King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 030

November1981

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Spectral Representation of the Love Wave Operator for Two Layers over a Half-space

By M.H. Kazi and A.S.M. Abu-Safiya

Summary: In this paper we use the method described in Kazi (1976) to determine the spectral representation of the two-dimensional Love wave operator associated with the propagation of monochromatic SH waves in a structure consisting of two uniform layers overlying a uniform half-space. The representation is useful in tackling a class of Love wave diffraction problems in horizontally discontinuous structures involving three-layered models.

1. Introduction: In a series of papers (Kazi (1978a, b), Kazi (1979), Niazy and Kazi (1980)), the authors use a method, based upon an integral equation formulation together with the application of Schwinger-Levine variational principle, to investigate the two-dimensional problems of the propagation of plane, harmonic, monochromatic Love waves, incident normally upon the plane of discontinuity in laterally discontinuous structures involving step-wise change in surface topography or change in material properties. Diffraction of Love waves is described by means of a scattering matrix and approximate expressions for its elements are sought through the variational principle. Reflection and transmission coefficients are then obtained through a transmission matrix related to the scattering matrix. The method has the advantage that it takes into account the body-wave contributions. However, the method pre-supposes the existence of a complete set of proper or improper eigenfunctions, in terms of which the displacements on either side of the discontinuity may be expressed. In order to accomplish

this Kazi (1976) gave a general method for finding the spectral representation of the two-dimensional Love wave operator associated with the propagation of monochromatic SH waves in a laterally-uniform layered strip or half-space. However, specific spectral representations were found only for two-layer models. Recently, Kennett (1981) has discussed the spectral representation of the elastodynamic operator associated with coupled seismic waves.

In this paper, we follow the same procedure as in Kazi (1976) to determine the explicit spectral representation of the Love wave operator associated with monochromatic SH-waves for a three-layer model comprising two homogeneous, infinite strips overlying a uniform half-space. This representation will find usage in tackling a class of Love wave diffraction problems associated with three-layer models by the method described above.

2. Equations of motion: We wish to represent the two-dimensional motion of a laterally homogeneous structure consisting of two uniform layers over a uniform half-space in a general way; the motion will consist of waves propagating along the direction of the x-axis in the coordinate system shown in Fig. 1. We consider a layer of infinite depth, rigidity μ_3, shear velocity β_3 and density ρ_3, overlaid by two infinite strips, consisting of a layer of finite depth H_2, density ρ_2, rigidity μ_2 $(< \mu_3)$ and shear velocity β_2 $(< \beta_3)$, and another layer of depth H_1 $(< H_2)$, density ρ_1, rigidity μ_1 $(< \mu_2)$ and shear velocity β_1 $(< \beta_2)$ (see Fig. 1). We suppose the density and the rigidity of each layer to be constant, and the top plane surface to be stress-free.

We choose the axes in such a way that the upper free surface coincides with the plane $z = - H_1$ and the xy-plane coincides with the plane of welded contact between the two upper layers as shown in Fig. 1.

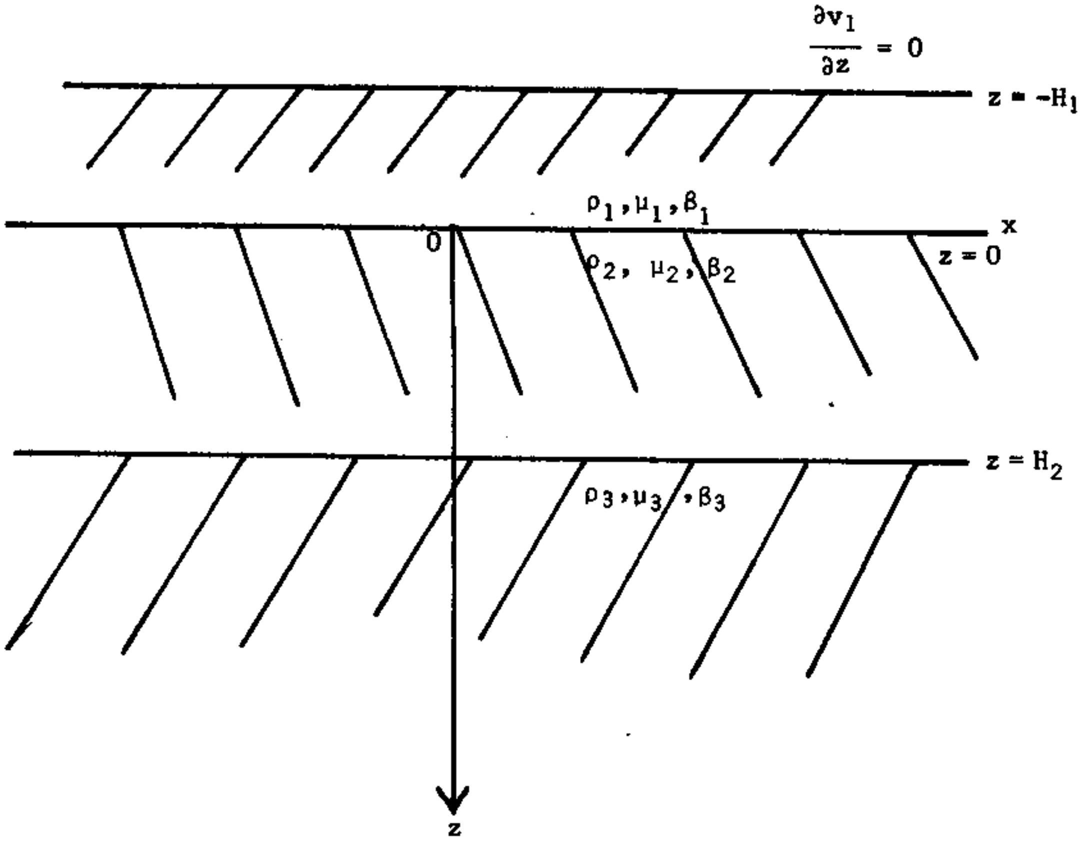

Figure 1. The geometry of the problem.

