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# Diffraction of Love Waves at a Vertical Step up and Step down Discontinuity I

M.H. Kazi and A. Niazy

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By M.H. KAZI AND A. NIAZY

## 1. INTRODUCTION:

In this report we consider the two-dimensional problem of diffraction of plane, harmonic, monochromatic Love waves, incident normally (from either side) upon the vertical plane of discontinuity in a structure consisting of a vertical step up and step down discontinuity (see Fig.1) in the surface layer and with lateral impedance contrasts in the surface layer and the substratum across the vertical plane of discontinuity. We use the method based on integral representation and Schwinger-Levine principle to describe the wave-field by means of a scattering matrix. In this method, approximate expressions for the elements of the scattering matrix are obtained through the plane-wave approximation and their variational improvement is sought through the Schwinger-Levine variational principle. The details of the method are given in Kazi (1978 a) and will not be given here. We shall confine ourselves to obtaining formulae for complex reflection and transmission coefficients through a transmission matrix related to the scattering matrix.

The authors have used the method for Love wave diffraction problems associated with laterally discontinuous structures involving a vertical surface step (Kazi 1978 a,b), welded layered quarter-spaces with plane top surface (Niazy and Kazi 1980, 1982) and M-discontinuity step (Kazi and Niazy, 1981). The present model is the superposition of all the three models and is a more realistic model of the continental margin. Numerical computation of the results will be given in another report.

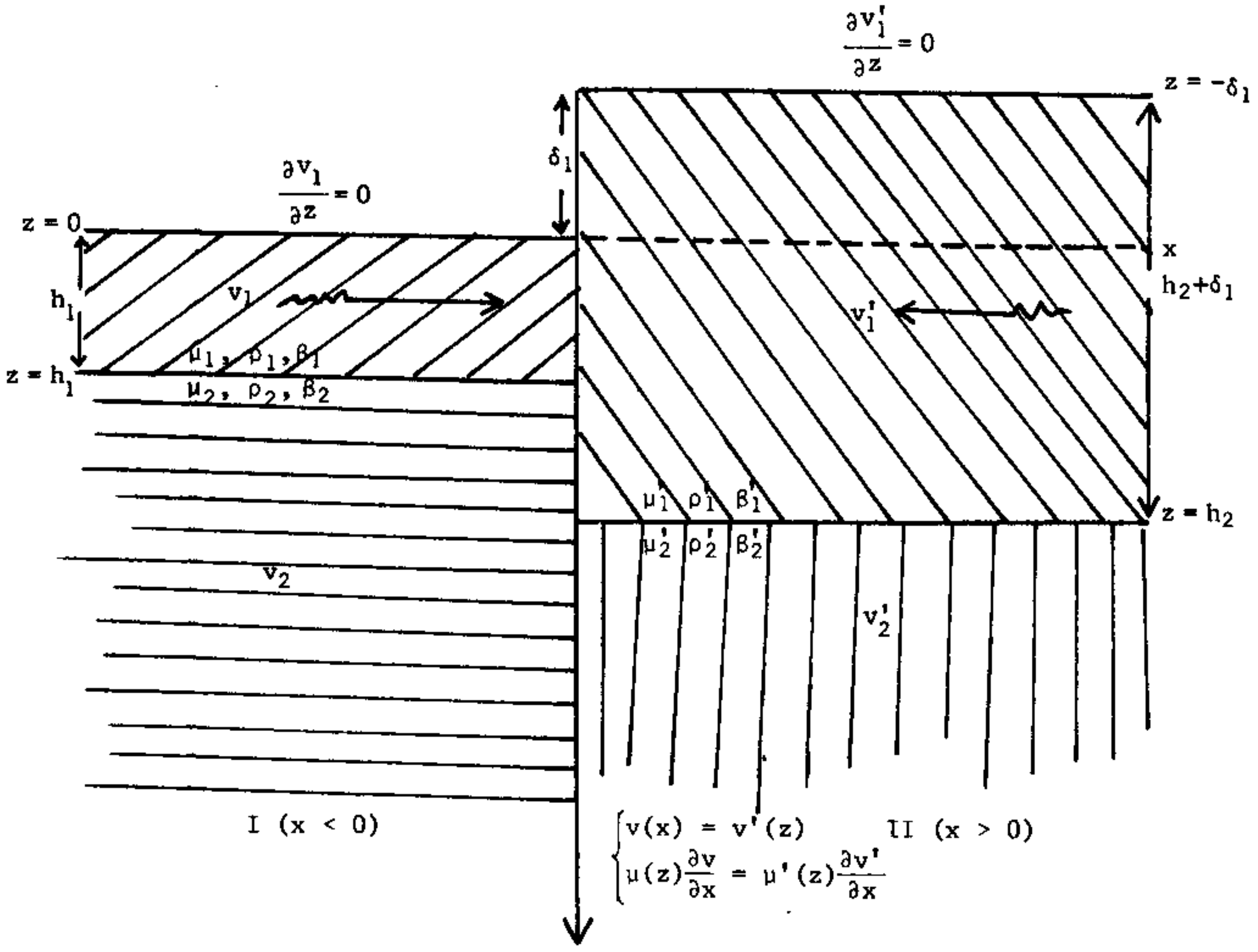


Figure 1. The geometry and elastic parameters of problem. Love waves are incident (from either side) on the vertical plane of discontinuity.

## 2- EQUATIONS OF MOTION

Let us suppose that a quarter-space consisting of a material of rigidity $\mu_2$, shear velocity $\beta_2$ and density $\rho_2$, overlain by a layer of depth $h_1$, density $\rho_1$, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$, is in welded contact with a quarter-space of material of rigidity $\mu_2'$, shear velocity $\beta_2'$ and density $\rho_2'$ overlain by a layer of depth $h_2+\delta_1\,(h_2>h_1)$, density $\rho_1'$, rigidity $\mu_1'(<\mu_2')$ and shear velocity $\beta_1'(<\beta_2')$, in such a way that we obtain a surface step of thickness $\delta_1$ and an M-discontinuity step of thickness $\delta_2 = h_2-h_1$ (See Fig. 1). We shall consider the two-dimensional problems of reflection, transmission and diffraction of time-harmonic Love waves incident normally (from either side) upon the vertical plane of discontinuity. We shall, therefore, confine ourselves to SH wave motion. Displacement fields to the left and right of the vertical plane of discontinuity are denoted by $e^{-i\omega t}v(x,z)$ and $e^{-i\omega t}v'(x,z)$ respectively, where

$$
\begin{aligned}
e^{-i\omega t}\,v(x,z) &= e^{-i\omega t}\,v_1(x,z),\quad 0\le z\le h_1,\quad x\le 0\\
&= e^{-i\omega t}\,v_2(x,z),\quad h_1\le z,\quad x\le 0
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-i\omega t}\,v'(x,z) &= e^{-i\omega t}\,v_1'(x,z),\quad -\delta_1\le z\le h_2,\quad x\ge 0\\
&= e^{-i\omega t}\,v_2'(x,z),\quad z\ge h_2,\quad x\ge 0
\end{aligned}
$$

($\omega$ being the angular frequency) are the solutions of the Love wave differential equation

$$\rho(z)\,\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu(z)\,\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu(z)\frac{\partial v}{\partial z}\right)$$

where $\mu(z) = \mu_1, \quad 0 \leq z \leq h_1, \ x < 0$

$\qquad = \mu_2, \ h_1 \leq z, \ x < 0$

and

$\mu'(z) = \mu_1', \ -\delta_1 \leq z \leq h_2, \ x > 0$

$\qquad = \mu_2', \ z \geq h_2, \ x > 0$

in the coordinate system (see Fig. 1) chosen in such a way that the z-axis is directed vertically downward, the upper boundary of the bounding surface is given by $z = 0, \ x < 0$; $z = -\delta_1, \ x > 0$; $0 \leq z \leq \delta_1, \ x = 0$ and the lower boundary including the M-discontinuity step is given by $z = h_1, \ x < 0$; $h_1 \leq z \leq \delta_2, \ x = 0$; $z = h_2, \ x > 0$ so that both the vertical discontinuities lie in the plane $x = 0$. It may be pointed out that in addition to the step discontinuities we have lateral discontinuities in the elastic parameters across the vertical plane of discontinuity $x = 0$.

