King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 031

November1981

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ON THE NORMAL APPROXIMATION OF AN ESTIMATE OF THE MEAN RESIDUAL LIFE OF A MULTICOMPONENT SYSTEM

By

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Abstract

An estimate of the mean residual life function of a complex system of k independent identically distributed components is proposed and studied with emphasis being on the order of normal approximation.

AMS 1980 Subject Classification: Primary: 60F05 Secondary: 60K10, 62G05

Key words and phrases: Mean residual life function, multicomponent system, nonparametric, U-statistics, normal approximation.

* Research supported by a University Faculty Research Grant No. MS-STAT-42, University of Petroleum & Minerals.

1. Introduction

Consider a system of k independent identically distributed components whose lives are denoted by $T_1, \ldots, T_k$. Let $\tau(T_1, \ldots, T_k)$ be an increasing function in each argument, and since all components are identically distributed, we may assume that $\tau(\cdot)$ is symmetric. Let F denote the distribution function of T_1 and $F^{[k]}$ denote the distribution function of $\tau(T_1, \ldots, T_k)$. For any fixed t, $0 \leq t < \infty$, the mean residual life function of $\tau(T_1, \ldots, T_k)$ is defined by:

$$(1.1) \qquad \mu(t) = \int_t^{\infty}(s - t)\, dF^{[k]}(s) \,/\, \bar{F}^{[k]}(t),$$

where $\bar{F}^{[k]}(t) = 1 - F^{[k]}(t)$. Thus in order to estimate $\mu(t)$, the mean ingredient would be to estimate $F^{[k]}(t)$ or equivalently, $\bar{F}^{[k]}(t)$. Let $T_{i_1}, \ldots, T_{in_i}$ denote a random sample of the i-th component $i = 1, \ldots, k$, and since all components are identical, let us put all observations into one set denoted by $T_1, \ldots, T_N$, where $N = \Sigma_{i=1}^{k} n_i$. Let $I(x,y) = 1$ if $x \geq y$ and is 0 otherwise. Then a nonparametric unbiased estimate of $\bar{F}^{[k]}(t)$ is given by:

$$(1.2) \qquad \bar{F}_N^{[k]}(t) = U_N(t) = \binom{N}{K}^{-1} \Sigma_c\, I(\tau(T_{i_1}, \ldots, T_{i_k}), t),$$

where Σ_c extends over all indices $1 \leq i_1 < \ldots < i_k \leq N$. Using (1.2), an estimate of $\mu(t)$ is given by:

$$(1.3) \qquad \hat{\mu}_N(t) = \Sigma_c[\tau(T_{i_1}, \ldots, T_{i_k}) - t]\, I(\tau(T_{i_1}, \ldots, T_{i_k}), t) / \Sigma_c\, I(\tau(T_{i_1}, \ldots, T_{i_k}, t)).$$

If we put:

(1.4) $\phi_1^t(t_1, \ldots, t_k) = [\tau(t_1, \ldots, t_k) - t]\, I\,(\tau(t_1, \ldots, t_k), t)$

and

(1.5) $\phi_2^t(t_1, \ldots, t_k) = I(\tau(t_1, \ldots, t_k), t).$

Set:

(1.6) $U_1(t) = \binom{N}{k}^{-1} \Sigma_c\, \phi_1^t(T_{i_1}, \ldots, T_{i_k})$

and

(1.7) $U_2(t) = \binom{N}{k}^{-1} \Sigma_c\, \phi_2^t(T_{i_1}, \ldots, T_{i_k}).$

Then we easily see that $\hat{\mu}_N(t) = U_1(t)/U_2(t)$, $0 \leq t < \infty$. Hence it follows from the standard theory of U-statistics (cf. Hoeffding (1948), Berk (1966), and others) that:

(i) $\hat{\mu}_N(t)$ converges with probability one to $\mu(t)$, provided that $E|\phi_1^t(T_1, \ldots, T_k)| < \infty$, and using a result of Berk (1966) and Slutsky's lemma.

(ii) $[\hat{\mu}_N(t) - \mu(t)]/(\sigma(t)/\sqrt{N})$ is approximately standard normal, where $\sigma^2(t) = \lim_{N\to\infty} N \operatorname{Var} U_1(t)/(\bar{F}^{[k]}(t))^2$.

The purpose of the present note is to establish the order of normal approximation in (ii) above. For the balance of this paper we shall assume that $k = 2$, extension to arbitrary case k is immediate but somewhat tedeous calculations may be needed. In this case $k = 2$ $\hat{\mu}(t)$ reduces to $\Sigma_{1\leq i<j\leq N}\, \phi_1^t(T_i, T_j)/\, \Sigma_{1\leq i<j\leq N}\, \phi_2^t(X_i, X_j)$. Let

(1.8) $g(T_1) = E[\phi_1^t(T_1, T_2)|\, T_1]$

and set $\sigma_g^2 = \text{Var } g(T_1)$. Assume that $\sigma_g^2 > 0$.

