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ON SOME GENERALIZED MIXED DISCRETE DISTRIBUTIONS*

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Abstract

The probability generating function derived by Bhalerao and Gurland (1976) has been re-written in terms of confluent hyper-geometric series functions and generalized to four and five parameter families. The generalized family of distributions is different from the Katz (1963) and Kemp (1968) families. The moment method has been employed to obtain the estimation of parameters. An example is given to illustrate the method.

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1. Introduction

Bhalerao and Gurland (1976) introduced the probability generating function (p.g.f.)

$$g(z) = \exp\{\lambda[\{1 - \frac{\beta}{1-\beta}(z-1)\}^{-\alpha/\beta} - 1]\} \tag{1}$$

where $\lambda > 0$, $\alpha > 0$ and $\beta < 1$. When $\beta < 0$, $-\alpha/\beta$ is a positive integer. The function (1) is the p.g.f. of a three parameter family of generalized Poisson distribution and was named as Poisson V POLPAB as it was a mixture of Poisson, Logarithmic, Pascal and Binomial distributions. This family will be referred to as the B-G family.

In this paper, a generalization of the B-G family is given. In addition, explicit formulae for the density function, moment generating function, and moments will be presented.

2. Generalization of B-G Family

The 3-parameter p.g.f. (1) can be rewritten in terms of confluent hypergeometric functions as

$$g(z) = \frac{{}_1F_1[1;\ 1;\ \lambda f(z)]}{{}_1F_1[1;\ 1;\ \lambda]}, \quad \lambda > 0 \tag{2}$$

where $f(z) = [1 - \frac{\beta}{1-\beta}(z-1)]^{-\alpha/\beta}$, ${}_1F_1[a;b;t] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{t^n}{n!}$,

$(r)_n = r(r+1)\ \ldots\ (r+n-1)$, $r > 0$. This has motivated the generalization to 4 or 5-parameter families of discrete distributions. This is done by introducing two more parameters in the arguments of the confluent hypergeometric function. The new p.g.f. will, therefore, be given by

$$g_*(z) = k \; {}_1F_1[a;\; \theta;\; \lambda f(z)], \tag{3}$$

where $k = ({}_1F_1[a;\; \theta;\; \lambda])^{-1}$ and $a > 0$, $\lambda > 0$ and $\theta > 0$. The function $g_*(z)$ is evidently a p.g.f. for $g_*(z)$ converges absolutely at least for $\left|\frac{a}{\theta} \cdot \frac{\lambda}{n}\right| \leq 1$, since $f^{(n)}(z)$ constitutes a bounded sequence of real numbers and $g_*(1) = 1$.

The probability function as the coefficient of z^x in the expansion of $g_*(z)$ is

$$f(x) = k \sum_{n=0}^{\infty} \frac{(a)_n}{(\theta)_n} \frac{\lambda^n}{n!} (1 - \beta)^{n\alpha/\beta} \left[\binom{-n\alpha/\beta}{x} (-\beta)^x\right], \tag{4}$$

$$x = 0, 1, 2, 3, \ldots$$

If $a = \theta = 1$, $f(x)$ in (4) is the B-G probability function. The moment generating function for (4) is

$$M(t) = k \; {}_1F_1[a,\; \theta,\; \lambda f(e^t)] \tag{5}$$

The moments from (5) can easily be obtained using the following property of the confluent hypergeometric function (see Erdelyi et al, 1953 p. 283).

$${}_1F_1(r,\; \theta,\; \lambda x) = \sum_{n=0}^{\infty} \frac{(r)_n \; (x-1)^n \lambda^n}{(\lambda)_n \; n!} \; {}_1F_1(r + n,\; \theta + n,\; \lambda).$$

In the derivation of moments, the following identity, proof of which is simple, is also used

$$\sum_{n=j}^{\infty} \frac{(n)_j}{(\theta)_n} \lambda^n = \frac{j! \; \lambda^j}{(\theta)_j} \; {}_1F_1[j + 1,\; \theta + j,\; \lambda].$$