The surfaces $z = 0, \ x < 0$; $z = -\delta_1, \ x > 0$ and the vertical surface of the surface step are stress-free, and so

$$\frac{\partial v_1}{\partial z} = 0 \text{ at } z = 0, \ x < 0 \qquad (2.1a)$$

$$\frac{\partial v_1'}{\partial z} = 0 \text{ at } z = -\delta_1, x > 0 \qquad (2.1b)$$

and

$$\frac{\partial v_1'}{\partial x} = 0 \text{ at } x = 0, \ -\delta_1 \leq z < 0 \qquad (2.1c)$$

Moreover, the conditions of welded contact at $x = 0$, $z \geq 0$ lead to

$$v = v' \qquad (2.1d)$$

$$\mu(z)\,\frac{\partial v}{\partial x} = \mu'(z)\,\frac{\partial v'}{\partial x} \qquad (2.1e)$$

at $x = 0$, $z \geq 0$

Using our work on the spectral representation of the Love wave operator (Kazi (1976), we may write the complete solution for the displacement $v(x,z)$ in domain I $(x<0)$, in terms of the eigenfunctions and improper eigenfunctions for a surface layer of depth $h_1$, rigidity $\mu_1$ and shear velocity $\beta_1$, overlying a half-space of rigidity $\mu_2$ and shear velocity $\beta_2$. Similarly, we may write the complete solution for the displacement $v'(x,z)$ in domain II $(x>0)$ in terms of eigenfunctions and improper eigenfunction for a surface layer of depth $h_2+\delta_1$, rigidity $\mu_1'$ and shear velocity $\beta_1'$, overlying a half-space of rigidity $\mu_2'$ and shear velocity $\beta_2'$. These eigenfunctions and improper eigenfunctions may be written with the help of expressions obtained in Kazi (1976). Specifically:

in DOMAIN I $(x<0,\ z\geq 0)$

$$v(x,z) = -\Bigg[\sum_{m=1}^{r}\left[A_m \exp(-ik_m|x|) + B_m \exp(ik_m|x|)\right]\chi_m(z)$$

$$+ \int_0^{\omega/\beta_2}\left[C(k)\exp(-ik|x|) + D(k)\exp(ik|x|)\right]\phi(z,k)\,dk$$

$$+ \int_0^\infty E(k) \exp(-k|x|)\psi(z,k)dk \Bigg], \qquad (2.2)$$

and in DOMAIN II $(x>0, -\delta_1 \leq z)$

$$v'(x,z) = \Bigg[\sum_{m=1}^{s} [A'_m \exp(-ik'_m x) + B'_m \exp(ik'_m x)]\chi'_m(z)$$

$$+ \int_0^{\omega/\beta'_2} [C'(k') \exp(-ik'x) + D'(k') \exp(ik'x)]\phi'(z,k')dk'$$

$$+ \int_0^\infty E'(k') \exp(-k'x)\psi'(z,k') dk' \Bigg], \qquad (2.3)$$

where $\chi_m, \chi'_m$ are eigenfunctions and $k_m$, $k'_m$ correspond to real and positive eigenvalues for $x<0$ and $x>0$, respectively;

$$\text{where} \quad \chi_m(z) = \phi_1^{(m)}(x), 0 \leq z \leq h_1 \qquad (2.4)$$

$$= \phi_2^{(m)}(z), h_1 \leq z$$

$$\chi'_m(z) = \phi_1'^{(m)}(z), -\delta_1 \leq z \leq h_2$$

$$= \phi_2'^{(m)}(z), \ z \geq h_2 , \qquad (2.5)$$

$$\phi_1^{(m)}(z) = F_m[(\cos \sigma_1^{(m)} z)/(\cos \sigma_1^{(m)} h_1)], \qquad (2.6)$$

$$\phi_2^{(m)}(z) = F_m \exp[\sigma_2^{(m)}(h_1 - z)], \qquad (2.7)$$

$$F_m = \left[\frac{2\sigma_2^{(m)}}{\mu_2} \left\{\frac{\beta_1^{-2} - U_m^{-1} C_m^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right\}\right]^{\frac{1}{2}}, \tag{2.8}$$

$$\phi_1'^{(m)}(z) = F_m' \left[(\cos \sigma_1'^{(m)}(z+\delta_1))/(\cos \sigma_1'^{(m)}(h_2+\delta_1))\right], \tag{2.9}$$

$$\phi_2'^{(m)}(z) = F_m' \exp [\sigma_2'^{(m)}(h_2 - z)] \tag{2.10}$$

$$F_m' = \left[2 \frac{\sigma_2'^{(m)}}{\mu_2'} \left(\frac{\beta_1'^{-2} - U_m'^{-1} C_m'^{-1}}{\beta_1'^{-2} - \beta_2'^{-2}}\right)\right]^{\frac{1}{2}}; \tag{2.11}$$

$U_m$, $U_m'$ are the group velocities and $C_m$, $C_m'$ the phase velocities ($C_m = \omega/k_m$ , $U_m^{-1} = \frac{dk_m}{d\omega}$ etc.)

$$\sigma_1(\lambda) = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right)^{\frac{1}{2}}, \quad \sigma_2(\lambda) = \left(\lambda - \frac{\omega^2}{\beta_2^2}\right)^{\frac{1}{2}}, \qquad \lambda = k^2 \tag{2.12}$$

$$\sigma_1(\lambda_m) = \sigma_1^{(m)}, \quad \sigma_2(\lambda_m) = \sigma_2^{(m)}, \tag{2.13}$$

$$\lambda_m = k_m^2 , \quad k_m > 0 \tag{2.14}$$

and similarly for $\sigma_1'(\lambda')$, $\sigma_2'(\lambda')$, $\sigma_1^{(m)'}$, $\sigma_2^{(m)'}$, and $\lambda_m'$.

$\lambda = \lambda_m = k_m^2$ , $m = 1,2,$ - - - satisfy the dispersion equation

$$\mu_1\sigma_1 \tan \sigma_1 h_1 - \mu_2\sigma_2 = 0, \tag{2.15}$$

whereas

$$\lambda' = \lambda'_m = k'^2_m \text{ , } m = 1,2, - - ., s, \text{ satisfy the}$$

equation

$$\mu'_1\sigma'_1 \tan \sigma'_1 (h_2+\delta_1) - \mu'_2 \sigma'_2 = 0 \tag{2.16}$$

$\psi(z,k)$, the improper eigenfunctions in the domain $x<0$, corresponding to the continuum of improper eigenvalues $\lambda = (ik)^2$, $k>0$ and representing non-propagated modes, are given by

$$\begin{aligned}\psi(z,k) &= \psi_1(z,k),\ 0 \le z \le h_1 \\ &= \psi_2(z,k), h_1 \le z ,\end{aligned} \tag{2.17}$$

where

$$\psi_1(z,k) = G_k[(\cos(\sigma_1^{(k)}z))/(\cos(\sigma_1^{(k)}h_1))], 0 \le z \le h_1, \tag{2.18}$$

and

$$\psi_2(z,k) = G_k[\sin\{\theta^{(k)} - s_2^{(k)}(z-h_1)\}]/[\sin \theta^{(k)}], z \ge h_1 \tag{2.19}$$

with

$$G_k = [2k/(\pi\mu_2\ s_2^{(k)})]^{\frac{1}{2}} \ \sin \theta^{(k)} , \tag{2.20}$$

$$s_2^{(k)} = \left(\frac{\omega^2}{\beta_2^2} - \lambda_k\right)^{\frac{1}{2}} \text{ real and positive} \tag{2.21}$$

and

$$\theta^{(k)} = \tan^{-1} \left[(\mu_2\ s_2^{(k)} \cot \sigma_1^{(k)} h_1)/(\mu_1\sigma_1^{(k)})\right] . \tag{2.22}$$

Similarly $\psi'(z,k')$, the improper eigenfunctions in the domain x>0 corresponding to improper eigenvalues $\lambda'=(ik')^2$, $k'>0$ are given by :

$$\psi'(z,k') = \psi'_1(z,k'),\ -\delta_1\le z\le h_2\ ,$$
$$= \psi'_2(z,k'), h_2\le z, \qquad (2.23)$$

where

$$\psi'_1(z,k')=G'_{k'}\ [(\cos\{\sigma_1'^{(k')}(z+\delta_1\ )\}\ )/(\cos(\sigma_1'^{(k')}(h_2+\delta_1))\ ], \qquad (2.24)$$

and

$$\psi'_2(z,k')=G'_{k'}[(\sin\{\ \theta'^{(k')}-s_2'^{(k')}\ \ (z-h_2)\}\ )/(\sin\ \theta^{(k')'})] \qquad (2.25)$$

with

$$\theta^{(k')}= \tan^{-1}\ [(\mu_2'\ s_2'^{(k')}\ \cot(\sigma_1'^{(k')}(h_2+\delta_1))/(\mu_1'\sigma_1'^{(k')})] \qquad (2.26)$$

$G'_{k'}$ , $s_2'^{(\bar{k})}$ having expressions similar to $G_k$ and $s_2{}^{(k)}$ but in the primed notation.

The improper eigenfunctions $\phi(z,k)$ and $\phi'(z,k')$ corresponding to the improper eigenvalues $\lambda= k^2$, $\lambda=k'^2$, $(0<k\le\frac{\omega}{\beta_2}\ ,\ 0<k'\le\frac{\omega}{\beta_2'})$ respectively, have expressions similar to those for $\psi(z,k)$ and $\psi'(z,k')$ respectively and correspond to waves travelling in the x-direction.

