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Some Remarks on the Concentrations Functions of Sums of IID Random Vectors

by

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Abstract

In this note a result of Esseen (1968, Z. Wahrscheinlichkeitstheorie verw. Geb., 9, 290-308) about the rate of convergence of the concentration function of sums of independent random vectors is extended to the case of random sums. Also a new concept of conditioned concentration functions is defined and the rates of convergence for conditioned sums are presented.

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1. Introduction.

The concentration function $Q(X;x)$ of a random variable X is a function of the nonnegative value x defined by

$$Q(X;x) = \sup_{y} P[y \leq X \leq y + x]. \tag{1.1}$$

A multidemensional generalization of (1.1) can be done in several ways. Let $\underline{X}$ be a random vector in R^k. A straight forward generalization of (1.1) is given by:

$$Q(\underline{X},\underline{x}) = \sup_{\underline{y}} P[y_1 \leq X_1 \leq y_1 + x_1,\ldots,y_k \leq X_k \leq y_k + x_k], \tag{1.2}$$

for all $\underline{x} \geq \underline{0}$.

We can also define a "spherical" concentration function of a random variable X; viz.,

$$Q_o(X;x) = \sup_{y} P[|X - y| \leq x]. \tag{1.3}$$

A straightforward generalization of (1.3) to the multidimensional case is as follows; Let $S_\rho(\underline{y})$ be a sphere in R^k with radius ρ and center $\underline{y}$. Then the spherical concentration function of $\underline{X}$ is given by:

$$Q_o(\underline{X};S_\rho) = \sup_{\underline{y}} P[\underline{X} \in S_\rho(\underline{y})]. \tag{1.4}$$

Let $\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_n$ be n independent random vectors defined on some probability space (Ω,A,P). Further, let $\underline{S}_n = \Sigma_{i=1}^n \underline{X}_i$. It is well-known that $Q(\underline{S}_n;\underline{x}) \to 0$ as $n \to \infty$ and similarly $Q_o(\underline{S}_n;S_\rho) \to 0$ as $n \to \infty$. The rate at which $Q(\underline{S}_n;\underline{x})$ converges to 0 was discussed in Esseen (1966) while the analogous results for $Q_o(\underline{S}_n;S_\rho)$ is presented in Esseen (1968) where it is remarked that Q_o is more interesting and practical than Q for the multidimensional case. When the $\underline{X}_i$'s are nondegenerate identically distributed random vectors, a result of Esseen (1968) (see Corollary 2, p. 305) can be stated as:

$$Q_o(\underline{S}_n; S_\rho) \leq C(k)(\rho/\tau)^k[1-Q_o(\underline{X}_1, S_\tau)]^{-1/2}n^{-1/2}, \tag{1.5}$$

for some $\tau \leq \rho$, where $C(k)$ is a positive constant dependent on k.

The first purpose of this note is to obtain an analogue of (1.5) when the sample size is random. Let $\{N_n\}$ be a sequence of integer-valued random variables not necessarily independent of $\underline{X}_i$'s such that (N_n/n) converges in probability to a positive random variable N, independent of the $\underline{X}_i$'s. Further let $\{\varepsilon_n\}$ be a sequence of real numbers converging to 0 such that $\varepsilon_n = O(n^{-1})$ as $n \to \infty$. Under certain conditions on the tail behavior of (N_n/n) and N we shall establish that

$$Q_o(\underline{S}_n; S_\rho) = O(\varepsilon_n^{1/2}), \text{ for all } \rho \geq 0. \tag{1.5}$$

This result is given in Theorem 1.

Next, let $B \in A$ be an event such that $P(B) > 0$. We define the conditioned concentration function of $\underline{X}$ by

$$Q_o(\underline{X}; S_\rho | B) = \sup_{\underline{y}} P[\underline{X} \in S_\rho(\underline{y}) | B]. \tag{1.6}$$

An interesting question to ask would be whether we can establish the rates of $Q_o(\underline{S}_n; \underline{x} | B)$. We note that it is not possible to establish rates that are valid for $B \in A$. We are able, however, to obtain analogues of (1.5) when $B \in F_m$ such that $P(B) > 0$ where $F_m = \sigma(\underline{X}_1, \ldots, \underline{X}_m)$ is the σ-field generated by $\underline{X}_1, \ldots, \underline{X}_m$.