We shall consider horizontally polarized shear waves only, which means that there are no displacements in the x and z directions and the motion is in the y-direction only. Let $v(x, z, t)$ be the y-component of displacement. It must satisfy the differential equation

$$\rho(z) \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} (\mu(z) \frac{\partial v}{\partial x}) + \frac{\partial}{\partial z} (\mu(z) \frac{\partial v}{\partial z}), \qquad (2.1)$$

where

$$\begin{aligned} \mu(z) &= \mu_1, \; -H_1 < z < 0 \\ &= \mu_2, \; 0 < z < H_2 \\ &= \mu_3, \; H_2 < z \end{aligned} \qquad (2.2)$$

and

$$\begin{aligned} \rho(z) &= \rho_1, \; -H_1 < z < 0 \\ &= \rho_2, \; 0 < z < H_2 \\ &= \rho_3, \; H_2 < z \end{aligned} \qquad (2.3)$$

For convenience we label the intervals $\{z : -H_1 < z < 0\}$, $\{z : 0 < z < H_2\}$ and $\{z : z > H_2\}$ as I_1, I_2 and I_3 respectively.

In order to obtain a general representation, we first of all examine harmonic waves, travelling in x-direction with positive real frequency ω and wave numbers k:

$$v(x, z, t) = V(z) \exp [i(\omega t - kx)]. \qquad (2.4)$$

(We shall assume ω to be fixed and choose k to satisfy the propagation conditions).

Equation (2.1) becomes

$$L(V) = \frac{d}{dz}\left(\mu(z)\,\frac{dV}{dz}\right) + (\omega^2\rho(z) - k^2\mu(z))\,V = 0,$$

$$V(z) = V_i(z), \quad z \in I_i, \quad i = 1, 2, 3, \tag{2.5}$$

L being the <u>Love wave operator</u>.

$V_1(z)$, $V_2(z)$ and $V_3(z)$ satisfy the following equations:

$$\frac{d^2V_1}{dz^2} + \sigma_1^2 V_1 = 0, \quad \sigma_1^2 = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right), \ \lambda = k^2, \ \beta_1^2 = \frac{\mu_1}{\rho_1}, \ -H_1 < z < 0 \tag{2.6}$$

$$\frac{d^2V_2}{dz^2} + \sigma_2^2 V_2 = 0, \ \sigma_2^2 = \left(\frac{\omega^2}{\beta_2^2} - \lambda\right), \ \beta_2^2 = \frac{\mu_2}{\rho_2}, \ 0 < z < H_2 \tag{2.7}$$

$$\frac{d^2V_3}{dz^2} - \sigma_3^2 V_3 = 0, \ \sigma_3^2 = \left(\lambda - \frac{\omega^2}{\beta_3^2}\right), \ \beta_3^2 = \frac{\mu_3}{\rho_3}, \ H_2 < z \tag{2.8}$$

with the <u>interface conditions</u>

$$V_1(0) = V_2(0) \tag{2.9}$$

$$\mu_1 V_1'(0) = \mu_2 V_2'(0) \tag{2.10}$$

and

$$V_2(H_2) = V_3(H_2) \tag{2.11}$$

$$\mu_2 V_2'(H_2) = \mu_3 V_3'(H_2) \tag{2.12}$$

and the <u>boundary conditions</u>

$$V_1'(-H_1) = 0, \tag{2.13}$$

$$\int_{-H_1}^{\infty} \mu(z)\,|V(z)|^2 dz < \infty, \tag{2.14}$$

where $\mu(z)$ is given by (2.2) and (') denotes differentiation with respect to z.

The system (2.5), (2.9) - (2.14) is a <u>SINGULAR</u> Sturm-Liouville system with two points of discontinuity and corresponding interface conditions. Such systems have been discussed in detail in Kazi (1976). The boundary condition at infinity (2.14) is taken to be the requirement that the solution must be of finite μ-norm, so as to ensure the uniqueness of the solution as explained in Kazi (1976).

3. Green's function

Let $G(z, \zeta; \lambda)\big|_{z \in I_i, \zeta \in I_j} = G_{ij}$, where $i, j = 1, 2, 3$ (see Fig. 2). G_{ij} determine the Green function $G(z, \zeta; \lambda)$ completely, provided the following conditions are satisfied:

(G_1) $G_{ij}(z, \zeta; \lambda)$ is a continuous function of z for all $z \in I_i$.

(G_2) $G_{ij}(z, \zeta; \lambda)$ $(i \neq j)$ possesses a continuous first order derivative of z at each point of I_i; G_{ij} $(i = j)$ possesses a continuous first order derivative at each point of I_i except $z = \zeta$, where it has a jump discontinuity, given by:

$$G_{ii}'(\zeta^+, \zeta; \lambda) - G_{ii}'(\zeta^-, \zeta; \lambda) = \frac{1}{\mu_i(\zeta)}.$$

(G_3) If $i \neq j$, $L(G_{ij}) = 0$. If $i = j$, $L(G_{ij}) = 0$ for $z \neq \zeta$.

(G_4) $G(z, \zeta; \lambda)$ satisfy the <u>interface conditions</u> (2.9) - (2.12) and the <u>boundary conditions</u> (2.13) - (2.14).