The orthonormality relations amongst various proper and improper eigenfunctions are given by (cf.Kazi 1976)

$$\int_0^\infty \mu(z)\ \chi_m(z)\ \chi_n(z)\ dz = \delta_{mn}\ ,\ 1\le m,n\le r\ , \quad (2.27a)$$

$$\int_{-\delta_1}^\infty \mu'(z)\ \chi_m'(z)\ \chi_n'(z)\ dz = \delta_{mn},\ 1\le m,n\le s, \quad (2.27b)$$

$$\int_0^\infty \mu(z)\ \chi_m(z)\ \phi(z,k)dz = 0,\ 1\le m\le r,\ 0<k\le \frac{\omega}{\beta_2}\ , \quad (2.27c)$$

$$\int_{-\delta_1}^\infty \mu'(z)\ \chi_m'(z)\ \phi'(z,k)dz = 0,\ 1\le m\le s,\ 0<k\le \frac{\omega}{\beta_2'} \quad (2.27d)$$

$$\int_0^\infty \mu(z)\ \chi_n(z)\ \psi(z,k)dz = 0,\ 1\le n\le r,\ 0<k\le\infty, \quad (2.27e)$$

$$\int_{-\delta_1}^\infty \mu'(z)\chi_m'(z)\psi'(z,k')dz = 0,\ 1\le m\le s,\ 0<k\le \infty, \quad (2.27f)$$

$$\int_0^\infty \mu(z)\phi(z,k)\psi(z,k)\ dz=0,\quad \int_{-\delta}^\infty \mu'(z)\phi'(z,k')\psi'(z,k')dz=0 \quad (2.27g)$$

$$\int_0^\infty \mu(z)\psi(z,k)\psi(z,\iota)\ dz = \delta(k-\iota),\ 0\le k,\iota\le \infty, \quad (2.27h)$$

$$\int_{-\delta_1}^\infty \mu'(z)\psi'(z,k')\psi'(z,\iota')\ dz = \delta(k'-\iota'),\ 0<k',\iota'\le \infty \quad (2.27i)$$

$$\int_0^\infty \mu(z)\phi(z,k)\phi(z,\iota)dz = \delta(k-\iota),\ 0<k,\iota\le\frac{\omega}{\beta_2} \quad (2.27j)$$

$$\int_{-\delta_1}^\infty \mu'(z)\phi'(z,k')\phi'(z,\iota')dz = \delta(k'-\iota'),0<k',\iota'\le \frac{\omega}{\beta_2'} \quad (2.27k)$$

## 3. Integral equation formulation

Let $T(z)$ denote the component $T_{xy}$ of stress at any point of the plane x = 0 :

$$T(z) = T_{xy}\Big/_{x=0} = \mu(z)\frac{\partial v}{\partial x}\Big/_{x=0^-} = \mu'(z)\frac{\partial v'}{\partial x}\Big/_{x=0^+}, \text{ when } z\geq 0, \tag{3.1}$$

and

$$T(z) = 0, \text{ when } \quad -\delta_1 \leq z < 0 \tag{3.2}$$

(because of condition (2.1c)). From (2.2) and (2.3), we have both

$$T(z) = \mu(z)\frac{\partial v}{\partial x}\Big/_{x\to 0-} = -\mu(z)\Big[\sum_{m=1}^{r} ik_m(A_m - B_m)\,\chi_m(z) + \int_0^{\omega/\beta_2} ik\{C(k)-D(k)\}\phi(z,k)dk + \int_0^{\infty} k\,E(k)\,\psi(z,k)\,dk\Big] \tag{3.3}$$

and

$$T(z) = \mu'(z)\frac{\partial v'}{\partial x}\Big/_{x\to 0+} = -\mu'(z)\Big[\sum_{m=1}^{s} ik'_m(A'_m - B'_m)\chi'_m(z) + \int_0^{\omega/\beta'_2} ik'\{C'(k')-D'(k')\}\phi'(z,k')dk' + \int_0^{\infty} k'E'(k')\psi'(z,k')\,dk'\Big] \tag{3.4}$$

If we multiply equation (3.3) by $\chi_m(z)$, $m=1,2,\ldots,r$, $\phi(z,k)$, $0<k\leq\omega/\beta_2$ and $\psi(z,k)$, $0<k\leq\infty$ separately, integrate each side with respect to z from 0 to $\infty$ and use the orthonormality relations (2.27 a, c, e, g, h, j), we obtain the following

$$-ik_m(A_m-B_m) = \int_0^\infty T(\eta)\chi_m(\eta)d\eta \ , \ m = 1,2,\ldots r \qquad (3.5a)$$

$$-ik\{(C(k)-D(k)\} = \int_0^\infty T(\eta)\ \phi(\eta,k)d\eta \qquad (3.5b)$$

$$-k\ E(k) = \int_0^\infty T(\eta)\ \psi(\eta,k)d\eta \qquad (3.5c)$$

Likewise, on multiplying equation (3.4) by $\chi'_m(z)$, $m=1,2,..s$, $\phi'(z,k')$ and $\psi'(z,k')$ separately, integrating each expression with respect to z from $-\delta_1$ to $\infty$ and using the orthonormality relations (2.27 b, d, f, g, i, k) together with condition (3.2), we obtain

$$-ik'_m(A'_m - B'_m) = \int_0^\infty T(\eta)\chi'_m(\eta)d\eta \ , \ m=1,2,\ldots,s, \qquad (3.5d)$$

$$-ik'\{C'(k')- D'(k')\} = \int_0^\infty T(\eta)\phi'(\eta,k')\ d\eta, \qquad (3.5e)$$

$$-k'E'(k') = \int_0^\infty T(\eta)\psi'(\eta,k')d\eta \qquad (3.5f)$$

For incident Love waves, travelling from left to right, we can set $C(k) = C'(k') = 0$ in equations (2.2), (2.3), (3.3), (3.4) and (3.5 b,c) without any loss of generality. Invoking the matching conditions (2.1 d) and substituting for $E(k)$, $E'(k')$, $D(k)$ and $D'(k')$ from equations (3.5 b,c,e) and (f) (with $C(k) = C'(k') = 0$) into (2.2) and (2.3) we obtain

$$\sum_{m=1}^{r} (A_m+B_m)\chi_m(z)+\sum_{m=1}^{s} (A'_m+B'_m)\chi'_m(z)=\int_0^\infty T(\eta)G(z,\eta)d\eta + i\int_0^\infty T(\eta)g(z,\eta)d\eta, z>0 \tag{3.6}$$

$$=\int_0^\infty \zeta(\eta)\, \mathcal{G}(z,\eta)d\eta, \tag{3.7}$$

where

$$\mathcal{G}(z,\eta) = G(z,\eta) + ig(z,\eta), \tag{3.8}$$

$$G(z,\eta)=\int_0^\infty \frac{\psi(z,k)\psi(\eta,k)}{k}dk + \int_0^\infty \frac{\psi'(z,k')\psi'(\eta,k')}{k'}dk' \quad z>0 \tag{3.9}$$

and

$$g(z,\eta) = \int_0^{\omega/\beta_2} \frac{\phi(z,k)\phi(\eta,k)}{k}dk+\int_0^{\omega/\beta'_2} \frac{\phi'(z,k')\phi'(\eta,k')}{k'}dk'. \tag{3.10}$$

We remark that $G(z,\eta)$ is a Green's function type symmetric kernel, whose real and imaginary parts correspond to non-propagated and propagated modes (respectively) arising out of the continuous part of the spectrum. The integral equation formulation of the problem is given by equations (3.5 a,d) and (3.7). If the amplitudes ($A_m$, m = 1,2,---r, $A_m'$, m = 1,2,...,s) of incident Love waves are specified, we have to find the amplitudes of the transmitted and reflected waves from the above-mentioned (r+s+1) integral equations. Using matrix formalism the problem can be stated in terms of a 'scattering matrix' as in Kazi (1978 a). The approximate formulae for the elements of the scattering matrix and the resulting reflection and transmission coefficients arising from the approximation based upon the neglect of modes corresponding to continuous spectrum and the variational approximation based upon the Schwinger-Levine variable, are identical in form to the formulae derived for the step problem in Kazi (1978 a). We shall, therefore, omit the details already covered in Kazi (1978 a) and confine ourselves to the derivation of explicit expressions for the reflection and transmission coefficients for various cases under both approximations.

# 4. FORMULAE FOR REFLECTION AND TRANSMISSION COEFFICIENTS

## (i) Approximation Based upon the Neglect of Modes Corresponding to the Continuous Spectrum

The transmission matrix obtained in Kazi (1978a) is given by the formula

$$\underline{T} = \begin{pmatrix} -\underline{Q} \\ \underline{R} \end{pmatrix}^{-1} \begin{pmatrix} \underline{Q} \\ \underline{R} \end{pmatrix} , \qquad (4.1)$$

where $\underline{Q}$ and $\underline{R}$ are given by

$$\underline{Q} = \left( \underbrace{\begin{matrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{matrix}}_{r \times r} \;\middle|\; \underbrace{\begin{matrix} \lambda_{11} P_{11} & \lambda_{21} P_{21} & \cdots & \lambda_{s1} P_{s1} \\ \lambda_{12} P_{12} & \lambda_{21} P_{22} & \cdots & \lambda_{s2} P_{s2} \\ \vdots & & & \\ \lambda_{1r} P_{1r} & \lambda_{2r} P_{2r} & \cdots & \lambda_{sr} P_{sr} \end{matrix}}_{r \times s} \right) \qquad (4.2)$$

and

$$\underline{R} = \left( \underbrace{\begin{matrix} \frac{P_{11}}{\lambda_{11}} & \frac{P_{12}}{\lambda_{12}} & \cdots & \frac{P_{1r}}{\lambda_{1r}} \\ \frac{P_{21}}{\lambda_{21}} & \frac{P_{22}}{\lambda_{22}} & \cdots & \frac{P_{2r}}{\lambda_{2r}} \\ \vdots & \vdots & & \vdots \\ \frac{P_{s1}}{\lambda_{s1}} & \frac{P_{s2}}{\lambda_{s2}} & \cdots & \frac{P_{sr}}{\lambda_{sr}} \end{matrix}}_{s \times r} \;\middle|\; \underbrace{\begin{matrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{matrix}}_{s \times s} \right) , \qquad (4.3)$$

The Pim, given by

$$\lambda_{im} P_{im} = \int_0^\infty \mu(z)\ \chi_i'(z)\chi_m(z)dz, \quad i=1,2,..s, \quad m=1,2,.\ r, \tag{4.4}$$

where

$$\lambda_{im} = \left(\frac{k_i'}{k_m}\right)^{\frac{1}{2}}, \tag{4.5}$$

($\chi_i'(z)$ and $\chi_m(z)$ are given by (2.4) and (2.5)), are called <u>Coupling Coefficients</u>

$$\text{If } \underline{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_r \\ A_1' \\ \cdot \\ \cdot \\ A_s' \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_r \\ B_1' \\ \cdot \\ \cdot \\ B_s' \end{pmatrix}, \tag{4.6}$$

then $B = \underline{T}.\ \underline{A}$, ($A_i, A_j', B_i, B_j'$ $i=1,2....r$, $j=1,2..s$ are as in (2.2) and (2.3)), (4.7)

where $\underline{T}$ is given by (4.1), yields the reflection and transmission coefficients.