This is done in Theorem 2. Using this result we are able to show that for any $B \in A$ such that $P(B) > 0$, $Q_o(\underline{S}_n; S_\rho | B) \to 0$ as $n \to \infty$.

2. Main Results.

THEOREM 1. *Assume that* $E(N) < \infty$ *and that for some positive constants* C_1 *and* C_2 *the following conditions hold:*

(i) $P[|\frac{N_n}{nN} - 1| > C_1\varepsilon_n] = 0(\varepsilon_n{}^{1/2})$

(ii) $P[N < \frac{C_2}{n\varepsilon_n}] = 0(\varepsilon_n{}^{1/2})$

If $Q_o(\underline{S}_n; S_\rho) = 0(n^{-1/2})$, *then* $Q_o(\underline{S}_{N_n}; S_\rho) = 0(\varepsilon_n{}^{1/2})$.

PROOF. First we show that $Q_o(\underline{S}_{[nN]}; S_\rho) = 0(\varepsilon_n{}^{1/2})$, where $[x]$ denotes the largest integer in x. Since N and $\underline{X}_i$'s are independent

$$\begin{aligned} Q_o(\underline{S}_{[nN]}, S_\rho) &= \Sigma_{\ell=1}^{\infty} \sup_{\underline{y}} P[\underline{S}_\ell \in S_\rho(\underline{y}),\ [nN] = \ell] \\ &\le \Sigma_{\ell=1}^{\infty} Q_o(\underline{S}_\ell, S_\rho)\ P[[nN] = \ell] \\ &\le P[N < \frac{C_2}{n\varepsilon_n}] + C\ \Sigma_{\ell=[C_2/\varepsilon_n]}^{\infty} \ell^{-1/2} P[[nN] = \ell] \\ &= 0(\varepsilon_n{}^{1/2}), \end{aligned} \tag{2.1}$$

for some $C > 0$ by Condition (ii) and since for sufficiently large ℓ, $\varepsilon_\ell = 0(\ell^{-1})$. Next, let

$$I_n = \{\ell \in \mathbb{N}:\ [nN](1-C_1\varepsilon_n) \le \ell \le [nN](1+C_1\varepsilon_n)\}, \tag{2.2}$$

where $\mathbb{N}$ denotes the set of nonnegative integers. Hence,

$$\begin{aligned} Q_o(\underline{S}_{N_n}, S_\rho) &\le \sup_{\underline{y}} P[\underline{S}_{N_n} \in S_\rho(\underline{y}),\ N_n \in I_n] + P[N_n \notin I_n] \\ &= \sup_{\underline{y}} P[\underline{S}_{N_n} \in S_\rho(\underline{y}),\ N_n \in I_n] + 0(\varepsilon_n{}^{1/2}), \end{aligned} \tag{2.3}$$

by Condition (i). Let us deal with the first term in (2.3). Assume without loss of generality that $[nN](1-C_1\varepsilon_n)$ and $[nN](1+C_1\varepsilon_n)$

are integers, otherwise take the integer part in them. Therefore, for all $p \geq 0$, $\underline{y} \in R^k$,

$$\{\underline{S}_{N_n} \in S_\rho(\underline{y}),\ N_n \in I_n\} = \{\underline{S}_{[nN](1-C_1\varepsilon_n)} \in S_\rho(\underline{y})\} \cup \{\underline{S}_{[nN](1-C_1\varepsilon_n)+1} \in S_\rho(\underline{y})\} \cup \ldots \cup \{\underline{S}_{[nN](1+C_1\varepsilon_n)} \in S_\rho(\underline{y})\}. \tag{2.4}$$

