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On Complete Boolean Algebras of Projections

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Abstract. The aim of this paper is to prove that a bounded Boolean algebra of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections on X.

1. Notations. Throughout this paper X will be a complex Banach space with a dual space X*. The value of the functional x* in X* at x in X will be denoted by $\langle x, x^* \rangle$. We use L(X) to denote the algebra of all linear (bounded) operators on X. The zero and the identity operators in L(X) will be denoted by 0 and I, respectively. C(X) will denote the algebra of all continuous, complex-valued functions on X. If A is a subset of X then clm(A) is the closed linear manifold spanned by A. The field of complex numbers will be denoted by $\underset{\sim}{C}$.

2. Preliminaries.

2.1. Definition. A Boolean algebra, $\underset{\sim}{B}$, is an abstract set in which two binary operations, $\vee$ and $\wedge$, are defined, satisfying the following conditions:

(1) $\vee$ and $\wedge$ are commutative, associative and distributive operations.

(2) There exist zero and identity elements, 0 and 1, respectively, such that for any x in $\underset{\sim}{B}$, $x \vee 0 = x$, $x \wedge 1 = x$.

(3) For any x in $\mathbf{B}$, there is an element x' in $\mathbf{B}$, called the complement of x, such that

$$x \vee x' = 1, \quad x \wedge x' = 0.$$

The above axioms are symmetric in the operations $\vee$ and $\wedge$. If $\mathbf{A}$ and $\mathbf{B}$ are two Boolean algebras, then a mapping f from $\mathbf{A}$ to $\mathbf{B}$ is called a Boolean homomorphism if

(1) $f(p \wedge q) = f(p) \wedge f(q)$

(2) $f(p \vee q) = f(p) \vee f(q)$

(3) $f(p') = (f(p))'$

where p and q are in $\mathbf{A}$. If f is both one-to-one and onto, then it is an isomorphism.

2.2. Theorem. Every abstract Boolean algebra is isomorphic to the Boolean algebra of all open-and-closed subsets, of some totally disconnected compact Hausdorff space Ω, called the Stone representation space.

Proof. ([4], Theorem 1.12.1, p. 41).

2.3. A Boolean algebra of projections on X is a commutative subset, $\mathbf{B}$, of $L(X)$ such that

(1) $E^2 = E \quad (E \in \mathbf{B})$;

(2) $0 \in \mathbf{B}$;

(3) if $E \in \mathbf{B}$ then $I - E \in \mathbf{B}$;

(4) If $E, F \in \underset{\sim}{B}$ then

$$E \vee F = E + F - EF \text{ and}$$

$$E \wedge F = EF,$$

are in $\underset{\sim}{B}$.

A Boolean algebra of projections, $\underset{\sim}{B}$, on X is said to be bounded if $\|E\| \leq M$ for every $E \in \underset{\sim}{B}$, where M is a real number.

2.4. Definition. A Boolean algebra $\underset{\sim}{B}$ on X will be called complete (σ-complete) if for every subset (sequence) $\{E_i\} \subseteq \underset{\sim}{B}$, the greatest lower bound $\wedge E_i$ and the least upper bound $\vee E_i$ of $\{E_i\}$ exist in $\underset{\sim}{B}$ and $(\vee E_i) X = \text{clm} \{E_i X\}$; $(\wedge E_i)X = \cap E_i X$.

2.5. Lemma. If every increasing or decreasing net (sequence) of elements of a Boolean algebra $\underset{\sim}{B}$ of projections on X converges strongly to an element of $\underset{\sim}{B}$, then $\underset{\sim}{B}$ is complete (σ-complete).

Proof. ([4], Lemma 4, p.p. 2197-2198).

2.6. Lemma. A complete Boolean algebra of projections is strongly closed.

Proof. ([4], Corollary 7, p. 2201).

2.7. Definition. A family $\Gamma \subseteq X^*$ is called total if and only if $y \in X$ and $< y, f > = 0$, for all f in Γ, together imply that $y = 0$.