The particular forms of $\underline{T}$ in the special cases $r \geq 1$, $s = 1$, $r = 1$, $s \geq 1$ are given by

$$\underline{T} = \frac{1}{N}\begin{bmatrix} -N+2P_{11}^2 & \frac{2P_{11}P_{12}\ \lambda_{11}}{\lambda_{12}} \cdots & \frac{2P_{11}P_{1r}\ \lambda_{11}}{\lambda_{1r}} & -2P_{11}\lambda_{11} \\ \frac{2P_{11}\lambda_{12}P_{12}}{\lambda_{11}} & -N+2P_{12}^2 \cdots & \frac{2P_{12}P_{1r}\lambda_{12}}{\lambda_{1r}} & -2P_{12}\lambda_{12} \\ \vdots & & & \vdots \\ \frac{2P_{1r}\lambda_{1r}\ P_{11}}{\lambda_{11}} & \frac{2P_{12}P_{1r}\lambda_{1r}}{\lambda_{12}} & -N+2P_{1r}^2 \cdots & -2P_{11}\lambda_{1r} \\ \frac{-2P_{11}}{\lambda_{11}} & \frac{-2P_{12}}{\lambda_{12}} & \frac{-2P_{1r}}{\lambda_{1r}} \cdots & -N+2 \end{bmatrix} \tag{4.8}$$

where

$$N = 1 + P_{11}^2 + P_{12}^2 + \ . \ . \ . + P_{1r}^2 \tag{4.9}$$

and

$$\underline{T} = \frac{1}{N}\begin{bmatrix} -2+N & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \cdots & -2\lambda_{s1}P_{s1} \\ \frac{-2P_{11}}{\lambda_{11}} & -2P_{11}^2+N & \frac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \cdots & \frac{-2P_{11}\lambda_{s1}P_{s1}}{\lambda_{11}} \\ \frac{-2P_{21}}{\lambda_{21}} & \frac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -2P_{21}^2+N \cdots & \frac{-2P_{21}\lambda_{s1}P_{s1}}{\lambda_{21}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2P_{s1}}{\lambda_{s1}} & \frac{-2P_{s1}P_{11}\lambda_{11}}{\lambda_{s1}} & & -2P_{s1}^2+N \end{bmatrix}, \tag{4.10}$$

where $N = 1 + P_{11}^2 + P_{21}^2 + \ldots\ldots + P_{s1}^2$. (4.11)

All the formulae for the reflection and transmission coefficients, which were derived in terms of the coupling coefficients for various special cases in Kazi (1978 a) remain unchanged for the present problem. We must, however, re-evaluate the coupling coefficients given by the integral (4.4). We find (see equation (A8) in the appendix):

$$\lambda_{im} P_{im} = F_i' F_m \left[ \frac{\mu_1 \sigma_1'^{(i)} \sin(\sigma_1'^{(i)} (h_1+\delta_1)) - \mu_2 \sigma_2^{(m)} \cos(\sigma_1'^{(i)} (h_1+\delta_1))}{\cos(\sigma_1'^{(i)} (h_2+\sigma_1) \{ (k_m^2 - k_1'^2) + \frac{\omega^2}{b_1^2} \}} \right.$$

$$\left. + \frac{\mu_2 \{\sigma_2'^{(i)} - \sigma_2^{(m)}\} \bar{e}^{\sigma_2^{(m)} \delta_2}}{(k_i^2 - k_m^2) + \frac{\omega^2}{b_2^2}} \right]$$

$$+ \frac{\mu_2 F_i' F_m}{(k_m^2 - k_i'^2) + \frac{\omega^2}{b_3^2}} \left[ \bar{e}^{\sigma_2^{(m)} \delta_2} \left\{ \frac{\mu_2' \sigma_2'^{(i)}}{\mu_1'} - \sigma_2^{(m)} \right\} \right.$$

$$+ \frac{1}{\cos(\sigma_1'^{(i)} (h_2+\delta_1)} \left\{ \sigma_2^{(m)} \cos(\sigma_1'^{(i)} (h_1+\delta_1)) \right.$$

$$\left. \left. - \sigma_1'^{(i)} \sin(\sigma_1'^{(i)} (h_1+\delta_1)) \right\} \right] +$$

$$+ \frac{\mu_1 \sigma_1^{(i)'} F_i' F_m \sin(\sigma_1^{(i)'} \delta_1)}{(k_i'^2 - k_m^2) \cos(\sigma_1^{(m)} h_1) \cos(\sigma_1^{(i)'} (h_2+\delta_1))} , \qquad (A8)$$

where

$$\frac{1}{b_1^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2} \qquad (A9)$$

$$\frac{1}{b_2^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2} \qquad (A\ 10)$$

$$\frac{1}{b_3^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_2^2} \qquad (A\ 11)$$

and $F_m$, $F_i$ are given by (2.8) and (2.11) respectively).

SOME SPECIAL CASES

(I) when i = 1, s=1 and if the incident wave is travelling from right to left with

$$\underline{A} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

then equations (4.6) - (4.9) yield the transmission coefficient

$$\overleftarrow{B}_1 = \frac{-2\lambda_{11} P_{11}}{1+P_{11}^2} \qquad (4.12)$$

and the reflection coefficient

$$\overleftarrow{B}_1' = \frac{1-P_{11}^2}{1+P_{11}^2} , \qquad (4.13)$$

where $P_{11}$ can be obtained from (A8) and $\lambda_{11}$ is given by (4.5). If the incident wave is travelling from <u>left</u> to <u>right</u> with

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then the <u>transmission</u> <u>coefficient</u>

$$\vec{B}_1' = \frac{-2P_{11}}{\lambda_{11}(1+P_{11}^2)} = \frac{k_1}{k_1'} \overleftarrow{B}_1 \qquad (4.14)$$

and the <u>reflection</u> <u>coefficient</u>

$$\vec{B}_1 = \frac{P_{11}^2 - 1}{1 + P_{11}^2} = (-1)\ \overleftarrow{B}_1' \qquad (4.15)$$

(II) When r = 1, s = 2 in equations (2.2) and (2.3) (i.e. the frequency here is such that single (fundamental) mode exists in the left hand domain and the first two modes exist on the right) and if the incident wave is travelling from <u>right</u> to <u>left</u> with

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

then equations (4.6 - (4.9) yield the <u>transmission</u> <u>coefficient</u>

$$\overleftarrow{B}_1 = \frac{-2\lambda_{11} P_{11}}{1+P_{11}^2 + P_{21}^2} \qquad (4.16)$$

and the reflection coefficients

$$\overleftarrow{B}_1' = \frac{1+P_{21}^2 - P_{11}^2}{1+P_{11}^2 + P_{21}^2} , \tag{4.17}$$

$$\overleftarrow{B}_2' = \frac{-2P_{21}\, P_{11}\, \lambda_{11}}{\lambda_{21}(1+P_{11}^2 + P_{11}^2)} \tag{4.18}$$

$P_{11}$ and $P_{22}$ can be obtained from (A8) and $\lambda_{11}$ and $\lambda_{21}$ are given by (4.5). For an incident wave travelling from left to right with

$$A= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

we get the transmission coefficients

$$\overrightarrow{B}_1' = \frac{-\,2P_{11}}{\lambda_{11}(1+P_{11}^2 + P_{21}^2)} = \frac{k_1}{k_1'}\; \overleftarrow{B}_1 \tag{4.19}$$

$$\overrightarrow{B}_2' = \frac{-\,2P_{21}}{\lambda_{21}(1+P_{11}^2 + P_{21}^2)} \tag{4.20}$$

and the reflection coefficient

$$\overrightarrow{B}_1 = \frac{-1+P_{11}^2+P_{21}^2}{1+P_{11}^2 + P_{21}^2} \tag{4.21}$$

(III) When r=1, s=3 in equations (2.2) and (2.3) (single (fundamental) mode on the left and first three modes on the right) and if the incident wave is travelling from <u>left</u> to <u>right</u> with

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

then from equations (4.6)-(4.9), we obtain the <u>transmission coefficients</u>

$$\vec{B}_1' = \frac{-2P_{11}}{\lambda_{11}(1+P_{11}^2+P_{21}^2+P_{31}^2)} \qquad (4.22)$$

$$\vec{B}_2' = \frac{-2P_{21}}{\lambda_{21}(1+P_{11}^2+P_{21}^2+P_{31}^2)} \qquad (4.23)$$

$$\vec{B}_3' = \frac{-2P_{31}}{\lambda_{31}(1+P_{11}^2+P_{21}^2+P_{31}^2)} \qquad (4.24)$$

and the <u>reflection coefficient</u>

$$\vec{B}_1 = \frac{-1+P_{11}^2+P_{21}^2+P_{31}^2}{1+P_{11}^2+P_{21}^2+P_{31}^2}, \qquad (4.25)$$

where expressions for $P_{11}$, $P_{21}$, $P_{31}$ and $\lambda_{11}, \lambda_{21}, \lambda_{31}$ follow from (A8) and (4.5)

(ii) Variational approximation. The formulae based upon Schwinger-Levine variational principle can be derived in exactly the same manner as in Kazi (1978a). We shall, therefore, omit the details and quote the results only for some special cases.