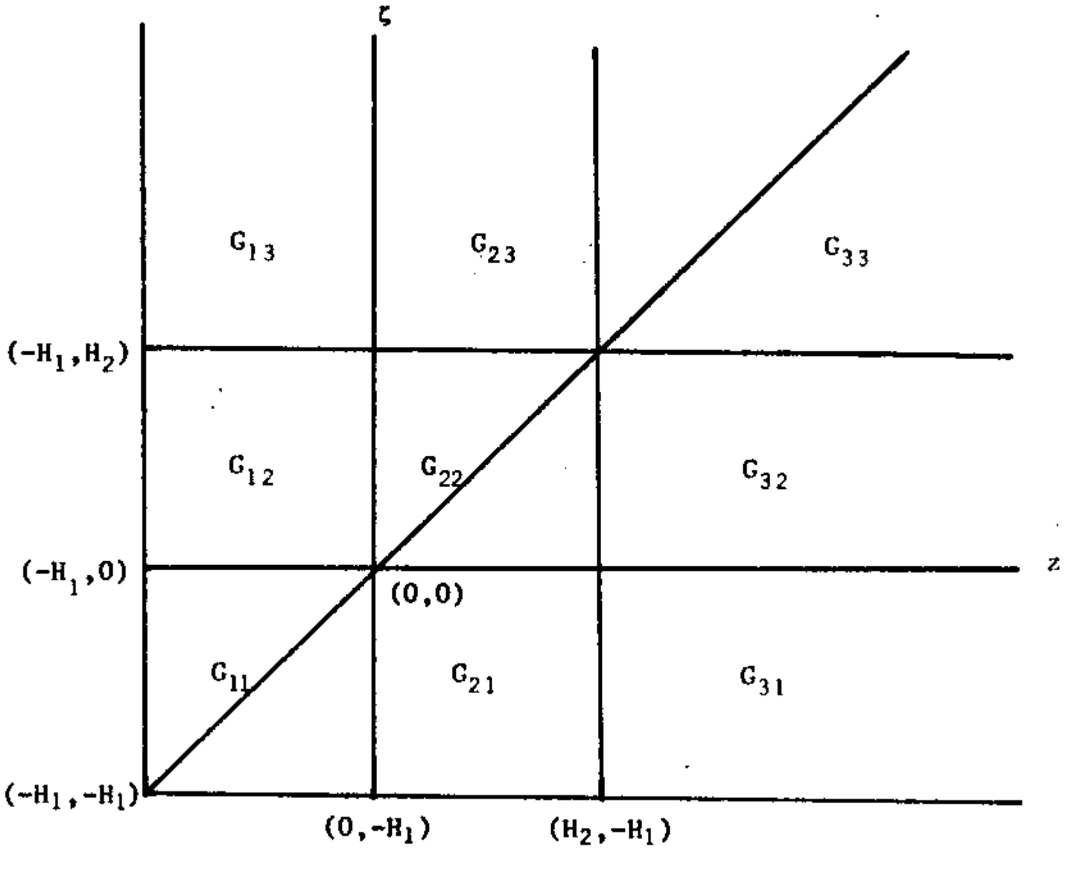

Figure 2. The character of the Green function.

The Green function is unique and symmetric. We now proceed to construct the Green function explicitly.

(i) If $\zeta \in I_1$, then G_{11}, G_{21} and G_{31} satisfy the differential equations:

$$\frac{\partial^2 G_{11}}{\partial z^2} + \sigma_1^2 G_{11} = \delta(z - \zeta) \tag{3.1}$$

$$\frac{\partial^2 G_{21}}{\partial z^2} + \sigma_2^2 G_{21} = 0 \tag{3.2}$$

and

$$\frac{\partial^2 G_{31}}{\partial z^2} - \sigma_3^2 G_{31} = 0 \tag{3.3}$$

together with the following conditions

$$G'_{11} = 0 \quad \text{at} \quad z = -H_1 \tag{3.4a}$$

$$G_{11} = G_{21} \quad \text{at} \quad z = 0 \tag{3.4b}$$

$$\mu_1 G'_{11} = \mu_2 G'_{21} \quad \text{at} \quad z = 0 \tag{3.4c}$$

$$G_{21} = G_{31} \quad \text{at} \quad z = H_2 \tag{3.4d}$$

$$\mu_2 G'_{21} = \mu_3 G'_{31} \quad \text{at} \quad z = H_2 \tag{3.4e}$$

$$G_{11}(\zeta^+, \zeta \,;\, \lambda) = G_{11}(\zeta^-, \zeta \,;\, \lambda) \tag{3.4f}$$

$$G'_{11}(\zeta^+, \zeta; \lambda) - G'_{11}(\zeta^-, \zeta; \lambda) = \frac{1}{\mu_1} \tag{3.4g}$$

and

$$\int_{-H_1}^{\infty} \mu(z) |G(z)|^2 dz < \infty. \tag{3.4h}$$

After considerable effort, we find

$$G_{11} = \frac{\mu_2\sigma_2\cos\sigma_1(\zeta+H_1)\cos\sigma_1(z+H_1)}{\Delta\cos^2\sigma_1 H_1}\left(1 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\tan\sigma_2 H_2\right)$$

$$+ \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1}\{\cos\sigma_1(z+H_1)\sin(\sigma_1\zeta)\cdot\theta(\zeta-z)$$

$$+ \cos\sigma_1(\zeta+H_1)\sin(\sigma_1 z)\cdot\theta(z-\zeta)\}, \quad z \in I_1 \tag{3.5}$$