Some special cases of $\mu(t)$ are as follows:

(i) If $k = 1$ then $\mu(t)$ is the usual mean residual life $\int_t^\infty \bar{F}(s)ds/\bar{F}(t)$ which is estimated (see Yang (1978) and Ahmad (1981a)) by $\Sigma_{i=1}^N (T_i - t)I(T_i - t)/\Sigma_{i=1}^N I(T_i - t)$.

(ii) If we consider the series system of two components, then $\tau(T_1, T_2) = \min(T_1, T_2)$ and $\hat{\mu}_N(t) = \Sigma_{i<j} (\min(T_i,T_j)-t)I(\min(T_i,T_j)-t)/\Sigma_{i<j} I(\min(T_i,T_j) - t)$. While, if the system is connected in parallel then $\tau(T_1, T_2) = \max(T_1, T_2)$ and $\hat{\mu}_N(t) = \Sigma_{i<j} (\max(T_i, T_j) - t) I(\max(T_i, T_j)-t)/\Sigma_{i<j} I(\max(T_i, T_j) - t)$.

In the next section we give the order of normal approximation in two cases; the usual case when σ_g^2 is known and the studentized case when $\lim_{N\to\infty} N \text{ Var}(U_1(t)) = \sigma^2$ is unknown and is estimated by:

$$S_N^2 = 4(N - 1)(N - 2)^{-2} \Sigma_{i=1}^N [V^t(T_i) - U_1(t)]^2, \tag{1.9}$$

where $V^t(T_i) = (N - 1)^{-1} \sum_{j\neq i}^N \phi_1^t(T_i, T_j)$, $i = 1, \ldots, N$.

In the first case we assume that $E|\phi_1^t(T_1, T_2)|^3 < \infty$ while in the second we need a stronger assumption, viz., $E|\phi_1^t(T_1, T_2)|^4 < \infty$.

Finally, we mention that various relations between components lives and systems' lives have been investigated in Marshall & Proschan (1970) for the two important special cases mentioned in (ii) above namely the series and parallel systems.

In Section 2 some results concerning the order of normal approximations of U-statistics are given. These results while instrumental in the proofs of our main results of Section 3, are of independent interest and aside from their use in proving our main result, other applications are possible. Section 3 contains our main results giving the exact order of normal approximation of the usual version and a studentized (when the variance is not known) version of the standardized $[\hat{\mu}_N(t)-\mu(t)]$.

2. Preliminary results.

Let $\{T_N\}_{N=2}^{\infty}$ be a sequence of independent identically distributed random variables with common distribution function F. Let $\phi(t_1, t_2)$ be a real-valued symmetric function of two variables such that $E\phi(T_1, T_2) = \theta$. Define a U-statistics by:

$$(2.1) \qquad U_N = \binom{N}{2}^{-1} \Sigma_{1\leq i<j\leq N}\, \phi(T_i, T_j),$$

and assume that $g(T_1) = E[\phi(T_1, T_2)|T_1]$ has a positive variance σ_g^2.

The rate of normal approximation of U-statistics has been recently investigated. The most important is the result of Callaert and Janssen (1978) where they showed that if $E|\phi(T_1, T_2)|^3 < \infty$ then there is a positive constant C such that

$$(2.2) \qquad \sup_x |P[\sqrt{N}\,(U_N-\theta) \leq z\,\sigma_g\, x] - \Phi(x)| \leq C\, \frac{E|\phi(T_1, T_2) - \theta|^3}{\sigma_g^3\, N^{\frac{1}{2}}} = O(N^{-\frac{1}{2}}).$$

When one assumes only that $E|h(T_1, T_2)|^{2+\delta} < \infty$, $0 < \delta \leq 1$, then the upper bound in (2.2) becomes (Theorem 2.1 of Ahmad (1981b)) $C\, \dfrac{E|\phi(T_1, T_2) - \theta|^{2+\delta}}{\sigma_g^{2+\delta}\, N^{-\delta/2}} = O(N^{-\delta/2})$.

Other types of convergence rates and an ergodic theorem for U-statistics are given in Ahmad (1981b).