2.8. Definition. Let $\sum$ be a σ-algebra of subsets of an arbitrary set Ω. Suppose that a mapping $E(\cdot)$ from $\sum$ into a Boolean algebra of projections on X satisfies the following conditions:

(1) $E(M_1) + E(M_2) = E(M_1)\, E(M_2) - E(M_1 \cup M_2)$;

(2) $E(M_1)\, E(M_2) = E(M_1 \cap M_2)$, $(M_1, M_2 \in \sum)$;

(3) $E(\Omega) = I$;

(4) $E(\Omega \backslash M) = I - E(M)$, $(M \in \sum)$;

(5) there is $K > 0$ such that $\|E(M)\| \leq K$ for all M in $\sum$.

(6) there is a total linear subspace Γ of X^* such that $< E(\cdot)\, x, y >$ is countably additive on $\sum$, for each x in X and each y in Γ.

Then $E(\cdot)$ is called a spectral measure of class $(\sum, \Gamma)$. A spectral measure $E(\cdot)$ is called regular if for each Borel subset N,

$$E(N)X = \text{clm} \bigcup_j E(N_j)X,$$

for all closed subsets N_j of N.

2.9. Theorem. If $\underset{\sim}{B}$ is a σ-complete Boolean algebra of projections, then the strong closure, $\underset{\sim}{B}^s$, of $\underset{\sim}{B}$ is complete.

Proof. ([1], Theorem 2.7, p. 350).

2.10. Theorem. If a generalized sequence of projections in a σ-complete Boolean algebra of projections in a Banach space converges weakly to a projection, then it converges strongly.

Proof. ([4], Theorem 27, p. 2218).

2.11. Theorem. Let $\underset{\sim}{B}$ be a σ-complete Boolean algebra of projections on X. Then the following statements are equivalent:

(1) $\underset{\sim}{B}$ is complete.

(2) $\underset{\sim}{B}$ is strongly closed.

(3) The uniformly closed operator algebra generated by $\underset{\sim}{B}$, $u(\underset{\sim}{B})$, is weakly closed.

Proof. ([1], Theorem 4.5, p. 358).

Now we prove Theorem 2.13 by taking $\underset{\sim}{B}$, rather than $u(\underset{\sim}{B})$, to be weakly closed.

2.12. Theorem. Let $\underset{\sim}{B}$ be a σ-complete Boolean algebra of projections on X. Then the following statements are equivalent:

(1) $\underset{\sim}{B}$ is complete.

(2) $\underset{\sim}{B}$ is strongly closed.

(3) $\underset{\sim}{B}$ is weakly closed.

Proof. (1) $\longrightarrow$ (2): Lemma 2.6.

(2) $\longrightarrow$ (3): Since $\underset{\sim}{B}$ is strongly closed, $\underset{\sim}{B} = \underset{\sim}{B}^s$. Since $\underset{\sim}{B}$ is σ-complete then by Theorem 2.10, $\underset{\sim}{B}^s \supseteq \underset{\sim}{B}^w$ (the weak closure of $\underset{\sim}{B}$). Since always $\underset{\sim}{B}^s \subseteq \underset{\sim}{B}^w$, it follows that $\underset{\sim}{B} = \underset{\sim}{B}^s = \underset{\sim}{B}^w$. Hence $\underset{\sim}{B}$ is weakly closed.

(3) $\longrightarrow$ (2): Since $\underset{\sim}{B}$ is σ-complete, $\underset{\sim}{B}^s = \underset{\sim}{B}^w$. Since B is weakly closed, it follows that $\underset{\sim}{B} = \underset{\sim}{B}^w = \underset{\sim}{B}^s$ which means that $\underset{\sim}{B}$ is strongly closed.

(2) $\longrightarrow$ (1): Since $\underset{\sim}{B}$ is σ-complete, then by Theorem 2.9, $\underset{\sim}{B}^s$ is complete. Since $\underset{\sim}{B}$ is strongly closed, $\underset{\sim}{B} = \underset{\sim}{B}^s$ which implies that $\underset{\sim}{B}$ is complete.