Then

$$\sup_{\underline{y}} P[\underline{S}_{N_n} \in S_\rho(\underline{y}), N_n \in I_n] = \sup_{\underline{y}} P[\bigcup_{q=[nN](1-C_1\varepsilon_n)}^{[nN](1+C_1\varepsilon_n)} \{\underline{S}_q \in S_\rho(\underline{y})\}]$$

$$= \sup_{\underline{y}} \{\Sigma_{\ell=0}^{\infty} P[\bigcup_{q=\ell(1-C_1\varepsilon_n)}^{\ell(1+C_1\varepsilon_n)} \{\underline{S}_q \in S_\rho(\underline{y})\}] P[[nN] = \ell]\}$$

$$\leq P[N < \frac{C_2}{n\varepsilon_n}] + \Sigma_{\ell=[C_2/\varepsilon_n]}^{\infty} P[[nN] = \ell]\ \Sigma_{q=\ell(1-C_1\varepsilon_n)}^{\ell(1+C_1\varepsilon_n)} Q_o(\underline{S}_q, S_\rho)$$

$$\leq P[N < \frac{C_2}{n\varepsilon_n}] + \Sigma_{\ell=[C_2/\varepsilon_n]}^{\infty} 2\ell C_1\varepsilon_n 0(\varepsilon^{1/2}_{\ell(1-C_1\varepsilon_n)}) P[[nN] = \ell]$$

$$\leq P[N < \frac{\varepsilon_2}{n\varepsilon_n}] + C\varepsilon_n^{3/2} E([nN]) = 0(\varepsilon_n^{1/2}), \tag{2.5}$$

since $E(N) < \infty$, $\varepsilon_n = 0(n^{-1})$, $n \to \infty$, and $Q_o(\underline{S}_n, S_\rho) = 0(n^{-1/2})$ (by assumption). Thus the theorem is completely proved. ||

<u>REMARKS</u>. (i) Following exactly the same proof as that of Theorem 1 above, one can show that under the conditions of Theorem 1, if $Q(\underline{S}_n;\underline{x}) = 0(n^{-1/2})$, then $Q(\underline{S}_{N_n};\underline{x}) = 0(\varepsilon_n^{1/2})$. (ii) It follows from Corollary 2 of Esseen (1968), that if $\underline{X}_1$ is nondegenerate, then $Q_o(\underline{S}_n, S_\rho) = 0(n^{-1/2})$ and hence under conditions of Theorem 1, $Q_o(\underline{S}_{N_n}, S_\rho) = 0(\varepsilon_n^{1/2})$. Esseen (1966), proved that if $\underline{X}_1$ is nondegenerate, then $Q(\underline{S}_n;\underline{x}) = 0(n^{-1/2})$,

thus in view of Remark (i) we have that under conditions of Theorem 1 and if $\underline{X}_1$ is nondegenerate then $Q(\underline{S}_{N_n};\underline{x}) = O(\varepsilon_n^{1/2})$.

Next, we consider the rates of convergence of the concentration function of conditioned sums of iid random vectors. As mentioned in the introduction it is not possible to establish a rate valid for all $B \in A$ with $P(B) > 0$. However, we can show the following:

THEOREM 2. *If $\underline{X}_1$ is nondegenerate, then for any $B \in F_m$ with* $P(B) > 0$,

$$Q_o(\underline{S}_n, S_\rho | B) = O((n-m)^{-1/2}).$$

PROOF. Consider $Q_o(\underline{S}_n, S_\rho, B) \equiv \sup_{\underline{y}} P[\underline{S}_n \in S_\rho(\underline{y}), B]$. Thus

$$Q_o(\underline{S}_n, S_\rho, B) = \sup_{\underline{y}} E\{P[\underline{S}_n \in S_\rho(\underline{y}) | F_m] I_B\}, \tag{2.6}$$

where I_B denote the indicator function of B. But note that if $\underline{S}_{m+1,n} = \Sigma_{i=m+1}^n \underline{X}_i$, we get

$$E\{P[\underline{S}_n \in S_\rho(\underline{y}) | F_m] I_B\} = E\{P[\underline{S}_{m+1,n} \in S_\rho(\underline{y} - \underline{S}_m)] I_B\}$$

$$\leq Q_o(\underline{S}_{n-m}, S_\rho) P(B). \tag{2.7}$$

Hence it follows from (2.6) and (2.7) that

$$Q_o(\underline{S}_n, S_\rho, B) \leq Q_o(\underline{S}_{n-m}, S_\rho) P(B) \tag{2.8}$$

Hence

$$Q_o(\underline{S}_n, S_\rho | B) \leq Q_o(\underline{S}_{n-m}, S_\rho) = O((n-m)^{-1/2}). \quad || \tag{2.9}$$

The following result is a conditional analogue of the result that $Q_o(\underline{S}_n, S_\rho) \to 0$ as $n \to \infty$.