where

$$\Delta = \mu_1\sigma_1\mu_2\sigma_2\tan(\sigma_1 H_1) + \mu_1\sigma_1\mu_3\sigma_3\tan(\sigma_2 H_2)\tan(\sigma_1 H_1)$$

$$- \mu_3\sigma_3\mu_2\sigma_2 + \mu_2^2\sigma_2^2\tan(\sigma_2 H_2), \tag{3.6}$$

and

$$\theta(\zeta - z) = 1, \quad \zeta > z$$

$$= 0, \quad \zeta < z$$

is the Heaviside unit function;

$$G_{21} = \frac{\cos\sigma_1(\zeta+H_1)}{\Delta\cos(\sigma_2 H_2)\cos(\sigma_1 H_1)}\{\mu_2\sigma_2\cos(\sigma_2(z-H_2))$$

$$- \mu_3\sigma_3\sin(\sigma_2(z-H_2))\}, \quad z \in I_2, \tag{3.7}$$

and

$$G_{31} = \frac{\mu_2\sigma_2\cos(\sigma_1(\zeta+H_1))}{\Delta\cos(\sigma_2 H_2)\cos(\sigma_1 H_1)}\, e^{-\sigma_3(z-H_2)}, \quad z \in I_3. \tag{3.8}$$

(ii) If $\zeta \in I_2$, then G_{12}, G_{22} and G_{32} satisfy the differential equations

$$\frac{\partial^2 G_{12}}{\partial z^2} + \sigma_1^2 G_{12} = 0 \tag{3.9}$$

$$\frac{\partial^2 G_{22}}{\partial z^2} + \sigma_2^2 G_{22} = \delta(z - \zeta) \tag{3.10}$$

and

$$\frac{\partial^2 G_{32}}{\partial z^2} - \sigma_3^2 G_{32} = 0 \tag{3.11}$$

together with the conditions (3.4a) - (3.4h) suitably modified. We obtain

$$G_{12} = \frac{\cos\{\sigma_1(z+H_1)\}\cdot B(\zeta)}{\Delta\cos(\sigma_1 H_1)\cos(\sigma_2 H_2)}, \quad z \in I_1 \tag{3.12}$$

where Δ is given by (3.6) and

$$B(\zeta) = \mu_2\sigma_2\cos(\sigma_2(\zeta-H_2)) - \mu_3\sigma_3\sin(\sigma_2(\zeta-H_2)), \tag{3.13}$$

$$G_{22} = \frac{B(\zeta)\ B(z)}{M\Delta\cos^2(\sigma_2 H_2)} - \left\{\frac{B(\zeta)\sin(\sigma_2 z)\cdot\theta(\zeta-z) + B(z)\cdot\sin(\sigma_2\zeta)\cdot\theta(z-\zeta)}{M\mu_2\sigma_2\cos(\sigma_2 H_2)}\right\}, \quad z \in I_2 \tag{3.14}$$

where Δ, $B(z)$ are given by (3.6), (3.13) and

$$M = \mu_2\sigma_2 + \mu_3\sigma_3\tan\sigma_2 H_2, \tag{3.15}$$

$$G_{32} = \frac{1}{\Delta\cos\sigma_2 H_2}\{\mu_2\sigma_2\cos(\sigma_2\zeta) - \mu_1\sigma_1\tan(\sigma_1 H_1)\sin(\sigma_2\zeta)\}\cdot e^{-\sigma_3(z-H_2)}, \quad z \in I_3 \tag{3.16}$$

(iii) If $\zeta \in I_3$, then G_{13}, G_{23} and G_{33} satisfy the differential equations

$$\frac{\partial^2 G_{13}}{\partial z^2} + \sigma_1^2 G_{13} = 0 \tag{3.17}$$

$$\frac{\partial^2 G_{23}}{\partial z^2} + \sigma_1^2 G_{23} = 0 \tag{3.18}$$

and

$$\frac{\partial^2 G_{33}}{\partial z^2} - \sigma_3^2 G_{33} = \delta(z - \zeta) \tag{3.19}$$

together with the conditions (3.4a) - (3.4h) suitably modified.

We obtain

$$G_{13} = \frac{\mu_2\sigma_2\cos\{\sigma_1(z+H_1)\}}{\Delta\cos(\sigma_1 H_1)\cos(\sigma_2 H_2)}\, e^{-\sigma_3(\zeta-H_2)}, \quad z \in I_1, \tag{3.20}$$

$$G_{23} = \frac{B(z)\mu_2\sigma_2 e^{-\sigma_2(\zeta-H_2)}}{M\Delta\cos^2(\sigma_2 H_2)} - \frac{e^{-\sigma_3(\zeta-H_2)}}{M\cos\sigma_2 H_2}, \quad z \in I_2 \tag{3.21}$$

and

$$G_{33} = \frac{\mu_2^2\sigma_2^2 e^{-\sigma_3(\zeta-H_2)}\; e^{-\sigma_3(z-H_2)}}{M\Delta\cos^2(\sigma_2 H_2)}$$

$$- \frac{e^{-\sigma_3(\zeta-H_2)}\; e^{-\sigma_3(z-H_2)}\,\{\mu_3\sigma_3\tan(\sigma_2 H_2) - \mu_2\sigma_2\}}{2\mu_3\sigma_3 M}$$

$$- \left\{\frac{e^{-\sigma_3(\zeta-z)}}{2\mu_3\sigma_3}\cdot\theta(\zeta - z) + \frac{e^{-\sigma_3(z-\zeta)}}{2\mu_3\sigma_3}\cdot\theta(z-\zeta)\right\}, \quad z \in I_3, \tag{3.22}$$

where Δ, $B(z)$ and M are given by (3.6), (3.13) and (3.15) respectively.

