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A Condition Implying Separability of a Reflexive Banach Space

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A condition implying separability of a reflexive Banach space

by

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Abstract. We prove that if X is a reflexive Banach space then X is separable if there exist a Banach space Y and a compact one-to-one operator T in $L(X, Y)$.

1. Introduction. In [5] J.D. Weston proves that given any separable Banach space Y, there exists a normed linear space X and a compact one-to-one linear operator which maps the conjugate space X^*, of X onto a subspace dense in Y. In [3], Goldberg and Kruse gave a condition under which there exists a one-to-one compact linear operator from a normed linear space X onto a dense subspace of a normed linear space Y. In the present paper we prove that a reflexive Banach space X is separable if there exist a Banach space Y and a compact one-to-one linear operator T from X into Y. We use the result of Goldberg and Kruse to show - by an example - that the reflexivity of X is necessary.

Let X and Y be two Banach spaces. Then $L(X, Y)$ (respectively, $K(X, Y)$) is the space of all bounded (respectively, compact) linear operators from X to Y. $L_0(X, Y)$ (respectively, $K_0(X, Y)$) is the set of all one-to-one operators in $L(X, Y)$ (respectively, $K(X, Y)$). $L_d(X, Y)$ (respectively, $K_d(X, Y)$) is the set of all operators in

$L(X, Y)$ (respectively, $K(X, Y)$) with range dense in Y. $L_{o,d}(X, Y) = L_o(X, Y) \cap L_d(X, Y)$ and $K_{o,d}(X, Y) = K_o(X, Y) \cap K_d(X, Y)$.

2. Preliminaries.

2.1. Definition. A family $\Gamma \subseteq X^*$ is called total if and only if $y \in X$, $f(y) = 0$ for all f in Γ imply $y = 0$.

2.2. The following remarks are easily verified.

(i) If X is a separable normed vector space, then each of the dual spaces X^* and X^{**} contains a countable total subset.

(ii) If X and Y are normed vector spaces, $L_o(X, Y) \neq \phi$ and Y^* has a countable total subset, then X^* has a countable total subset.

2.3. Lemma. If X is a Banach space and (X_n) is a sequence of elements of X converging weakly to x, then some sequence of convex combinations of the elements x_n converges to x in the metric topology.

Proof. ([2], Corollary 14, p. 422).

2.4. Theorem. Let T be a linear operator of a Banach space X into a Banach space Y. Then T is continuous with respect to the metric topologies in X and Y if and only if it is continuous with respect to the weak topologies.

Proof. ([2], Theorem 15, p. 422).

3. Some characterizations of separable Banach spaces.

3.1. **Theorem.** Suppose that X is an infinite-dimensional normed linear space and Y is an infinite-dimensional Banach space. Then.

(i) $K_o(X, Y) \neq \phi$ if and only if X^* has a countable total subset.

(ii) $K_d(X, Y) \neq \phi$ if and only if Y is separable.

(iii) $K_{o,d}(X, Y) \neq \phi$ if and only if $K_o(X, Y) \neq \phi$ (alternatively, $L_o(X, Y) \neq \phi$) and $K_d(X, Y) \neq \phi$ i.e. if and only if Y is separable and X^* has a countable total subset.

Proof. ([3], p. 810-811).

3.2. **Theorem.** Let X and Y be two Banach spaces such that X is reflexive. Suppose that T, in $L(X, Y)$, is compact and one-to-one. Then X is separable.

Proof. Since T is compact, it is continuous ([1], Theorem 2.7, p. 46). By Theorem 2.4, T is continuous when X and Y are endowed with their weak topologies. Let B be the closed unit ball of X and let $T|B$ denote the restriction of T to B. Since X is reflexive, B is weakly compact ([2], Theorem 7, p. 425). Since T is continuous and one-to-one, $T|B$ is a continuous and one-to-one operator from B to Y. By ([2], Lemma 8, p. 18), $T|B$ is a weak homeomorphism which implies that $(T|B)^{-1}$ is weakly continuous. The compactness of T implies that $T(B)$ is separable ([4], Theorem 5.5-A), hence $T(B)$ contains a countable dense subset, A. Since the weak closure of a set contains its norm closure, A is weakly dense in T(B). Hence the set $J = (T|B)^{-1}(A)$ is weakly dense in B and countable. Let H be the set of all convex combinations with rational coefficients of elements of J, then H is countable and, by Lemma 2.3, norm dense in X. This implies that X is separable.

3.3 Example. Let $X = L_\infty$, be the Banach space of all bounded sequences $x = \{x_n\}_{n=1}^{\infty}$. Then X is not reflexive and not separable. Let L_1 be the Banach space of all sequences $y = \{y_n\}_{n=1}^{\infty}$ for which the norm

$$\|y\| = (\sum_{n=1}^{\infty} \|y_n\|)$$

is finite. Then

$$L_1^* = X \text{ and } L_1 \subseteq X^*.$$

Now consider the set $A = \{e_i \in L_1 : e_i$ has value one in the ith place and zero elsewhere$\}$. Then A is a countable subset of X^*. We show that it is total. Let $t = \{t_i\}_{i=1}^{\infty}$ be a nonzero element of X, then at least one of the entries of t is not zero, say t_j. Hence the element $e_j \in A$ satisfies $e_j(t) \neq 0$ since $e_j(t) = \sum_{i=1}^{\infty} e_i t_i$. Thus A is total and countable in X^*. Now if Y is any separable Banach space then by Theorem 3.1 (iii), $K_{o,d}(X, Y) \neq \phi$.

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