.22714	1.10441	1.03558	1.00069
2.90	2.06213	1.53039	1.24161	1.09448	1.02516	0.99576
2.95	2.28153	1.56656	1.23100	1.07598	1.01346	0.99160
3.00	2.30336	1.54169	1.19836	1.05168	1.00196	0.98865
3.05	2.14051	1.46387	1.14857	1.02465	0.99190	0.98712
3.10	1.84558	1.34592	1.08774	0.99797	0.98424	0.98703
3.15	1.48705	1.20282	1.02256	0.97438	0.97953	0.98821
3.20	1.12699	1.04973	0.95969	0.95604	0.97793	0.99037
3.25	0.80818	0.90070	0.90514	0.94434	0.97923	0.99316
3.30	0.55156	0.76805	0.86376	0.93987	0.98294	0.99620
3.35	0.36091	0.66212	0.83886	0.94237	0.98835	0.99914
3.40	0.23042	0.59106	0.83187	0.95091	0.99466	1.00168
3.45	0.15239	0.56045	0.84235	0.96398	1.00108	1.00363
3.50	0.12323	0.57264	0.86808	0.97977	1.00689	1.00489
3.55	0.14671	0.62614	0.90544	0.99637	1.01154	1.00545
3.60	0.23318	0.71528	0.94983	1.01195	1.01467	1.00539
3.65	0.39405	0.83048	0.99625	1.02500	1.01614	1.00483
3.70	0.63282	0.95919	1.03988	1.03439	1.01601	1.00395
3.75	0.93641	1.08745	1.07652	1.03951	1.01449	1.00293
3.80	1.27164	1.20170	1.10297	1.04024	1.01194	1.00193
3.85	1.59006	1.29048	1.11732	1.03693	1.00876	1.00109
3.90	1.84005	1.34582	1.11900	1.03031	1.00539	1.00049
3.95	1.98117	1.36389	1.10873	1.02140	1.00223	1.00020
4.00	1.99487	1.34503	1.08838	1.01134	0.99959	1.00021
4.05	1.88755	1.29334	1.06065	1.00130	0.99771	1.00050
4.10	1.68589	1.21577	1.02884	0.99230	0.99668	1.00100
4.15	1.42737	1.12119	0.99638	0.98519	0.99652	1.00164
4.20	1.15016	1.01941	0.96656	0.98051	0.99712	1.00233
4.25	0.88573	0.92029	0.94218	0.97851	0.99833	1.00299
4.30	0.65546	0.83297	0.92533	0.97913	0.99994	1.00357
4.35	0.47112	0.76517	0.91716	0.98204	1.00170	1.00402
4.40	0.33785	0.72262	0.91791	0.98671	1.00342	1.00432
4.45	0.25798	0.70858	0.92688	0.99248	1.00489	1.00446
4.50	0.23400	0.72353	0.94261	0.99863	1.00599	1.00446
4.55	0.26966	0.76511	0.96305	1.00448	1.00665	1.00435
4.60	0.36839	0.82833	0.98582	1.00944	1.00684	1.00417
4.65	0.52986	0.90613	1.00847	1.01309	1.00659	1.00396
4.70	0.74598	0.99017	1.02874	1.01517	1.00600	1.00374
4.75	0.99841	1.07174	1.04476	1.01560	1.00516	1.00355
4.80	1.25931	1.14274	1.05522	1.01451	1.00419	1.00342
4.85	1.49555	1.19655	1.05946	1.01216	1.00321	1.00335
4.90	1.67538	1.22859	1.05749	1.00890	1.00233	1.00334
4.95	1.77526	1.23674	1.04993	1.00517	1.00163	1.00339
5.00	1.78461	1.22132	1.03793	1.00140	1.00116	1.00349