(I) When r=1, s=1 in equations (2.2) and (2.3), then

$$\underline{A} = \begin{bmatrix} A_1 \\ A'_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B'_1 \end{bmatrix}$$

and

$$\underline{T} = \frac{1}{1+P^2_{11}-iI'_{11}} \begin{bmatrix} P^2_{11}-1-iI'_{11} & -2P_{11}\lambda_{11} \\ -\dfrac{2P_{11}}{\lambda_{11}} & 1-P^2_{11}-iI'_{11} \end{bmatrix} \qquad (4.26)$$

where $P_{11}$ and $\lambda_{11}$ are given as before and $I'_{11}$ is given by (A24) in the appendix.

For an incident wave travelling from right to left with $\underline{A} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we obtain the transmission coefficient

$$\overleftarrow{B}_1 = \frac{-2\lambda_{11}P_{11}}{1(1+P^2 11-iI'_{11})} \qquad (4.27)$$

and the reflection coefficient

$$\overleftarrow{B}_1' = \frac{1 - (P_{11}^2 + iI_{11}')}{1 + P_{11}^2 - iI_{11}'} \tag{4.28}$$

For an incident wave travelling from left to right with $\underline{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we obtain the transmission coefficient

$$\overrightarrow{B}_1' = \frac{-2P_{11}}{\lambda_{11}(1+P_{11}^2-iI_{11}')} = \frac{k_1}{k_1'} \overleftarrow{B}_1 \tag{4.29}$$

and the reflection coefficient

$$\overrightarrow{B}_1 = \frac{P_{11}^2-1-iI_{11}'}{\lambda_{11}(1+P_{11}^2-iI_{11}')} \tag{4.30}$$

(II) When r=1, s=2 in equations (2.2) and (2.3), then

$$\underline{A} = \begin{bmatrix} A_1 \\ A_1' \\ A_2' \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} B_1 \\ B_1' \\ B_2' \end{bmatrix}$$

$$\underline{T} = \frac{1}{1+P_{11}^2+P_{21}^2-iI_{11}'} \begin{bmatrix} -1+P_{11}^2+P_{21}^2-iI_{11}' & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \\ \frac{-2P_{11}}{\lambda_{11}} & -P_{11}^2+1+P_{21}^2-iI_{11}' & \frac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \\ \frac{-2P_{21}}{\lambda_{21}} & \frac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P_{21}^2+1+P_{11}^2-iI_{11}' \end{bmatrix},$$

Whence for an incident Love wave travelling from left to right with

$$\underline{A} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

we get the transmission coefficients

$$\vec{B}_1' = \frac{-2P_{11}}{\lambda_{11}(1+P_{11}^2+P_{21}^2-iI_{11}')} \qquad (4.31)$$

$$\vec{B}_2' = \frac{-2P_{21}}{\lambda_{21}(1+P_{11}^2+P_{21}^2-iI_{11}')} \qquad (4.32)$$

and the reflection coefficient

$$\vec{B}_1 = \frac{-1+P_{11}^2+P_{21}^2-iI_{11}'}{1+P_{11}^2+P_{21}^2-iI_{11}'} \qquad (4.33)$$

For an incident Love wave travelling from right to left with

$$\underline{A} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ,$$

we obtain the transmission coefficient

$$\overleftarrow{B}_1 = \frac{-2\lambda_{11}P_{11}}{1+P_{11}^2+P_{21}^2-iI_{11}'} = \frac{k_1'}{k_1} \vec{B}_1' \qquad (4.34)$$

and the reflection coefficients

$$\overset{\leftarrow}{B}{}'_1 = \frac{(1+P_{21}^2-P_{11}^2) - iI'_{11}}{(1+P_{11}^2+P_{21}^2) - iI'_{11}} \qquad (4.35)$$

$$\overset{\leftarrow}{B}{}'_2 = \frac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}(1+P_{11}^2+P_{21}^2-iI'_{11})} \qquad (4.36)$$

(III) When r = 1, s = 3 in equations (2.2) and (2.3), then

$$\underline{A} = \begin{bmatrix} A_1 \\ A'_1 \\ A'_2 \\ A'_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B'_1 \\ B'_2 \\ B'_3 \end{bmatrix}$$

$$\underline{T} = \frac{1}{1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11}} \begin{bmatrix} -1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11} & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} & -2\lambda_{31}P_{31} \\ \frac{-2P_{11}}{\lambda_{11}} & P_{31}^2-P_{11}^2+1+P_{21}^2-iI'_{11} & \frac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} & \frac{-2P_{11}\lambda_{31}P_{31}}{\lambda_{11}} \\ \frac{-2P_{21}}{\lambda_{21}} & \frac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P_{21}^2+P_{11}^2+P_{31}^2+1-iI'_{11} & \frac{-2P_{21}\lambda_{31}P_{31}}{\lambda_{21}} \\ \frac{-2P_{31}}{\lambda_{31}} & \frac{-2P_{31}P_{11}\lambda_{11}}{\lambda_{31}} & \frac{-2P_{31}}{\lambda_{31}}\lambda_{21}P_{21} & -P_{31}^2+1+P_{11}^2+P_{12}^2-iI'_{11} \end{bmatrix}$$

whence for an incident wave travelling from <u>left</u> to <u>right</u>

with $A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we get the <u>transmission</u> <u>coefficients</u>

$$\overset{\rightarrow}{B}{}'_1 = \frac{-2P_{11}}{\lambda_{11}(1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11})} \qquad (4.37)$$

$$\vec{B}'_2 = \frac{-2P_{21}}{\lambda_{21}(1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11})} \tag{4.38}$$

$$\vec{B}'_3 = \frac{-2P_{31}}{\lambda_{31}(1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11})} \tag{4.39}$$

and the <u>reflection</u> <u>coefficient</u>

$$\vec{B}_1 = \frac{-1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11}}{1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11}} \, . \tag{4.40}$$

For an incident wave travelling from <u>right</u> to <u>left</u>

with

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ,$$

we obtain the <u>transmission</u> <u>coefficient</u>

$$\overleftarrow{B}_1 = \frac{-2\lambda_{11}P_{11}}{1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11}} = \frac{k'_1}{k_1}\vec{B}'_1 \tag{4.41}$$

and the <u>reflection</u> <u>coefficients</u>

$$\overleftarrow{B}'_1 = \frac{(1+P_{21}^2+P_{31}^2-P_{11}^2)-iI'_{11}}{(1+P_{11}^2+P_{21}^2+P_{31}^2)-iI'_{11}} \tag{4.42}$$

$$\overleftarrow{B}'_2 = \frac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}(1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11})} \tag{4.43}$$

$$\overleftarrow{B}'_3 = \frac{-2P_{31}\;P_{11}\;\lambda_{11}}{\lambda_{31}(1+P_{11}^2+P_{21}^2+P_{31}^2-iI'_{11})} \tag{4.44}$$

All the formulae for reflection and transmission coefficients obtained under the first approximation can be recovered from the corresponding formulae under the variational approximation on setting $I_{11}' = 0$. Thus the parameter $I_{11}'$ incorporates the effects of propagated and non-propagated modes which arise out of continuous part of the spectrum. The variational approximation, therefore, takes into account the contributions from body waves and the non-propagated modes.

In another report, we shall present numerical computation of our results under both approximations for several models of geophysical interest.

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Department of Mathematical Sciences
University of Petroleum and Minerals,
Dhahran, Saudi Arabia (M.H.K.)

Department of Earth Sciences
University of Petroleum and Minerals,
Dhahran, Saudi Arabia (A.N)

## APPENDIX

(i) Evaluation of $P_{im}$ :

Consider

$$I = \lambda_{im} P_{im} = \int_0^{\infty} \mu(z) \chi_i'(z) \chi_m(z) dz, \tag{A1}$$

$$\lambda_{im} = \left(\frac{k_i'}{k_m}\right)^{\frac{1}{2}} ; \tag{A2}$$

$i = 1,2,\ldots,s; \quad j = 1,2,\ldots,r.$

We write

$$I = \mu_1 \int_0^{h_1} \phi_1'^{(i)}(z) \phi_1^{(m)}(z) dz + \mu_2 \int_{h_1}^{h_2} \phi_1'^{(i)}(z) \phi_2^{(m)}(z) dz$$

$$+ \mu_2 \int_{h_2}^{\infty} \phi_2'^{(i)}(z) \phi_2^{(m)}(z) dz$$

$$= I_1 + I_2 + I_3 , \tag{A3}$$

where

$$I_1 = \frac{\mu_1 F_i' F_m}{\cos\{\sigma_1'^{(i)}(h_2+\delta_1)\}\cos(\sigma_1^{(m)} h_1)} \int_0^{h_1} \cos(\sigma_1^{(m)} z)\cos\{\sigma_1'^{(i)}(z+\delta_1)\} dz$$