We note that $G_{ij}(z, \zeta; \lambda) = G_{ji}(\zeta, z; \lambda)$, $i, j = 1, 2, 3$ i.e. the Green function is symmetric.

4. Spectral representation: The essential step in obtaining the spectral representation is to integrate the Green function $G(z, \zeta; \lambda)$ obtained in the previous section around a large circle $|\lambda| = R$ in the complex λ-plane. The Green function has, in addition to simple poles, a branch-point singularity. The spectrum is the disjoint union of the point-spectrum, giving rise to proper eigenfunctions, and the continuous spectrum, which yields improper eigenfunctions. The continuous spectrum will be the set of points on the branch-cut along a portion of the real axis and the discrete spectrum will be the set of poles lying on the real axis. The sum of residues at the poles and the contribution from the branch-cut yield the representation of the delta function in terms of proper eigenfunctions $\{\phi^{(n)}(z)\}$ and improper eigenfunctions $\{\psi(z, \lambda)\}$ (see Kazi (1976)):

$$\lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G(z, \zeta; \lambda)d\lambda = \Sigma_n \phi^{(n)}(z)\overline{\phi^{(n)}(\zeta)} + \int \psi(z,\lambda)\overline{\psi(\zeta,\lambda)}d\lambda$$

$$= \frac{\delta(z-\zeta)}{\mu(\zeta)} \qquad (4.1)$$

(i) First, we consider

$$I_{11} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{11}(z, \zeta; \lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} [M \frac{\cos\{\sigma_1(\zeta+H_1)\}\cos\{\sigma_1(z+H_1)\}}{\Delta\cos^2\sigma_1 H_1}$$

$$+ \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1} \{\cos(\sigma_1(z+H_1))\sin(\sigma_1\zeta)\, \theta(\zeta-z)$$

$$+ \cos(\sigma_1(\zeta+H_1))\sin(\sigma_1 z)\theta(z-\zeta)\}]d\lambda \quad \text{(using 3.5)}, \qquad (4.2)$$

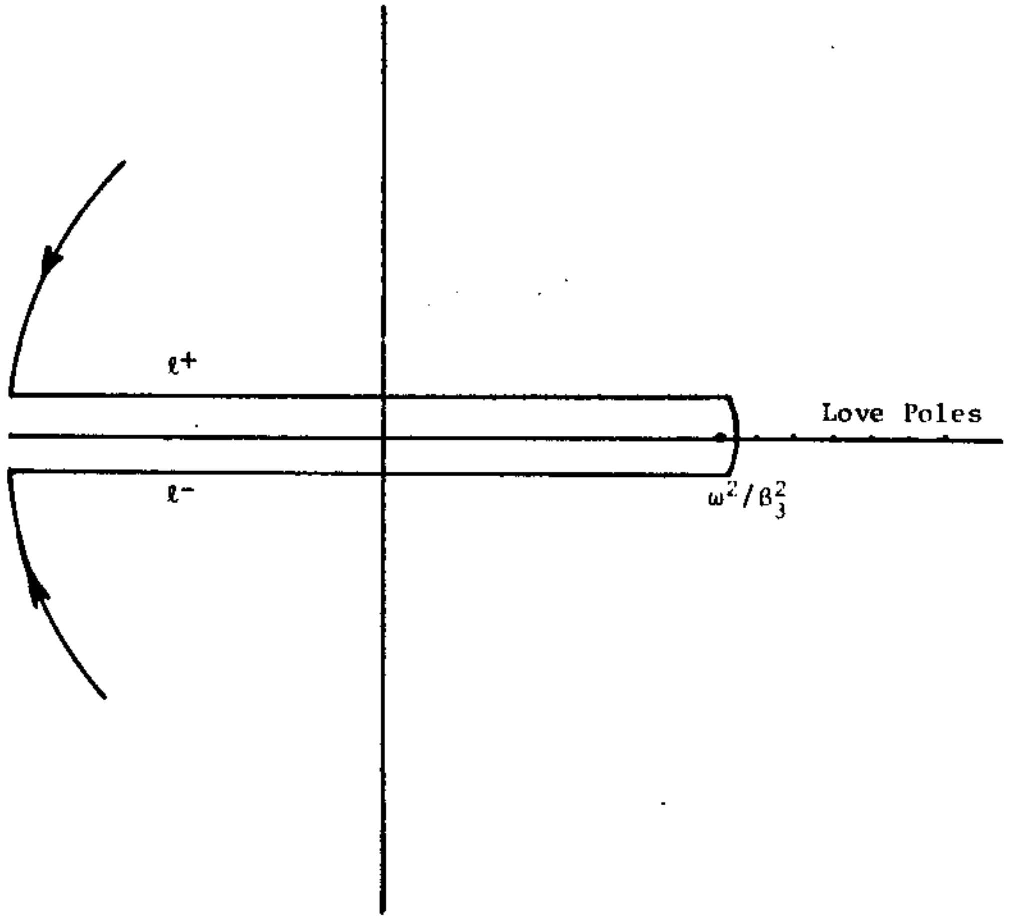

Figure 3. The contour of integration in the complex λ-plane.