2.13. Theorem. A Boolean algebra is complete if and only if its Stone representation space, Ω, is extremely disconnected in the sense that the closure of every open subset is open.

Proof. ([4], Exercise 16, p. 2225).

In this case and if $\sum_\Omega$ denotes the Borel field of Ω, there corresponds to each Borel set e_0 in $\sum_\Omega$, a unique open-and-closed set e, such that

$$(e_0 \setminus e) \cup (e \setminus e_0)$$

is a set of first category. Moreover, each Borel function differs from a unique continuous function, on a Borel set of the first category.

Let $\underset{\sim}{B}$ be a complete Boolean algebra of projections in X and let Ω be its Stone representation space. If e is an open-and-closed subset of Ω, denote by $E(e)$ the element of $\underset{\sim}{B}$ corresponding to e. This mapping is extended to the Borel field $\sum_\Omega$ by setting

$$E(e_0) = E(e)$$

for each Borel set e_0, where e is the open-and-closed subset of Ω which differs from e_0 by a set of the first category, in the sense above. It follows from the definition of completeness and Theorem 2.5 that the vector and scalar-valued measures $E(\cdot)\,x$ and $< E(\cdot)x, x^* >$ associated with $x \in X$ and $x^* \in X^*$ are countably additive on $\sum_\Omega$. By Theorem 2.2 of [1], $\underset{\sim}{B}$ is uniformly bounded. Comparison of the above with the definition of a spectral measure shows that we may regard a complete Boolean algebra of projections as a spectral measure defined on the Borel field of the Stone representation space.

2.14. Let $y \subseteq X$. We denote by Y^w the closure of Y in the weak operator topology.

Definition. Let $\{T_i\}$ be a net of operators on X and let $x \in X_w$. Then z_x is called a weak x-cluster point of $\{T_i\}$ if $z_x \in \cap \{T_i(x)\}^w \;(i \geq j)$; that is if z_x is a weak cluster point of the net of vectors $\{T_i x\}$.

2.15. Definition. A net $\{T_i\}$ of operators on X is said to be naturally ordered if $T_i = T_i\, T_j = T_j\, T_i$ whenever $i \leq j$.

2.16. Theorem. Let $\{T_i\}$ be a naturally ordered uniformly bounded net of operators on X. Then $\{T_i\}$ converges in the strong operator topology if and only if $\{T_i\}$ has a weak x-cluster point for each x in X.

Proof. ([3], Theorem 6.4, p.p. 159-160).

2.17. Theorem. Let A be an algebra of operators in X such that A is the homomorphic image under a continuous homomorphism S of the algebra $C(Z)$ of all complex continuous functions on a compact space Z. Then there is a unique spectral measure E in X^* defined on the Borel sets in Z such that $xE(\cdot)x^*$ is regular and countably additive for x in X and x^* in X^*, and for which

$$S(f)^* = \int_Z f(\lambda)\ E\ (d\lambda), \quad f \in C(Z).$$

Proof. ([4], Theorem 4, p. 2184).

2.18. Theorem. A bounded Boolean algebra $\underset{\sim}{B}$ of projections on a weakly complete Banach space X can be embedded in a σ-complete Boolean algebra of projections.

Proof. Let Ω be the Stone representation space of $\underset{\sim}{B}$. Let $K(\Omega)$ be the set of all characteristic functions of open-and-closed subsets of Ω, and let

$$\psi : K(\Omega) \to \underset{\sim}{B}, \quad \text{such that}$$

$$\psi(K_e) = \underset{\sim}{B}(e)$$

be the representation isomorphism for $\underset{\sim}{B}$. Let $K'(\Omega)$ be the algebra of all finite sums $\sum c_j K_{e_j}$ ($c_j \in \underset{\sim}{C}$, e_j open-and-closed in Ω), and let $\underset{\sim}{B}'$ be the corresponding algebra of sums $\sum c_j \underset{\sim}{B}(e_j)$. Then ψ extends to an algebra isomorphism $\psi' : k'(\Omega) \to \underset{\sim}{B}'$ such that