On the other hand if N is an integer-valued random variable such that $EN = m$ and $Var(N) = m_2$, then Ahmad (1980) showed that if N and $\{T_N\}$ are independent, then the right hand side of (2.2) becomes $C\left[\frac{E|\phi(T_1, T_2) - \theta|^3}{\sigma_g^3} m^{-\frac{1}{2}} + \frac{m_2^{\frac{1}{2}}}{m} + \left(\frac{m_2^{\frac{1}{2}}}{m}\right)^{\frac{1}{2}}\right] = O\left(m^{-\frac{1}{2}} + \left(\frac{m_2^{\frac{1}{2}}}{m}\right)^{\frac{1}{2}}\right)$. Similar rate with the leading term $m^{-\delta/2}$ is obtained when only $E|\phi(T_1, T_2)|^{2+\delta} < \infty$. This extension finds many applications among them the application to branching processes given in Ahmad (1980) (see his Theorem 3.1). When σ^2 is unknown and is estimated by S_N^2 (an estimate based on the data) we still would like to know the rate of convergence. One such estimate may be given by (see (1.9) above):

(2.3) $$S_N^2 = 4(N-1)(N-2)^{-2} \Sigma_{i=1}^N [V(T_i) - U_N]^2,$$

where $V(T_i) = (N-1) \Sigma_{j \neq i} \phi(T_i, T_j)$, $i = 1, \ldots, N$. Then we can prove the following result:

Theorem 2.1. *If* $E|\phi(T_1, T_2)|^4 < \infty$, *then*

(2.4) $$\sup |P[\sqrt{N}(U_N - \theta) \leq x S_N] - \Phi(x)| = O(N^{-\frac{1}{2}}).$$

Note: Before proving this result, we note here that it has been proved by Callaert and Veraverbeke (1981) assuming that $E|\phi(T_1, T_2)|^{9/2} < \infty$. Thus our theorem is an improvement.

Proof. Instead of decomposing S_N^{-1} as in Callaert and Veraverbeke (1981) we proceed using the following simple and well-known device (see Michel and Pfanzagl (1971), Lemma 1): If $\{\xi_N\}$ and $\{\eta_N\}$ are two sequences of random variables, then for any sequence of positive numbers $\{\varepsilon_N\}$,

$$\sup_x |P[\xi_N \leq x\ \eta_N] - \Phi(x)| \leq \sup_x |P[\xi_N \leq x] - \Phi(x)| + P[|\eta_N - 1| \geq \varepsilon_N] + O(\varepsilon_N). \tag{2.5}$$

Applying this device we have that

$$\sup_x |P[\sqrt{N}\ (U_N - \theta) \leq x\ S_N] - \Phi(x)| \leq \sup_x |P[\sqrt{N}(U_N - \theta) \leq r\,\sigma_g\ x] - \Phi(x)| + P[|\frac{S_N}{2\sigma_g} - 1| \geq \varepsilon_N] + O(\varepsilon_N). \tag{2.6}$$

But the first term in the right-hand-side of (2.6) is $O(N^{-\frac{1}{2}})$, see (2.2) above, and choosing $\varepsilon_N = N^{-\frac{1}{2}}$, the third term is $O(N^{-\frac{1}{2}})$. Thus it remains only to check the rate on the middle term

$$P[|\frac{S_N}{2\sigma_g} - 1| \geq \varepsilon_N] \leq P[|\frac{S_N^2}{4\sigma_g^2} - 1| \geq \varepsilon_n] = P[|S_N^2 - 4\sigma_g^2| \geq a_N],$$

where $a_N = 4\sigma_g^2\ \varepsilon_N$. Now, using the decomposition of Callaert and Veraverbeke(1981) $S_N^2 - 4\sigma_g^2$ we get

$$\begin{aligned} S_N^2 - 4\sigma_g^2 = {} & N^{-1}\Sigma_{i=1}^N [4(g^2(T_i) - \sigma_g^2) + 8\tilde{g}(T_i)] - 4\binom{N}{2}^{-1}\Sigma_{1 \leq i < j \leq N}[(g(T_i) + g(T_j)) \\ & \cdot\ \xi(T_i, T_j) - \tilde{g}(T_i) - \tilde{g}(T_j)] - 8N^{-1}\Sigma_{i=1}^N [g(T_i)\binom{N-1}{2}\Sigma_{k<m}^{(i)}\xi(T_k, T_m)] \\ & + 4(N-2)^{-1}\Sigma_{i=j}^N [\binom{N-1}{2}^{-1}\Sigma_{k<m}^{(i)}\xi(T_i, T_k)\ \xi(T_i, T_m) - 4N(N-1)(N-2)^{-2} \\ & \cdot\ [\binom{N}{2}^{-1}\Sigma_{i<j}\xi(T_i, T_j)]^2 + 4N(N-2)^{-2}\ \binom{N}{2}^{-1}\ \Sigma_{i<j}\ \xi^2(T_i, T_j) \\ = {} & T_N + \Sigma_{\ell=1}^6\ R_{\ell N}, \quad \text{say,} \end{aligned} \tag{2.7}$$

where $\xi(T_1, T_2) = \phi(T_1, T_2) - E\phi(T_1, T_2) - g(T_1) - g(T_2)$ is the orthogonal complement of $\phi(T_1, T_2)$ and $\tilde{g}(t) = \int g(u)\, \phi(t, u)\, dF(u)$. Hence,

$$(2.8) \qquad P[|S_N^2 - 4\sigma_g^2| \geq a_n] \leq P[|T_N| \geq a_N/7] + \Sigma_{\ell=1}^{6} P[|R_{\ell N}| \geq a_N/7].$$