where Δ and M are given by (3.6) and (3.15) respectively. We note that $\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (4.2) and the poles are the roots of

$$\Delta = \mu_1\sigma_1\mu_2\sigma_2\tan(\sigma_1 H_1) + \mu_1\sigma_1\mu_3\sigma_3\tan(\sigma_2 H_2)\tan(\sigma_1 H_1)$$

$$- \mu_3\sigma_3\mu_2\sigma_2 + \mu_2^2\sigma_2^2\tan(\sigma_2 H_2) = 0 \qquad (4.3)$$

which is the <u>dispersion equation</u> for Love wave propagation in two layers over a half-space (see Ewing et al. 1957 p. 229). The poles are all simple, finite in number and are located in the interval $(\omega^2/\beta_3^2, \omega^2/\beta_1^2]$. The continuous spectrum is related to the integral over the branch-lines ℓ^+, ℓ^- in the complex λ-plane, and the path of integration is shown in Fig. 3. These remarks are valid for all the integrands we shall encounter. We assume that $\mathrm{Re}(\sigma_3) > 0$ for $I(\lambda) \neq 0$. This means that on the branch-line ℓ^+, $\sigma_3 = is_3$ and on ℓ^-, $\sigma_3 = -is_3$, where $s_3 = (\omega^2/\beta_3^2 - \lambda)^{\frac{1}{2}}$ is real and positive for $\lambda < \omega^2/\beta_3^2$.

Let

$$\gamma_1 = \frac{M}{\Delta} = \frac{\mu_2\sigma_2 + \mu_3\sigma_3\tan(\sigma_2 H_2)}{\Delta}.$$

Then

$$\gamma_1^+ - \gamma_1^- = 2i\, I(\gamma_1^+) = \frac{2i\mu_2^2\sigma_2^2\mu_3 s_3 \sec^2(\sigma_2 H_2)}{p^2 + q^2} \qquad (4.4)$$

where

$$p = \mu_1\sigma_1\mu_2\sigma_2\tan(\sigma_1 H_1) + \mu_2^2\sigma_2^2\tan(\sigma_2 H_2) \qquad (4.5)$$

$$q = \mu_1\sigma_1\mu_3 s_3\tan(\sigma_2 H_2)\tan(\sigma_1 H_1) - \mu_2\sigma_2\mu_3 s_3, \qquad (4.6)$$

and the superscripts + and − refer to the values at the branches ℓ^+ and ℓ^- respectively.

The contribution to I_{11} from ℓ^+ and ℓ^- is

$$-\frac{1}{2\pi i}\left(\int_{\ell^+} G_{11}d\lambda - \int_{\ell^-} G_{11}d\lambda\right) = \frac{+1}{2\pi i}\int_{-\infty}^{\omega^2/\beta_3^2}(G_{11}^+ - G_{11}^-)\,d\lambda$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{(\gamma_1^+ - \gamma_1^-)\cos(\sigma_1(\zeta+H_1))\cos(\sigma_1(z+H_1))}{\cos^2\sigma_1 H_1}\,d\lambda$$

$$= -\frac{\mu_3\mu_2^2}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{s_3\sigma_2^2\cos(\sigma_1(\zeta+H_1))\cos(\sigma_1(z+H_1))}{(p^2+q^2)\cos^2(\sigma_2 H_2)\cos^2(\sigma_1 H_1)}\,d\lambda, \quad \text{(using (4.4))}$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2}\psi_1(z,\lambda)\,\psi_1(\zeta,\lambda)\,d\lambda, \tag{4.7}$$

where

$$\psi_1(z,\lambda) = \frac{\mu_2\sigma_2\mu_3 s_3\cos(\sigma_1(z+H_1)\cos\theta}{p\sqrt{\pi\mu_3 s_3}\,\cos(\sigma_1 H_1)\,\cos(\sigma_2 H_2)} \tag{4.8}$$

and

$$\theta = \tan^{-1} q/p, \tag{4.9}$$

where p and q are given by (4.5) and (4.6) respectively. The sum of the residues at the poles $\{\lambda_n\}$ is given by

$$-\sum_{n=1}^{N}\frac{\cos(\sigma_1^{(n)}(\zeta+H_1))\cos(\sigma_1^{(n)}(z+H_1))\,(M)_{\lambda=\lambda_n}}{\cos^2(\sigma_1^{(n)}H_1)\,[\frac{\partial\Delta}{\partial\lambda}]_{\lambda=\lambda_n}} = \sum_{n=1}^{N}\phi_1^{(n)}(z)\,\phi_1^{(n)}(\zeta) \tag{4.10}$$

where

$$\sigma_1^{(n)} = \left(\frac{\omega^2}{\beta_1^2}-\lambda_n\right)^{1/2},\ \sigma_2^{(n)} = \left(\frac{\omega^2}{\beta_2^2}-\lambda_n\right)^{1/2},\quad \sigma_3^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_3^2}\right)^{1/2} \text{ and}$$

$$\phi_1^{(n)}(z) = \frac{\cos\sigma_1^{(n)}(z+H_1)}{\cos(\sigma_1^{(n)}H_1)}\left[\left\{\frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)}\right\}_{\lambda=\lambda_n}\right]^{1/2} \tag{4.11}$$

From (4.2), (4.7) and (4.10) we obtain

$$I_{11} = \sum_{n=1}^{N} \phi_1^{(n)}(z)\phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_2^2} \psi_1(z,\lambda)\psi_1(\zeta,\lambda)\, d\lambda, \tag{4.12}$$

where ψ_1 is given by (4.8) and $\phi_1^{(n)}$ is given by (4.11).

(ii) Next, we consider

$$I_{21} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{21}(z,\ \zeta;\ \lambda)\, d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{B(z)\cos\sigma_1(\zeta+H_1)}{\Delta\cos(\sigma_2 H_2)\cos(\sigma_1 H_1)}\, d\lambda, \quad \text{(using (3.7))} \tag{4.13}$$

where $B(z)$ is given by (3.13).