(using (2.6) and (2.9))

$$= \frac{\mu_1 F_i' F_m}{2\cos\{\sigma^{(i)'}(h_2+\delta_1)\}\cos(\sigma_1^{(m)}h_1)} \left| \frac{\sin\{(\sigma_1^{(i)'}+\sigma_1^{(m)})z+\sigma_1^{(i)'}\delta_1\}}{\sigma_1'^{(i)}+\sigma_1^{(m)}} \right.$$

$$\left. + \frac{\sin\{(\sigma_1^{(i)'}-\sigma_1^{(m)})z+\sigma_1^{(i)'}\delta_1\}}{\sigma_1^{(i)'}-\sigma_1^{(m)}} \right|_0^h$$

$$= \frac{\mu_1 F_i' F_m}{2\cos\{\sigma_1^{(i)'}(h_2+\delta_1)\}\cos(\sigma_1^{(m)}h_1)} \left[ \frac{1}{\sigma_1^{(i)'}+\sigma_1^{(m)}} \{\sin[\sigma_1^{(i)'}(h_1+\delta_1)+\sigma_1^{(m)}h_1]\right.$$

$$\left. - \sin(\sigma_1^{(i)'}\delta_1\} + \frac{1}{\sigma_1^{(i)'}-\sigma_1^{(m)}} \{\sin[\sigma_1^{(i)'}(h_1+\delta_1)-\sigma_1^{(m)}h_1]-\sin\sigma_1^{(i)'}\delta_1\} \right]$$

$$= \tfrac{1}{2}\mu_1 F_i' F_m \frac{\cos(\sigma_1^{(i)'}(h_1+\delta_1))}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \left[ \frac{\tan(\sigma_1^{(i)'}(h_1+\delta_1))+\tan\sigma_1^{(m)}h_1}{\sigma_1^{(i)'}+\sigma_1^{(m)}} + \frac{\tan(\sigma_1^{(i)'}(h_1+\delta_1))-\tan\sigma_1^{(m)}h_1}{\sigma_1^{(i)'}-\sigma_1^{(m)}} \right]$$

$$- \frac{\mu_1 F_i' F_m \sin(\sigma_1^{(i)'}\delta_1)}{2\cos\{\sigma_1^{(i)'}(h_2+\delta_1)\}\cos(\sigma_1^{(m)}h_1)} \left[ \frac{1}{\sigma_1^{(i)'}+\sigma_1^{(m)}} + \frac{1}{\sigma_1^{(i)'}-\sigma_1^{(m)}} \right]$$

$$= F_i' F_m \frac{\cos(\sigma_1^{(i)'}(h_1+\delta_1))}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \left[ \frac{\mu_1 \sigma_1^{(i)'}\tan(\sigma_1^{(i)'}(h_1+\delta_1)) - \mu_1\sigma_1^{(m)}\tan\sigma_1^{(m)}h_1}{(\sigma_1^{(i)'})^2 - (\sigma_1^{(m)})^2} \right]$$

$$- \frac{\mu_1 \sigma_1^{(i)'} F_i' F_m \sin(\sigma_1^{(i)'}\delta_1)}{\{(\sigma_1^{(i)'})^2 - (\sigma_1^{(m)})^2\}\cos(\sigma_1^{(i)'}(h_2+\delta_1))\cos(\sigma_1^{(m)}h_1)}$$

$$= \frac{F_i' F_m}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \frac{1}{\{(k_m^2-k_1^2)+\omega^2(\frac{1}{\beta_1'^2}-\frac{1}{\beta_1^2})\}} \left[ \mu_1\sigma_1^{(i)'}\sin(\sigma_1^{(i)'}(h_1+\delta_1))-\mu_2\sigma_2^{(m)}\cos(\sigma_1^{(i)'}(h_1+\delta_1)) \right]$$

$$+ \frac{\mu_1\sigma_1^{(i)'} F_i' F_m \sin(\sigma_1^{(i)'}\delta_1)}{(k_i'^2-k_m^2)\cos(\sigma_1^{(m)}h_1)\cos\sigma_1^{(i)'}(h_2+\delta_1)} \qquad \text{(A4)}$$

(using the period equation (2.15) and the relations :

$$(\sigma_1^{(m)})^2 = \frac{\omega^2}{\beta_1^2} - k_m^2\ ,\ (\sigma_1^{(i)'})^2 = \frac{\omega^2}{\beta_1'^2} - k_1'^2)$$

$$I_2 = \frac{\mu_2 F_i' F_m}{\cos\{\sigma_1^{(i)'}(h_2+\delta_1)} \int_{h_1}^{h_2} \cos\{\sigma_1^{(i)'}(z+\delta_1)\} e^{\sigma_2^{(m)}(h_1-z)} dz$$

$$= \frac{\mu_2 F_i' F_m}{\cos\{\sigma_1^{(i)'}(h_2+\delta_1)\}} \cdot e^{\sigma_2^{(m)}(h_1+\delta_1)} \int_{h_1+\delta_1}^{h_2+\delta_1} e^{-\sigma_2^{(m)} z} \cos(\sigma_1^{(i)'} z) dz$$

$$= \frac{\mu_2 F_i' F_m}{\cos(\sigma_1^{(i)'}(h_2+\delta_1)\}} e^{\sigma_2^{(m)}(h_1+\delta_1)} \left| \frac{e^{-\sigma_2^{(m)} z}}{(\sigma_1^{(i)'})^2 + (\sigma_2^{(m)})^2} \{-\sigma_2^{(m)} \cos(\sigma_1^{(i)'} z) + \sigma_1^{(i)'} \sin \sigma_1^{(i)'} z\} \right|_{h_1+\delta_1}^{h_2+\delta_1}$$

$$= \frac{\mu_2 F_i' F_m}{\cos\{\sigma_1^{(i)'}(h_2+\delta_1)\}} \frac{e^{\sigma_2^{(m)}(h_1+\delta_1)}}{(\sigma_1^{(i)'})^2 + (\sigma_2^{(m)})^2} \Big[ \sigma_1^{(i)'} \{ e^{-\sigma_2^{(m)}(h_2+\delta_1)} \sin(\sigma_1^{(i)'}(h_2+\delta_1))$$

$$- e^{-\sigma_2^{(m)}(h_1+\delta_1)} \sin(\sigma_1^{(i)'}(h_1+\delta_1))\} - \sigma_2^{(m)} \{ e^{-\sigma_2^{(m)}(h_2+\delta_1)} \cos(\sigma_1^{(i)'}(h_2+\delta_1))$$

$$- e^{-\sigma_2^{(m)}(h_1+\delta_1)} \cos(\sigma_1^{(i)'}(h_1+\delta_1))\} \Big]$$

$$= \frac{\mu_2 F_i' F_m}{(\sigma_1^{(i)'})^2 + (\sigma_2^{(m)})^2} \Big[ \sigma_1^{(i)'} \{ e^{-\sigma_2^{(m)} \delta_2} \tan(\sigma_1^{(i)'}(h_2+\delta_1)) - \frac{\sin(\sigma_1^{(i)'}(h_1+\delta_1))}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \}$$

$$- \sigma_2^{(m)} \{ e^{-\sigma_2^{(m)} \delta_2} - \frac{\cos(\sigma_1^{(i)'}(h_1+\delta_1))}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \} \Big] ,$$

where $\delta_2 = h_2 - h_1$ , (A5)

$$= \frac{\mu_2 F_i' F_m}{(\sigma_1^{(i)'})^2 + (\sigma_2^{(m)})^2} \left[ e^{-\sigma_2^{(m)} \delta_2} \left\{ \frac{\mu_2' \sigma_2^{(i)'}}{\mu_1'} - \sigma_2^{(m)} \right\} \right.$$

$$\left. + \frac{1}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \{ \sigma_2^{(m)} \cos(\sigma_1^{(i)'}(h_1+\delta_1)) - \sigma_1^{(i)'} \sin(\sigma_1^{(i)'}(h_1+\delta_1)) \} \right],$$

(using dispersion relation (2.16))

(A6)

$$I_3 = \mu_2 F_i' F_m \int_{h_2}^{\infty} e^{\{\sigma_2^{(m)}(h_1 - z) + \sigma_2^{(i)'}(h_2 - z)\}} dz$$

$$= \mu_2 F_i' F_m \, e^{(\sigma_2^{(m)} h_1 + \sigma_2^{(i)'} h_2)} \cdot \frac{e^{-(\sigma_2^{(m)} + \sigma_2^{(i)'}) h_2}}{\sigma_2^{(m)} + \sigma_2^{(i)'}}$$

$$= \frac{\mu_2 F_i' F_m \, e^{-\sigma_2^{(m)} \delta_2}}{\sigma_2^{(m)} + \sigma_2^{(i)'}}$$

$$= \frac{\mu_2 F_i' F_m \{\sigma_2^{(i)'} - \sigma_2^{(m)}\} \, e^{-\sigma_2^{(m)} \delta_2}}{\{(k_i'^2 - k_m^2) + \omega^2 (\frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2})\}}, \quad \text{(A7)}$$