But Callaert and Veraverbeke (1981) showed that if $E\phi^4(T_1, T_2) < \infty$ then $ER_{\ell N}^2 = O(N^{-2})$, $\ell = 1, \ldots, 6$ and thus with $\epsilon_N = N^{-\frac{1}{2}}$ we get that each of the six terms in the upper bound of (2.8) is $O(N^{-1})$. All is left is to consider $P[|T_N| \geq a_N/7]$. First note that by the c_r-inequality (Loeve (1963), p. 155),

$$(2.9) \qquad \mathrm{Var}\,[g^2(T_1) - \sigma_g^2) + \tilde{g}(T_1)] \leq 2\{\mathrm{Var}\{g^2(T_1) - \sigma_g^2\} + \mathrm{Var}\,\tilde{g}^2(T_1)\}$$

$$\leq 4\{Eg^4(T_1) + \sigma_g^4 + E\tilde{g}^2(T_1)\}.$$

But from Lemma 1 of Callaert and Veraverbeke (1981), if $E|\phi(T_1, T_2)|^4 < \infty$ then $Eg^4(T_1) < \infty$ and obviously $E\tilde{g}^2(T_1) < \infty$, thus the extreme right-hand-side of (2.9) is finite. Now, taking N large enough we need only to consider (see Callaert and Veraverbeke (1981)) $P\,[T_N^2 > a_N/7]$. But

$$(2.10) \qquad P[T_N^2 > a_N/7] \leq \frac{N^{\frac{1}{2}}\,\mathrm{Var}(T_N)}{(4\sigma_g^2/7)} = N^{-\frac{1}{2}} \frac{7\mathrm{Var}[(g^2(T_1) - \sigma_g^2) + \tilde{g}(T_1)]}{4\sigma_g^2} = O(N^{-\frac{1}{2}}).$$

This concludes the proof. QED.

Next, we give a random sample size version of Theorem 2.1 above, thus giving a studentized version of Theorem 1.1 of Ahmad (1980).

<u>Theorem 2.2</u> <u>Under the conditions of Theorem 2.1 and assuming that</u> $\{T_N\}$ <u>and</u> N <u>are independent, then</u>

(2.11) $\sup_x |P[\sqrt{N}(U_N - \theta) \le x\, S_N] - \Phi(x)| = O(m^{-1/2} + \frac{m_2^{1/2}}{m} + (\frac{m_2^{1/2}}{m})^{1/2})$,

<u>where</u> $EN = m$ <u>and</u> $\mathrm{Var}(N) = m_2$.

<u>Proof</u>. Since N and $\{T_N\}$ are independent and putting $p_{N,k} = P[N=k]$ we have

$$(2.12)\quad \sup_x |P[\sqrt{N}(U_N-\theta) \le x\, S_N] - \Phi(x)|$$

$$\le \Sigma_{k=1}^{\infty} p_{N,k} \sup_x |P[\sqrt{k}(U_k-\theta) \le x\, S_k] - \Phi(x)|$$

$$\le \Sigma_{k=1}^{\infty} p_{N,k} \sup_x |P[\sqrt{k}(U_k-t) \le 2\sigma_g\, x] - \Phi(x)| + \Sigma_{k=1}^{\infty} p_{N,k}$$

$$P[|\frac{S_k}{2\sigma_g} - 1| \ge \varepsilon_k] + \Sigma_{k=1}^{\infty} p_{N,k}\, O(\varepsilon_k) = J_1 + J_2 + J_3, \text{ say.}$$

But it follows from Theorem 1.1 of Ahmad (1980) that $J_1 = O(m^{-1/2} + (\frac{m_2^{1/2}}{m})^{1/2}) + (\frac{m_2^{1/2}}{2})^{1/2})$. Next, choosing $\varepsilon_k = k^{-1/2}$,

$$(2.13)\quad J_3 \le C\, \Sigma_{k=1}^{\infty} p_{N,k}\, k^{-1/2} = C\, \{\Sigma_{k:|m-k|\le m/2}\, p_{N,k}\, k^{-1/2} + \Sigma_{k:|m-k|>m/2}\, p_{N,k}\, k^{-1/2}$$

$$\le C\{(\frac{m}{2})^{-1/2} + \Sigma_{k:|m-k|>m/2}\, p_{N,k} \frac{2|k-m|}{k^{1/2} m}\} \le C\{(\frac{m}{2})^{-1/2} + \frac{C}{m} E|N-m|\}$$

$$\le C\{m^{-1/2} + \frac{m_2^{1/2}}{m}\}.$$

Finally we evaluate J_2.