Let $\gamma_2 = \frac{B(z)}{\Delta}$. Then

$$\gamma_2^+ - \gamma_2^- = 2i\ I(\gamma_2^+)$$

$$= \frac{2i\mu_2\sigma_2\mu_3 s_3 C(z)}{(p^2+q^2)\cos(\sigma_2 H_2)}, \tag{4.14}$$

where

$$C(z) = \mu_2\sigma_2\cos(\sigma_2 z) - \mu_1\sigma_1\sin(\sigma_2 z)\tan(\sigma_1 H_1) \tag{4.15}$$

and p and q are given by (4.5) and (4.6) respectively.

The branch-line contribution to the integral is given by

$$-\frac{\mu_2\mu_3}{\pi} \int_{-\infty}^{\omega^2/\beta_3^2} \frac{\sigma_2 s_3\cos(\sigma_1(\zeta+H_1))\ C(z)}{(p^2+q^2)\cos^2(\sigma_2 H_2)\cos(\sigma_1 H_1)}\, d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \psi_2(z,\ \lambda)\psi_1(\zeta,\ \lambda)\, d\lambda \tag{4.16}$$

where $\psi_1(\zeta, \lambda)$ is given by (4.8) and

$$\psi_2(z, \lambda) = \frac{\mu_3 s_3\ C(z)\ \cos\theta}{p\ \cos(\sigma_2 H_2)\ \sqrt{\pi\mu_3 s_3}} \tag{4.17}$$

where θ is given by (4.9) and $C(z)$ is given by (4.15).

Contribution from the poles is given by

$$-\sum_{n=1}^{N} \frac{\cos\{\sigma_1^{(n)}(\zeta+H_1)\}\ [B(z)]_{\lambda=\lambda_n}}{\cos(\sigma_2^{(n)}H_2)\cos(\sigma_1^{(n)}H_1)[\frac{\partial}{\partial\lambda}\ \Delta]_{\lambda=\lambda_n}} = \sum_{n=1}^{N} \phi_2^{(n)}(z)\ \phi_1^{(n)}(\zeta), \tag{4.18}$$

where $\phi_1(\zeta)$ is given by (4.11) and

$$\phi_2^{(n)}(z) = \frac{1}{\cos(\sigma_2^{(n)}H_2)}\ [\{\frac{1}{M\ \frac{\partial}{\partial\lambda}(-\Delta)}\}_{\lambda=\lambda_n}]^{\frac{1}{2}}\ \{B(z)\}_{\lambda=\lambda_n}. \tag{4.19}$$

From (4.13), (4.16) and (4.18), we obtain

$$I_{21} = \sum_{n=1}^{N} \phi_2^{(n)}(z)\ \phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_2^2} \psi_2(z, \lambda)\ \psi_1(\zeta, \lambda)\ d\lambda, \tag{4.20}$$

where $\phi_1^{(n)}$, $\phi_2^{(n)}$, ψ_1 and ψ_2 are given by (4.11), (4.19), (4.8) and (4.17) respectively.

(iii) Next, we consider

$$I_{31} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{31}(z, \zeta; \lambda)\ d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\mu_2\sigma_2\cos(\sigma_1(\zeta+H_1))\ e^{-\sigma_3(z-H_2)}}{\Delta\cos(\sigma_2 H_2)\ \cos(\sigma_1 H_1)}\ d\lambda. \tag{4.21}$$

Let

$$\gamma_3 = \frac{e^{-\sigma_3(z-H_2)}}{\Delta}\ .$$

Then

$$\gamma_3^+ - \gamma_3^- = 2i\ I(\gamma_3^+)$$

$$= \frac{-2i\ D(z)}{p^2 + q^2},$$

where

$$D(z) = p\ \sin\{s_3(z-H_2)\} + q\ \cos\{s_3(z-H_2)\} \tag{4.22}$$

and p, q are given by (4.5) and (4.6) respectively.

The branch-line contribution is given by

$$\frac{\mu_2}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2} \frac{\sigma_2\cos\sigma_1(\zeta+H_1)\cos\theta\cdot\sin\{\theta+s_3(z-H_2)\}}{p\ \cos(\sigma_2 H_2)\ \cos(\sigma_1 H_1)}\,d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \psi_3(z,\ \lambda)\ \psi_1(\zeta,\ \lambda)\ d\lambda, \tag{4.23}$$

where

$$\psi_3(z,\ \lambda) = \frac{-\sin\{\theta + s_3(z-H_2)\}}{\sqrt{\pi\mu_3 s_3}} \tag{4.24}$$

and ψ_1 is given by (4.8).

Contribution from the poles is given by

$$-\sum_{n=1}^{N} \frac{\mu_2\sigma_2^{(n)}\cos\{\sigma_1^{(n)}(\zeta+H_1)\}\ e^{-\sigma_3^{(n)}(z-H_2)}}{\cos(\sigma_2^{(n)}H_2)\cos(\sigma_1^{(n)}H_1)\,[\frac{\partial}{\partial\lambda}\Delta]_{\lambda=\lambda_n}} = \sum_{n=1}^{N} \phi_3^{(n)}(z)\phi_1^{(n)}(\zeta) \tag{4.25}$$

where

$$\phi_3^{(n)}(z) = \frac{\mu_2\sigma_2^{(n)}\ e^{-\sigma_3^{(n)}(z-H_2)}}{\cos(\sigma_2^{(n)}H_2)}\ [\{\frac{1}{M\ \frac{\partial}{\partial\lambda}(-\Delta)}\}_{\lambda=\lambda_n}]^{\frac{1}{2}} \qquad (4.26)$$

and $\phi_1^{(n)}(z)$ is given by (4.11). From (4.21), (4.23) and (4.25), we obtain

$$I_{31} = \sum_{n=1}^{N} \phi_3^{(n)}(z)\phi_1^{n}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \psi_3(z,\ \lambda)\ \psi_1(\zeta,\ \lambda)\ d\lambda, \qquad (4.27)$$

where $\phi_1^{(n)}$, $\phi_3^{(n)}$, ψ_1, ψ_3 are given by (4.11), (4.26), (4.8) and (4.24) respectively.