(using the relations $(\sigma_2^{(i)'})^2 = k_i'^2 - \dfrac{\omega^2}{\beta_2'^2}$ and $(\sigma_2^{(m)})^2 = k_m^2 - \dfrac{\omega^2}{\beta_2^2}$)

whence from (A1), (A3), (A4), (A6) and (A7), we obtain

$$\lambda_{im} P_{im} = F_i' F_m \left[ \frac{\mu_1 \sigma_1^{(i)'} \sin(\sigma_1^{(i)'}(h_1+\delta_1)) - \mu_2 \sigma_2^{(m)} \cos(\sigma_1^{(i)'}(h_1+\delta_1))}{\cos(\sigma_1^{(i)'}(h_2+\delta_1)) \{(k_m^2 - k_1'^2) + \frac{\omega^2}{b_1^2}\}} \right.$$

$$\left. + \frac{\mu_2 \{\sigma_2^{(i)'} - \sigma_2^{(m)}\} \, e^{-\sigma_2^{(m)} \delta_2}}{(k_i'^2 - k_m^2) + \frac{\omega^2}{b_2^2}} \right]$$

$$+ \frac{\mu_2 F_i' F_m}{(k_m^2 - k_i'^2) + \frac{\omega^2}{b_3^2}} \left[ e^{-\sigma_2^{(m)}\delta_2} \{ \frac{\mu_2' \sigma_2^{(i)'}}{\mu_1'} - \sigma_2^{(m)} \} \right.$$

$$\left. + \frac{1}{\cos(\sigma_1^{(i)'}(h_2+\delta_1))} \{ \sigma_2^{(m)} \cos(\sigma_1^{(i)'}(h_1+\delta_1)) - \sigma_1^{(i)'} \sin(\sigma_1^{(i)'}(h_1+\delta_1) \} \right]$$

$$+ \frac{\mu_1 \sigma_1^{(i)'} F_i' F_m \sin(\sigma_1^{(i)'}\delta_1)}{(k_i'^2 - k_m^2)\cos(\sigma_1^{(m)} h_1)\cos(\sigma_1^{(i)'}(h_2+\delta_1))}, \quad \text{(A8)}$$

with $\frac{1}{b_1^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2}$, (A9), $\frac{1}{b_2^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2}$ (A10), $\frac{1}{b_3^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_2^2}$ (A11)

We note that in the limit as $\delta_2 \to 0$, $\mu_1' \to \mu_1$, $\mu_2' \to \mu_2$, $\beta_1' \to \beta_1$ and $\beta_2' \to \beta_2$ the equation (A8) yields the formulae for coupling coefficients for the step problem discussed by Kazi (1978a). Moreover, as $\delta_1 \to 0$, $\delta_2 \to 0$, we obtain the same formulae for the coupling coefficients for the welded quarter space problem discussed by Niazy and Kazi (1980). The formulae for the coupling coefficients for the M-discontinuity step (Kazi and Niazy (1981)) can be obtained from (A8) in the limit as $\delta_1 \to 0$.

(ii) Evaluation of $I_{11}'$

$$I_{11}' = k_1 I_{11} = k_1 \int_0^\infty \frac{dk'}{k'} \left[ \int_0^\infty \mu(\eta)\psi'(\eta,k')\chi_1(\eta)d\eta \int_0^\infty \mu(z)\psi'(z,k')\chi_1(z)dz \right.$$

$$\left. + i\int_0^{\omega/\beta_2'} \frac{dk'}{k'} \int_0^\infty \mu(\eta)\phi'(\eta,k')\chi_1(\eta)d\eta \int_0^\infty \mu(z)\phi'(z,k')\chi_1(z)dz \right] \quad \text{(A12)}$$

We first compute

$$I_1 = \int_0^\infty \mu(z)\phi'(z,k')\chi_1(z)dz$$

$$= \mu_1 \int_0^{h_1} \phi_1'(z,k')\phi_1^{(1)}(z)dz + \mu_2\int_{h_1}^{h_2} \phi_1'(z,k')\phi_2^{(1)}(z)dz$$

$$+ \mu_2\int_{h_2}^{\infty} \phi_2'(z\ ,k')\phi_2^{(1)}(z)dz$$

$$= I_2 + I_3 + I_4, \tag{A13}$$

where

$$I_2 = \frac{\mu_1 G_{k'}' F_1}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))\cos(\sigma_1^{(1)}h_1)} \int_0^{h_1} \cos(\sigma_1^{(1)}z)\ \cos(\sigma_1^{(k')'}(z+\delta_1))dz$$

(using (2.6) and expression for $\phi_1'(z,k)$)

$$= F_1 G_{k'}' \frac{\cos(\sigma_1^{(k')'}(h_1+\delta_1))}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \left[\frac{\mu_1\sigma_1^{(k')'}\tan(\sigma_1^{(k')'}(h_1+\delta_1)) - \mu_1\sigma_1^{(1)}\tan\sigma_1^{(1)}h_1}{(\sigma_1^{(k')'})^2 - (\sigma_1^{(m)})^2}\right]$$

$$- \frac{\mu_1\sigma_1^{(k')'} F_1 G_{k'}' \sin(\sigma_1^{(k')'}\delta_1)}{\{(\sigma_1^{(k')'})^2 - (\sigma_1^{(1)})^2\}\cos(\sigma_1^{(k')'}(h_2+\delta_1))\cos(\sigma_1^{(1)}h_1)}$$

$$= F_1 G_{k'}' \frac{\cos(\sigma_1^{(k')'}(h_1+\delta_1))}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \left[\frac{\mu_1\sigma_1^{(k')'}\tan(\sigma_1^{(k')'}(h_1+\delta_1)) - \mu_2\sigma_2^{(1)}}{(\sigma_1^{(k')'})^2 - (\sigma_1^{(m)})^2}\right]$$

$$- \frac{\mu_1\sigma_1^{(k')'} F_1 G_{k'}' \sin(\sigma_1^{(k')'}\delta_1)}{\{(\sigma_1^{(k')'})^2 - (\sigma_1^{(1)})^2\}\cos(\sigma_1^{(k')'}(h_2+\delta_1))\cos(\sigma_1^{(1)}h_1)}, \tag{A14}$$

$$I_3 = \frac{\mu_2 G'_{k'} F_1}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \int_{h_1}^{h_2} \cos(\sigma_1^{(k')'}(z+\delta_1))\, e^{\sigma_2^{(1)}(h_1-z)} dz$$

(using (2.7) and expression for $\phi_1'(z,k')$)

$$= \frac{\mu_2 F_1 G'_{k'}}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} e^{\sigma_2^{(1)}(h_1+\delta_1)} \int_{h_1+\delta_1}^{h_2+\delta_1} e^{-\sigma_2^{(1)} z} \cos(\sigma_1^{(k')'} z) dz$$

$$= \frac{\mu_2 F_1 G'_{k'}}{(\sigma_1^{(k')'})^2 + (\sigma_2^{(1)})^2} \Bigg[ \sigma_1^{(k')'} \{ e^{-\sigma_2^{(1)}\delta_2} \tan(\sigma_1^{(k')'}(h_2+\delta_1)) - \frac{\sin(\sigma_1^{(k')'}(h_1+\delta_1))}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))}$$

$$- \sigma_2^{(1)} \{ e^{-\sigma_2^{(1)}\delta_2} - \frac{\cos(\sigma_1^{(k')'}(h_1+\delta_1))}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \} \Bigg] \quad , \ \delta_2 = h_2 - h_1$$

$$= \frac{\mu_2 F_1 G'_{k'}}{(\sigma_1^{(k')'})^2 + (\sigma_2^{(1)})^2} \Bigg[ e^{-\sigma_2^{(1)}\delta_2} \{ \sigma_1^{(k')'} \tan(\sigma_1^{(k')'}(h_2+\delta_1)) - \sigma_2^{(1)} \}$$

$$+ \frac{1}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \{ \sigma_2^{(1)} \cos(\sigma_1^{(k')'}(h_1+\delta_1)) - \sigma_1^{(k')'} \sin(\sigma_1^{(k')'}(h_1+\delta_1)) \} \Bigg] \tag{A15}$$

and

$$I_4 = \frac{\mu_2 G'_{k'} F_1}{\sin\theta^{(k')'}} \int_{h_2}^{\infty} e^{\sigma_2^{(1)}(h_1-z)} \sin\{\theta^{(k')'} - s_2^{(k')'}(z-h_2)\} dz$$

(using (2.7) and the expression for $\phi_2'(z\ \ k)$)

$$= - \frac{\mu_2 F_1 G'_{k'}}{(\sigma_2^{(1)})^2 + (s_2^{(k')'})^2} \cdot \frac{1}{\sin\theta^{(k')'}} \{ s_2^{(k')'} \cos\theta^{(k')'} - \sigma_2^{(1)} \sin\theta^{(k')'} \}$$

$$= \frac{\mu_2 F_1 G'_{k'}}{(\sigma_2^{(1)})^2 + (s_2^{(k')'})^2} \{ \sigma_2^{(1)} - \frac{\mu_1'}{\mu_2'} \sigma_1^{(k')'} \tan(\sigma_1^{(k')'}(h_2+\delta_1)) \} \tag{A16}$$

(using relation (2.26) : $\theta'^{(k')} = \tan^{-1}\left(\frac{\mu_2' s_2^{(k')'} \cot \sigma_1^{(k')'}(h_2+\delta_1)}{\mu_1' \sigma_1^{(k')'}}\right)$