$$(2.14)\quad J_2 \le \Sigma_{k=1}^{\infty} p_{N,k}\, P[|S_k^2 - 4\sigma_g^2| > a_k] \le \Sigma_{k=1}^{\infty} p_{N,k}\, P[|T_k| > a_k/7] +$$

$$\Sigma_{\ell=1}^{6} \Sigma_{k=1}^{\infty} p_{N,k}\, P[|R|_{\ell k} > a_k/7].$$

First, let us evaluate

$$(2.15)\qquad \Sigma_{k=1}^{\infty} p_{N,k}\, P[|R_{\ell k}| > a_k/7] = \Sigma_{k:|m-k|<m/2}\, p_{N,k}\, P[|R_{\ell k}| > a_k/7]$$

$$+ \Sigma_{k:|m-k|>m/2}^{\infty}\, p_{N,k}\, P[|R_{\ell k}| > a_k/7] = I_1 + I_2, \text{ say.}$$

But using Theorem 2.1 above

$$(2.16)\qquad I_1 \le C\, \Sigma_{k:|m-k|\le m/2}\, p_{N,k}\, k^{-1} \le C(\tfrac{m}{2})^{-1},$$

and

$$(2.17)\qquad I_2 \le C\, \Sigma_{k:|m-k|>m/2}\, p_{N,k}\, k^{-1} \le C\, \Sigma_{k:|m-k|>m/2}\, p_{N,k}\, \frac{2|m-k|}{km}$$

$$\le 2C\, \frac{E|N-M|}{m} \le C^{*}\, \frac{m_2^{1/2}}{m}.$$

Hence the right-hand-side of (2.15) in $O(m^{-1/2} + (\frac{m_2^{1/2}}{m}))$.

Finally choosing m large enough we only consider $\Sigma_{k=1}^{\infty} p_{N,k}\, P[T_k^2 > a_k/7]$. But

$$(2.18)\qquad \Sigma_{k=1}^{\infty} p_{N,k}[P[T_k^2 > a_k\, /\, 7] \le C\, \Sigma_{k:|k-m|\le m/2}\, p_{N,k}\, k^{-1/2} + \Sigma_{k:|m-k|>m/2}\, \frac{2|m-k|}{mk^{-1/2}}$$

$$\le C(m^{-1/2} + \frac{m_2^{1/2}}{m}) = O(m^{-1/2} + \frac{m_2^{1/2}}{m}).$$

This concludes the proof of the theorem. QED.

The results of Callaert and Janssen (1978) and Ahmad (1980) along with Theorem 2.1 and Theorem 2.2 above, will be used, respectively to establish Theorems 3.1, 3.2, 3.3, and 3.4 of the next section.

Aside from its use in proving Theorem 3.4, Theorem 2.2 has a nice application in branching processes: Let z_0 be a fixed number and define the sequence of stochastic processes:

$$Z_0 = z_0,\quad Z_1 = T_1 + \ldots + T_{Z_0},\ Z_2 = T_{Z_0+1} + \ldots + T_{Z_0+Z_1},\ \ldots,\ Z_N = T_{Z_0+\ldots+Z_{N-2}+1} + \ldots + T_{Z_0+\ldots+Z_{N-1}}.$$

Assume that $Z_1 = m > 0$. Let Z_N^* denote the r.v. Z_N under the conditional probability measure that $Z_N > 0$. Since $m^N Z_N^{-\frac{1}{2}}(W - W_N)$ (where $W_N = m^{-N} Z_N$ and W is the a.s. limit of W_N) has the same distribution as that of $(Z_N^*)^{\frac{1}{2}} [T_1 + \ldots + T_{Z_N^*}]$, then $\text{Var } T_1 = \sigma^2/(m^2-m)$. In many cases this quantity is unknown ($\sigma^2 = \text{Var } T_1$) and may be estimated by $S^2_{Z_N^*} = Z_N^{*-1} \Sigma_{i=1}^{Z_N^*}(T_i - \bar{T}_{Z_N^*})^2$, with $\bar{T}_{Z_N^*} = \frac{1}{Z_N^*} \Sigma_{i=1}^{Z_N^*} T_i$. Since $\{T_N\}$ and Z_N^* are independent, Theorem 2.2 applies and gives under the assumption that $E|Z_N|^4 < \infty$, that

$$\text{(2.19)} \qquad \sup_x |P[\Sigma_{i=1}^{Z_N^*} T_i \leq x\, S_{Z_N^*} \sqrt{Z_N^*}] - \Phi(x)| = O\Big(\frac{m^{\frac{1}{2}}}{m} + \Big(\frac{m^{\frac{1}{2}}}{m}\Big)^{\frac{1}{2}}\Big).$$