All the other integrals can be manipulated in the same manner as in (i) - (iii). The final result is

$$\begin{aligned} I_{ij} &= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{ij}(z,\ \zeta;\ \lambda)\ d\lambda \\ &= \sum_{i=1}^{N} \phi_i^{(m)}(z)\ \phi_j^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \psi_i(z,\ \lambda)\ \psi_j(\zeta,\ \lambda)\ d\lambda, \end{aligned} \qquad (4.28)$$

$(i,j = 1,2,3)$ with G_{ij} given by (3.5), (3.7), (3.8), (3.14), (3.16), (3.22).

From (4.1) and (4.28), we obtain the following representation of the delta function:

$$\delta(z-\zeta) = \sum_{n=1}^{N} \mu(\zeta)\ \phi^{(n)}(z)\phi^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_3^2} \mu(\zeta)\psi(z,\ \lambda)\psi(\zeta,\ \lambda)d\lambda, \qquad (4.29)$$

where $\mu(\zeta)$ is given by (2.2),

$$\phi^{(n)}(z) = \phi_1^{(n)}(z), \quad -H_1 \leq z \leq 0,$$

$$= \phi_2^{(n)}(z), \quad 0 \leq z \leq H_2,$$

$$= \phi_3^{(n)}(z), \quad H_2 \leq z, \qquad (4.30)$$

are the normalized eigenfunctions, and

$$\psi(z, \lambda) = \psi_1(z, \lambda), \quad -H_1 \leq z \leq 0,$$

$$= \psi_2(z, \lambda), \quad 0 \leq z \leq H_2,$$

$$= \psi_3(z, \lambda), \quad H_2 \leq z \qquad (4.31)$$

are normalized improper eigenfunctions.

If $f(z)$ is of finite μ-norm over the interval $(-H_1, \infty)$, then the representation of $f(z)$ in terms of eigenfunctions $\{\phi^{(n)}(z)\}$ and improper eigenfunctions $\{\psi(z, \lambda)\}$ can be obtained on multiplying (4.29) by $f(\zeta)$ and integrating with respect to ζ from $-H_1$ to ∞. We get

$$\int_{-H_1}^{\infty} f(\zeta)\delta(z-\zeta)d\zeta = \sum_{n=1}^{N} \phi^{(n)}(z) \int_{-H_1}^{\infty} \mu(\zeta)f(\zeta)\phi^{(n)}(\zeta)d\zeta$$

$$- \int_{-\infty}^{\omega^2/\beta_3^2} \psi(\lambda,z)\ dz \int_{-H_1}^{\infty} \mu(\zeta)\psi(\zeta,\lambda)f(\zeta)d\zeta,$$

whence

$$f(z) = \sum_{n=1}^{N} f_n\, \phi^{(n)}(z) - \int_{-\infty}^{\omega^2/\beta_3^2} f_\lambda\, \psi(\lambda,z)\ dz, \qquad (4.32)$$

where

$$f_n = \langle f, \phi^{(n)} \rangle = \int_{-H_1}^{\infty} \mu(\zeta)f(\zeta)\ \phi^{(n)}(\zeta)\ d\zeta, \qquad (4.3\ )$$

and

$$f_\lambda = \langle f, \psi(\zeta, \lambda)\rangle = \int_{-H_1}^{\infty} \mu(\zeta)\, \psi(\zeta, \lambda)\, f(\zeta)\, d\zeta. \tag{4.34}$$

In particular, if $f(z) = \phi^{(m)}(z)$ or $\psi(z, \lambda')$, then (4.35) - (4.37) yield the following orthonormality relations:

$$\int_{-H_1}^{\infty} \mu(z)\phi^{(m)}(z)\phi^{(n)}(z)dz = \delta_{mn} = \langle\phi^{(m)}, \phi^{(n)}\rangle, \quad 1 \leq m, n \leq N \tag{4.35a}$$

$$\int_{-H_1}^{\infty} \mu(z)\psi(z,\lambda)\psi(z,\lambda')dz = \delta(\lambda-\lambda') = \langle\psi(z,\lambda), \psi(z,\lambda')\rangle, \quad -\infty < \lambda, \lambda' < \omega^2/\beta_3^2 \tag{4.35b}$$

$$\int_{-H_1}^{\infty} \mu(z)\phi^{(m)}(z)\psi(z,\lambda)dz = 0 = \langle\phi^{(m)},\psi\rangle, \; 1 \leq m \leq N, \quad -\infty < \lambda < \omega^2/\beta_3^2 \tag{4.35c}$$

5. Conclusion: We have obtained the spectral representation of the two-dimensional Love wave operator associated with monochromatic SH-waves in a structure comprising two homogeneous layers overlying a uniform half-space. The spectral representation enables us to tackle classes of problems associated with the transmission and reflection of Love waves at a horizontally discontinuous change either in elevation or in material properties of three-layered models, using the method based on an integral equation formulation together with the application of Schwinger-Levine variational principle as in Kazi (1978a, b) and Niazy and Kazi (1980).

Acknowledgements

The authors wish to thank Dr. David Willmer for several valuable discussions and suggestions, and also to the University of Petroleum and Minerals for providing excellent facilities.

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