From equation (A13)—(A16), we obtain

$$I_1 = F_1 G'_{k'} \left[ \frac{\cos(\sigma_1^{(k')'}(h_1+\delta_1))}{\cos(\sigma_1^{(k')'}(h_2+\delta_1))} \cdot \frac{\mu_1 \sigma_1^{(k')'} \tan(\sigma_1^{(k')'}(h_1+\delta_1)) - \mu_2 \sigma_2^{(1)}}{\{(\sigma_1^{(k')'})^2 - (\sigma_1^{(1)})^2\}} \right.$$

$$- \frac{\mu_1 \sigma_1^{(k')'} \sin(\sigma_1^{(k')'} \delta_1)}{\{(\sigma_1^{(k')'})^2 - (\sigma_1^{(1)})^2\} \cos(\sigma_1^{(k')'}(h_2+\delta_1)) \cos \sigma_1^{(1)} h_1}$$

$$+ \frac{\mu_2 e^{-\sigma_2^{(1)} \delta_2} \{\sigma_1^{(k')'} \tan(\sigma_1^{(k')'}(h_2+\delta_1)) - \sigma_2^{(1)}\}}{(\sigma_2^{(1)})^2 + (\sigma_1^{(k')'})^2}$$

$$+ \frac{\mu_2 \sec(\sigma_1^{(k')'}(h_2+\delta_1)) \{\sigma_2^{(1)} \cos(\sigma_1^{(k')'}(h_1+\delta_1)) - \sigma_1^{(k')'} \sin(\sigma_1^{(k')'}(h_1+\delta_1))\}}{(\sigma_1^{(k')'})^2 + (\sigma_2^{(1)})^2}$$

$$\left. + \frac{\mu_2 \sigma_2^{(1)} - \frac{\mu_1' \mu_2}{\mu_2'} \sigma_1^{(k')'} \tan(\sigma_1^{(k')'}(h_2+\delta_1))}{(\sigma_2^{(1)})^2 + (s_2^{(k')'})^2} \right] \qquad \text{(A17)}$$

Using the relations

$$(\sigma_1^{(k')'})^2 - (\sigma_1^{(1)})^2 = (k_1^2 - k'^2) + \frac{\omega^2}{b_1^2}, \quad \frac{1}{b_1^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2}$$

$$(\sigma_2^{(1)})^2 + (s_2^{(k')'})^2 = (k_1^2 - k'^2) + \frac{\omega^2}{b_2'^2}, \quad \frac{1}{b_2'^2} = \frac{1}{\beta_2'^2} - \frac{1}{\beta_2^2}$$

$$(\sigma_2^{(1)})^2 + (\sigma_1^{(k')'})^2 = (k_1^2 - k'^2) + \frac{\omega^2}{b_3^2}, \quad \frac{1}{b_3^2} = \frac{1}{\beta_1'^2} - \frac{1}{\beta_2^2},$$

we rewrite (A17) :

$$I_1 = F_1 G'_{k'} H_1(k'), \tag{A18}$$

where

$$H_1(k') = \Bigg[\frac{\cos\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_1+\delta_1)\}}{\cos\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_2+\delta_1)\}} \cdot \frac{\mu_1(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}\tan\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_1+\delta_1)\} - \mu_2(k_1^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}}}{k_1^2 - k'^2 + \frac{\omega^2}{b_1^2}}$$

$$- \frac{\mu_1(\frac{\omega^2}{\beta_1'^2} - k'^2)^{\frac{1}{2}} \sin\{(\frac{\omega^2}{\beta_1'^2} - k'^2)^{\frac{1}{2}}\delta_1\}}{(k_1^2-k'^2+\frac{\omega^2}{b_1^2})\cos\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_2+\delta_1)\}\cos(\frac{\omega^2}{\beta_1^2}-k_1^2)^{\frac{1}{2}} h_1\}}$$

$$+ \frac{\mu_2(\exp\{-(k^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}}(h_2-h_1)\})[(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}\tan\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_2+\delta_1)\}-(k_1^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}}]}{k_1^2 - k'^2 + \frac{\omega^2}{b_3^2}}$$

$$+ \frac{\mu_2[(k_1^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}}\cos\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_1+\delta_1)\} - (\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}\sin\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_1+\delta_1)\}]}{\cos\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_2+\delta_1)\}(k_1^2-k'^2+\frac{\omega^2}{b_3^2})}$$

$$+ \frac{\mu_2\mu_2'(k_1^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}} - \mu_2\mu_1'(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}\tan\{(\frac{\omega^2}{\beta_1'^2}-k'^2)^{\frac{1}{2}}(h_2+\delta_1)\}}{\mu_2'(k_1^2-k'^2+\frac{\omega^2}{b_2'^2})}\Bigg] \tag{A19}$$

$$F_1 = \left\{\frac{2(k_1^2-\frac{\omega^2}{\beta_2^2})^{\frac{1}{2}}}{\mu_2} \; \frac{(\beta_1^{-2} - U_1^{-1}C_1^{-1})}{(\beta_1^{-2} - \beta_2^{-2})}\right\}^{\frac{1}{2}}, \tag{A20}$$

$$G'_{k'} = \left(\frac{2k'}{\pi\mu_2' s_2'(k')}\right)^{\frac{1}{2}} \sin\theta'(k')$$

$$= \left[ \frac{2k'\mu' s_2^{(k')'}}{\pi\{\mu_1'^2 (\sigma_1^{(k')'})^2 \tan^2\{\sigma_1^{(k')'}(h_2+\delta_1)\} + \mu_2'^2 (s_2^{(k')'})^2\}} \right]^{\frac{1}{2}}$$

$$= \frac{\{2k'\mu_2' (\frac{\omega^2}{\beta_2'^2} - k'^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{\sqrt{\pi}[\mu'^2 (\frac{\omega^2}{\beta_1'^2} - k'^2) \tan^2\{(\frac{\omega^2}{\beta_1'^2} - k'^2)^{\frac{1}{2}}(h_2+\delta_1)\} + \mu_2'^2 (\frac{\omega^2}{\beta_2'^2} - k'^2)]^{\frac{1}{2}}} \tag{A21}$$

We now consider the integral

$$I_5 = \int_0^\infty \mu(z)\psi'(z,k')\chi_1(z)dz \,, \tag{A22}$$

where $\psi'(z,k')$ is given by (2.23) and $\chi_1(z)$ is given by (2.4). Following the same procedure as for the evaluation of the integral $I_{11}$, we obtain

$$I_5 = \frac{F_1\{2k'\mu_2' (\frac{\omega^2}{\beta_2'^2} + k'^2)^{\frac{1}{2}}\}^{\frac{1}{2}} H_2(k')}{\sqrt{\pi}[\mu_1'^2 (\frac{\omega^2}{\beta_1'^2} + k'^2)\tan^2\{(\frac{\omega^2}{\beta_1'^2} + k'^2)^{\frac{1}{2}}(h_2+\delta_1)\} + \mu_2'^2 (\frac{\omega^2}{\beta_2'^2} + k'^2)]^{\frac{1}{2}}} \,,$$

where $F_1$ is given by (A20) and

$$H_2(k') = H_1(ik') \,, \tag{A23}$$

where $H_1(k')$ is given by (A19).

From equations (A12)—(A23), we finally obtain :

$$I_{11}' = k_1 \frac{4(k_1^2 - \frac{\omega^2}{\beta_2^2})\mu_2'(\beta_1^{-2} - U_1^{-1}C_m^{-1})}{\pi\mu_2(\beta_1^{-2} - \beta_2^{-2})} \left[\int_0^\infty \{H_3(k')\}^2 dk' + i\int_0^{\omega/\beta_2'} \{H_4(k')\}^2 dk'\right] \,, \tag{A24}$$

where

$$H_3(k') = \frac{\left(\frac{\omega^2}{\beta_2'^2} - k'^2\right)^{1/4} H_1(k')}{\left[\mu_1'^2\left(\frac{\omega^2}{\beta_1'^2} - k'^2\right)\tan^2\left\{\left(\frac{\omega^2}{\beta_1'^2} - k'^2\right)^{1/2}(h_2+\delta_1)\right\} + \mu_2'^2\left(\frac{\omega^2}{\beta_2'^2} - k'^2\right)\right]^{1/2}},$$

$$H_4(k') = \frac{\left(\frac{\omega^2}{\beta_2'^2} + k'^2\right)^{1/4} H_2(k')}{\left[\mu_1'^2\left(\frac{\omega^2}{\beta_1'^2} + k'^2\right)\tan^2\left\{\left(\frac{\omega^2}{\beta_1'^2} + k'^2\right)^{1/2}(h_2+\delta_1)\right\} + \mu_2'^2\left(\frac{\omega^2}{\beta_2'^2} + k'^2\right)\right]^{1/2}}$$

and $H_1(k')$ and $H_2(k')$ are given by (A19) and (A20) respectively.

A check on the above formulae is provided by the fact that in the limit as $\delta_2 \to 0$, $\mu_1' \to \mu_1$, $\mu_2' \to \mu_2$, $\beta_1' \to \beta_1$ and $\beta_2' \to \beta_2$, these formulae reduce to the formulae derived for the step problem in Kazi (1978a). Similarly in the limit as $\delta_1 \to 0$, $\delta_2 \to 0$ we recover the formulae for the welded quarter spaces problem (Niazy and Kazi (1980)) and in the limit as $\delta_1 \to 0$ we obtain the same formulae as for the M-discontinuity step problem (Kazi and Niazy (1981)). We remark that the real and imaginary parts of $I_{11}'$ in (A24) correspond to the non-propagated and propagated modes arising from the continuous part of the spectrum. The integrands in (A24) are of the order of $\frac{1}{k'^3}$ for large values of $k'$ and are regular at all points in their respective domains of definition. The integrals are, therefore, convergent. However, in view of the complicated forms of the integrands, the integrals have to be evaluated numerically.