3. Main Results.

Theorem 3.1. *If* $|E\phi_1^t(T_1, T_2)|^3 < \infty$, *then*

$$\text{(3.1)} \qquad \sup_x |P[\hat{\mu}_N(t) - \mu(t) \leq \bar{\sigma}_g\, x\, N^{-\frac{1}{2}}] - \Phi(x)| = O(N^{-\frac{1}{2}})$$

where $\bar{\sigma}_g^2 = \bar{\sigma}_g^2(t) = 4\text{Var }\{E[\phi_1^t(T_1, T_2)|\ T_1] - \mu(t)\ E\ [\phi_2^t(T_1, T_2)|T_1]\}$

Proof: Note that

$$\hat{\mu}_N(t) - \mu(t) = \frac{1}{U_2(t)} \{[U_1(t) - EU_1(t)] - \mu(t)[U_2(t) - EU_2(t)]\}$$

Then using Lemma 1 of Michel and Pfanzagl (1971) we have for any real x,

$$(3.2) \qquad |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t) \leq \bar{\sigma}_g x] - \Phi(x)|$$

$$\leq |P[\sqrt{N}[(W_N(t) - EW_N(t)] \leq \bar{\bar{\sigma}}_g x] - \Phi(x)| + P[\,|\frac{U_2(t)}{EU_2(t)} - 1| > \varepsilon_N] + O(\varepsilon_N),$$

where $W_N(t) = U_1(t) - \mu(t)\, U_2(t)$ is a U-statistics with kernel $\phi_1^t(T_1, T_2) - \mu(t)\, \phi_2^t(T_1, T_2)$ and $\bar{\bar{\sigma}}_g^2 = \bar{\sigma}_g^2\, EU_1(t)$ with $EU_1(t) = \bar{F}^{[2]}(t)$. But by a theorem of Callaert and Janssen (1978) (see 2.2 above),

$$(3.3) \qquad \sup_x |P[\sqrt{N}(W_N(t) - EW_N(t)) \leq \bar{\bar{\sigma}}_g x] - \Phi(x)|$$

$$\leq C \frac{E|\phi_1^t(T_1,T_2)-\mu(t)\phi_1^t(T_1, T_2) - E[\phi_1^t(T_1,T_2)-\mu(t)\phi_2^t(T_1,T_2)]|^3}{\bar{\sigma}_g^3 N^{\frac{1}{2}}}$$

$$= O(N^{-\frac{1}{2}}),$$

since $E|\phi_1^t(T_1, T_2)|^3 < \infty$ implies that the numerator of the right-hand-side in (3.3) is finite. Next, since $U_1(t) - EU_2(t) \to 0$ w.p.1 and since $EU_2(t) > 0$ we can choose n large enough that $|U_2(t) - EU_2(t)| > EU_1(t)\, \varepsilon_n$ implies that $|U_2(t) - EU_2(t)|^2 > EU_2(t)\, \varepsilon_n$. Thus it suffices to consider $P[|U_2(t) - EU_2(t)|^2 > EU_2(t)\, \varepsilon_n]$ which is bounded above by $\mathrm{Var}\, U_2(t)/EU_2(t)\, \varepsilon_n$, by Tchebychev's inequality. But (see Hoeffding (1948)) Var $U_2(t) = O(N^{-1})$. Hence the proof is terminated by choosing $\varepsilon_N = N^{-\frac{1}{2}}$. QED

Remark 3.1. If we assume the weaker condition $E|\phi_1^t(T_1, T_2)|^{2+\delta} < \infty$ for some $0 < \delta \leq 1$, then it follows from Ahmad (1981b), Theorem 2.1, that

$$(3.4) \qquad \sup_x |P[\sqrt{N}(W_N(t) - EW_N(t)) \leq \bar{\bar{\sigma}}_g x] - \Phi(x)| = O(N^{-\delta/2})$$

Hence under this condition we have

$$(3.5) \qquad \sup_x |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t)) \leq \bar{\sigma}_g x] - \Phi(x)| = O(N^{-\delta/2}), \quad 0 < \delta \leq 1.$$