If $a = 1$, the first four moments are

$$\mu_1' = \frac{k\alpha}{1 - \beta} \frac{\lambda}{\theta} F(2) \tag{6}$$

$$\mu_2' = \frac{\mu_1'}{(1 - \beta)} \left[\frac{2\alpha\lambda}{(\theta + 1)} F(3) + (1 + \alpha) F(2)\right] \quad (7)$$

$$\mu_3' = \frac{\mu_1'}{(1 - \beta)^2} \left[\frac{6\alpha^2\lambda^2}{(\theta + 1)_2} F(4) + \frac{6\lambda\alpha(\alpha + 1)}{(\theta + 1)} F(3)\right.$$

$$\left. + (\alpha^2 + 3\alpha + \beta + 1) F(1)\right] \quad (8)$$

and

$$\mu_4' = \frac{\mu_1'}{(1 - \beta)^3} \left[\frac{24\ \alpha^3\lambda^3}{(\theta + 1)_3} F(5) + 36\ \alpha^2(\alpha + 1) \frac{\lambda^2}{(\theta + 1)_2} F(4)\right.$$

$$+ 2(7\alpha^3 + 18\alpha^2 + 4\alpha\beta + 7\alpha) \frac{\lambda}{\theta + 1} F(3)$$

$$\left. + (\alpha^3 + 6\alpha^2 + 7\alpha + 4\alpha\beta + \beta^2 + 4\beta + 1)F(1)\right] \quad (9)$$

where $F(r) = {}_1F_1(r, \theta + r - 1, \lambda)$.

3. Estimation.

Using the sample moments, we obtain the moment estimators of the parameters. If θ and λ are known, the moment estimators of α and β can be obtained as:

$$\tilde{\alpha} = \theta(1 - \tilde{\beta})m_1/[k\lambda F(2)] \quad (10)$$

$$\tilde{\beta} = 1 - \frac{k\ F(2)}{k\ F(2) - a(\lambda,\ \theta)} \quad (11)$$

where

$a(\lambda, \theta) = \theta m_1'\left(\lambda^{-1} - \frac{2F(3)}{(\theta + 1)\ F(2)}\right)$ and m_i' is the ith sample moment about zero.

If $\theta = 1$ and λ unknown, the moment estimator of λ is given by

$$\tilde{\lambda} = \frac{(2s^2 - m_1') + \sqrt{[(2s^2 - m_1')^2 + 16m_1'^3(1 + m_1')\ (m_3' - m_1'^3)]}}{2m_1'(m_3' - m_1^3)}$$

where $s^2 = m_2' - m_1'^2$.

If all the four parameters, α, β, θ and λ are unknown, the moment estimating equations in (6) - (9) are solved by minimizing (using the least square method) $\sum_{i=1}^{4} (m_i' - g_i)^2$ with respect to $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\theta}$ and $\tilde{\lambda}$ where $g_i = g_i(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{\lambda})$, $i = 1, 2, 3, 4$ denotes the right hand sides of the moment equations (6) - (9). [See Fletcher et al (1963) and Fiacco and McCormick (1968)].

4. Example.

Williford and Price (1976) fitted various compound distributions to the data on two types of thunderstorm outcomes. The two types are the frequencies of the number of days that experienced x thunderstorm events and x thunderstorm hits at Cape Kennedy, Florida for the 10-year period 1957-66 for the months of June, July and August. They found that some of the modified distributions will provide a better fit than either the Poisson, binomial or negative binomial distributions. However, the modified negative binomial distribution seems to have fitted well for the thunderstorm hits with $\chi^2 = 0.28$ for 1 degree of freedom but the compound modified binomial and negative binomial distributions do not seem to have fitted well. We have used the probability model (4) for the data on the thunderstorm events and found $\chi^2 = 2.01$ for 1 degree of freedom which shows a better fit than those given by the compound modified binomial and the negative binomial distributions.

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