$$\psi'(\sum c_j K_{e_j}) = \sum c_j \underset{\sim}{B}(e_j)$$

and ψ' is an isometry ([2], Theorem 2.1). Since Ω is totally disconnected, $k'(\Omega)$ is norm dense in $C(\Omega)$. Hence ψ' can be extended to an isometric isomorphism (also denoted by ψ')

$$\psi' : C(\Omega) \to L(X)$$

with

$$\psi'(f)^* = \int_\Omega f(\lambda)\ E\ (d\lambda), \quad (f \in C(\Omega))$$

where $E(\cdot)$ is a spectral measure on X^* of class X defined on the Borel sets of Ω. Let $\sum$ be the algebra of Borel sets in Ω and let $\sum_0$ be the class of sets in $\sum$ such that there is a spectral measure $F(\cdot)$ defined by $F(e)^* = E(e)$ for every e in $\sum_0$. We show that $\sum_0$ is a Boolean algebra of sets.

(1) Let $e \in \sum_0$ then

$$\begin{aligned} E(\Omega \setminus e) &= E(\Omega) \setminus E(e) \\ &= I - E(e) \\ &= I^* - F(e)^* \\ &= (I - F(e))^* \end{aligned}$$

$$= (F(\Omega) \setminus F(e))^*$$

$$= (F(\Omega \setminus e))^*.$$

Thus Thus $e \in \sum_0$ if and only if $(\Omega \setminus e) \in \sum_0$.

(2) Let e_1 and e_2 be in $\sum_0$ then there are projections $F(e_1)$ and $F(e_2)$ such that $E(e_1) = F(e_1)^*$ and $E(e_2) = F(e_2)^*$. Hence we have

$$E(e_1 \cap e_2) = E(e_1)\,E(e_2) = F(e_1)^*\,F(e_2)^* = (F(e_2)\,F(e_1))^*$$
$$= F((e_2 \cap e_1))^*.$$

Hence $e_1 \cap e_2$ is in $\sum_0$.

(3) Let $e_1, \ldots, e_n \in \sum_0$ then $\bigcup_{i=1}^{n} e_i$ is in $\sum_0$. To prove this, suppose first that $n = 2$. Then by (1) and (2), we have $(\Omega \setminus e_1) \cap (\Omega \setminus e_2) = \Omega \setminus (e_1 \cup e_2)$ is in $\sum_0$. Hence by (1), $e_1 \cup e_2$ is in $\sum_0$. By induction it follows that $\bigcup_{i=1}^{n}$ is in $\sum_0$.

It follows from (1), (2), and (3) that $\sum_0$ is a field (or a Boolean algebra of sets).

Let $\{e_n\}$ be a sequence of sets in $\sum_0$ then

$$< x,\ E(\bigcup_{n=1}^{m} e_n)\ y > = < x,\ F(\bigcup_{n=1}^{m} e_n)^*\ y >$$

$$= < F(\bigcup_{n=1}^{m} e_n)\ x,\ y > \quad \text{for} \quad x \in X,\ y \in X^*.$$

Since $< x,\ E(\bigcup_{n=1}^{m} e_n)\ y >$ is a Cauchy sequence, $< F(\bigcup_{n=1}^{m} e_n)\ x,\ y >$ is a Cauchy sequence. Hence $F(\bigcup_{n=1}^{m} e_n)\ x$ is a weak Cauchy sequence.

Since X is weakly complete, every weak Cauchy sequence has a weak limit. Therefore, $F(\bigcup_{n=1}^{m} e_n)\,x$ is weakly convergent for each fixed x. Since $F(\bigcup_{n=1}^{m} e_n)$ is naturally ordered and uniformly bounded, then, by Theorem 2.16 the net $\{F(\bigcup_{n=1}^{m} e_n)\}$ is convergent in the strong operator topology. Hence, by Theorem 2.5, $\{F(\bigcup_{n=1}^{m} e_n)\}$ is σ-complete and B is embedded in it.

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