Remark 3.2. Another type of convergence rate of $\hat{\mu}_N(t)$ is possible. Write Δ_N for the left hand side of (3.1) and Δ_N^* for the left-hand-side of (3.3). Then it follows from Theorem 3.2 of Ahmad (1981b) $\Sigma_{N=1}^{\infty} N^{-1+\delta/2} \Delta_N^* < \infty$, $0 \leq \delta < 1$ and since $U_2(t)$ has moments of all orders and it follows from Sen (1973) that $E[U_2(t)]^r \leq C\, N^{-r/2}$ and the upper bound of $P[|U_2(t) - EU_2(t)|^2 > \varepsilon_n EU_2(t)] \leq E(U_2(t) - EU_2(t))^{2r}/(EU_2(t))^r \varepsilon_N^r \leq C\, N^{-r}/[E(U_2(t))]^r \varepsilon_N^r$. Thus with $r = 2$ and choosing $\varepsilon_N = N^{-\frac{1}{2}}$ we get that $\Sigma_{N=1}^{\infty} N^{-1+\delta/2} (N\varepsilon_N)^{-2} = \Sigma_{N=1}^{\infty} N^{-2+\delta/2} < \infty$ and $\Sigma_{N=1}^{\infty} N^{-1+\delta/2} \varepsilon_N = \Sigma_{N=1}^{\infty} N^{-3/2-\delta/2} < \infty$. Hence we arrive at the following result:

$$(3.6) \qquad \Sigma_{N=1}^{\infty} N^{-1+\delta/2} \Delta_N < \infty.$$

Next, we give a random version of Theorem 3.1. Assume that N is an integer-valued nonnegative random variable such that $EN = m$ and $\text{Var } N = m_2 < \infty$.

Theorem 3.2. If N is independent of $T_1, \ldots, T_N$, then under the conditions of Theorem 2.1

$$(3.7) \qquad \sup_x |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t)) \leq \bar{\sigma}_g x] - \Phi(x)| = O\left(m^{-\frac{1}{2}} + \frac{\sqrt{m_2}}{m} + \left(\frac{\sqrt{m_2}}{m}\right)^{\frac{1}{2}}\right).$$

Proof: Since N and $T_1, T_2, \ldots$ are independent, then if we denote by Δ_N the left-hand-side of (2.7), then

(3.8) $$\Delta_N = \Sigma_{k=1}^{\infty} \sup_x |P[\sqrt{k}(\hat{\mu}_k(t)-\mu(t)) \le \bar{\sigma}_g x] - \Phi(x)| \; P[N = k]$$

$$\le \Sigma_{k=1}^{\infty} \sup_x |P[\sqrt{k}(W_k(t) - EW_k(t)) \le \bar{\bar{\sigma}}_g x] - \Phi(x)| \; P[N=k] + \Sigma_{k=1}^{\infty} P[|U_2(t) -$$

$$EU_2(t)|^2 > EU_2(t)\varepsilon_k] P[N=k] + \Sigma_{k=1}^{\infty} O(\varepsilon_k) P[N=k] = J_1 + J_2 + J_3, \quad \text{say.}$$

But $J_1 = O(m^{-1/2} + \frac{\sqrt{m_2}}{m} + (\frac{\sqrt{m_2}}{m})^{1/2})$ by Theorem 1.1 of Ahmad (1980). Next, with $\varepsilon_k = k^{-1/2}$,

(3.9) $$J_3 \le C \; \Sigma_{k=1}^{\infty} k^{-1/2} P[N = k] \le \Sigma_{k:|m-k| \le m/2} k^{-1/2} P[N = k] + \Sigma_{k:|m-k|>m/2} k^{-1/2} P[N=k]$$

$$\le (\tfrac{m}{2})^{-1/2} + \Sigma_{k:|m-k|>m/2} P[N=k] \frac{2|k-m|}{k^{1/2} m} \le (\tfrac{m}{2})^{-1/2} + \frac{C}{m} E|N-m| \le O(m^{-1/2} + \frac{\sqrt{m_2}}{m}) .$$

Finally, we evaluate J_2.

(3.10) $$J_2 \le \Sigma_{k=1}^{\infty} \frac{E(U_2(t) - EU_2(t))^2}{EU_2(t)\varepsilon_k} P[N = k] \le \frac{C}{EU_2(t)} \Sigma_{k=1}^{\infty} k^{-1/2} P[N = k]$$

$$\le O(m^{-1/2} + \frac{\sqrt{m_2}}{m}),$$

as shown in (3.9). Hence the conclusion follows from (3.8) - (3.10). QED

In most reliability applications the variance of the kernel $(\bar{\sigma}_g)$ is not known and has to be estimated from the data. One such estimate is that given in (1.9). The next two theorems are studentized analogues of Theorems 3.1 and 3.2 above.

Theorem 3.3. If $E|\phi_1^t(T_1, T_2)|^4 < \infty$, then

(3.11) $$\sup_x |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t) \leq x\, S_N] - \Phi(x)| = O(N^{-\frac{1}{2}}).$$

Proof. Again using Lemma 1 of Michel and Pfanzagl we get that

(3.12) $$\sup_x |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t) \leq x\, S_N] - \Phi(x)|$$

$$\leq \sup_x |P\ [N(\hat{\mu}_N(t)-\mu(t)) \leq \bar{\sigma}_g x] - \Phi(x)| + P\ [|\frac{S_N}{\bar{\sigma}_g} - 1| > \varepsilon_N] + O(\varepsilon_N)$$

$$\leq O(N^{-\frac{1}{2}}) + P[|S_N^2 - \bar{\sigma}_g^2| > \varepsilon_N\, \bar{\sigma}_g] + O(\varepsilon_N),$$

where $\sup_x |P[\sqrt{N}(\hat{\mu}_N-(t) - \mu(t) \leq \bar{\sigma}_g\ x] - \Phi(x)| = O(N^{-\frac{1}{2}})$ follows from Theorem 3.1. Choosing $\varepsilon_n = N^{-\frac{1}{2}}$ we get $O(\varepsilon_N) = O(N^{-\frac{1}{2}})$. The proof that the middle term is $O(N^{-\frac{1}{2}})$ proceeds exactly as in the proof of Theorem 2.1 (see (2.7)-(2.10) and is not repeated. This completes the proof. QED.

Theorem 3.4. Under the conditions of Theorem 2.3 and if N is a non-negative integer-valued random variable independent of T_i's and such that $EN = m$ and $\mathrm{Var}\ (N) = m_2$, then

(3.13) $$\sup_x\ |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t)) \leq x\ S_N] - \Phi(x)| = O(m^{-\frac{1}{2}} + \frac{\sqrt{m_2}}{m} + (\frac{\sqrt{m_2}}{m})^{\frac{1}{2}}).$$

Proof. Again since N is independent of T_i's we get (using Lemma 1 of Michel and Pfanzagl (1971)

(3.14) $$\sup\ |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t)) \leq x\ S_N] - \Phi(x)|$$

$$\leq \Sigma_{k=1}^{\infty} \sup_x |\ P[\sqrt{k}\ (\hat{\mu}_k - \mu(t)) \leq S_k\ x] - \Phi(x)|\ P[N = k]$$

$$\leq \Sigma_{k=1}^{\infty} \sup_x |P[\sqrt{k}(\hat{\mu}_k(t)-\mu(t)) \leq \bar{\sigma}_g x] - \Phi(x)| \; P[N = k]$$

$$+ \Sigma_{k=1}^{\infty} P[|\frac{S_k}{\bar{\sigma}_g} - 1| > \varepsilon_k] \; P[N = k] + \Sigma_{k=j}^{\infty} P[N = k]\varepsilon_n$$

$$= 0(m^{-\frac{1}{2}} + \frac{\sqrt{m_2}}{m} + (\frac{\sqrt{m_2}}{m})^{\frac{1}{2}}) + 0(m^{-\frac{1}{2}} + \frac{\sqrt{m_2}}{m}) + 0(m^{-\frac{1}{2}} + \frac{\sqrt{m_2}}{m}),$$

where the first term in the last bound follows from Theorem 2.2, while the second term follows from steps (2.14) - (2.18) of Theorem 2.2 above, while the last term follows from (2.13) of Theorem 2.2 above. QED.

Remark 3.3. It is possible to utilize Theorems 2.1 and 2.2 directly in proving Theorems 3.3 and 3.4 above by using the decomposition of $\hat{\mu}_N(t) - \mu(t)$ used in Theorems 3.1 and 3.2, i.e.

$$\hat{\mu}_N(t) - \mu(t) = \frac{1}{U_2(t)} [(U_1(t) - U_2(t)) - E(U_1(t) - \mu(t) \; U_2(t))]$$

$$= \frac{1}{U_2(t)} [W_N(t) - EW_N(t)].$$

Thus we get that

$$(3.15) \qquad \sup |P[\sqrt{N}(\hat{\mu}_N(t) - \mu(t)) \leq x \; S_N] - \Phi(x)|$$

$$\leq \sup_x |P[\sqrt{N}(W_n(t)-EW_n(t)) \leq xEU_2(t)S_N]-\Phi(x)| + P[|\frac{U_2(t)}{EU_2(t)} - 1| \geq \varepsilon_N] + 0(\varepsilon_N).$$

Hence Theorem 2.1 (or 2.2) gives the bound $0(N^{-\frac{1}{2}})$ (or $0(m^{-\frac{1}{2}} + \frac{m_2^{\frac{1}{2}}}{m} + (\frac{m_2^{\frac{1}{2}}}{m})^{\frac{1}{2}})$) and we handle the remaining two terms as done in Theorems 3.1 and 3.2.

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