COROLLARY 3. *If $B \in A$ is such that $P(B) > 0$ and if $\underline{X}_1$ is nondegenerate, then*

$$Q_o(\underline{S}_n, S_\rho | B) \to 0 \quad \text{as} \quad n \to \infty.$$

PROOF. Let $F_\infty = \sigma(\underline{X}_1, \underline{X}_2, \ldots)$ be the σ-field generated by $\underline{X}_1, \underline{X}_2, \ldots$. Then there exists random variables $\{\xi_n\}$ such that $0 \le \xi_n \le 1$, ξ_n is F_n-measurable, $n \ge 1$, and

$$E | P(B | F_\infty) - \xi_n | \to 0 \quad \text{as} \quad n \to \infty, \tag{2.10}$$

i.e., given $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $E|P(B|F_m) - \xi_m| < \varepsilon$ for all $m > N$. Now,

$$\begin{aligned} Q_o(\underline{S}_n, S_\rho, B) &= \sup_{\underline{y}} P[\underline{S}_n \in S_\rho(\underline{y}) | B] P(B) \\ &\le \sup_{\underline{y}} \{P[\underline{S}_n \in S_\rho(\underline{y}), E(B|F_\infty)] - P[\underline{S}_n \in S_\rho(\underline{y}), \xi_m] \\ &\qquad + P[\underline{S}_n \in S_\rho(\underline{y}), \xi_m]\} \\ &\le \varepsilon/2 + \sup_{\underline{y}} E\{P[\underline{S}_n \in S_\rho(\underline{y}) | F_m] \xi_m\} \\ &\le \varepsilon/2 + Q_o(\underline{S}_{n-m}, S_\rho)\ E(\xi_m) \\ &\le \varepsilon/2 + C/(n-m)^{1/2} < \infty, \end{aligned} \tag{2.11}$$

for sufficiently large n and m such that $n > m$ and $(n-m) \to \infty$ as $m, n \to \infty$. Note that the second inequality of (2.11) follows from (2.10) and the last 2 follow from Theorem 2.||

REMARKS. (i) Using an argument exactly similar to that of Theorem 2, we can establish that if $\underline{X}_1$ is nondegenerate and if $B \in F_m$, with $P(B) > 0$, then $Q(\underline{S}_n; \underline{x} | B) = 0((n-m)^{-1/2})$. Thus using this result and following the proof of Corollary 3 we can easily establish that if $B \in A$ with $P(B) > 0$, then $Q(\underline{S}_n; \underline{x} | B) \to 0$ as $n \to \infty$.

(ii) Let $N_n > m$ for all $n \ge 1$ be such that $\frac{N_n - m}{n}$ converges in probability to a positive random variable N independent of $\underline{X}_i$'s, let $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n \to 0$ and $\varepsilon_n = 0(n^{-1})$ as $n \to \infty$, and let $B \in F_m$ be such that $P(B) > 0$. Assume further that the following conditions hold:

(i*) $P[|\frac{N_{n-m}}{nN} - 1| > C_1 \varepsilon_{n-m}] = 0(\varepsilon_{n-m}^{1/2})$

(ii*) $P[N < \frac{C_2}{n\varepsilon_{n-m}}] = 0(\varepsilon_{n-m}^{1/2})$

Then $Q_o(\underline{S}_{N_n}, S_\rho | F_m) = 0(\varepsilon_{n-m}^{1/2})$ whenever $Q_o(\underline{S}_n, S_\rho) = 0(n^{-1/2})$. The proof of this result follows directly from the fact that $Q_o(\underline{S}_{N_n}, S_\rho | F_m) \le Q_o(\underline{S}_{N_n - m}, S_\rho) = Q_o(\underline{S}_{N_n^*}, S_\rho)$, say, where $N_n^* = N_n - m$. Thus if $Q_o(\underline{S}_{n-m}, S_\rho) = 0((n-m)^{-1/2})$ and under conditions (i*) and (ii*) it follows from Theorem 1, that $Q_o(\underline{S}_{N_n^*}, S_\rho) = 0(\varepsilon_{n-m}^{1/2})$. Whether we can use this result to establish the rate of $Q_o(\underline{S}_{N_n}, S_\rho | B)$ for any $B \in A$ with $P(B) > 0$ is an interesting and open question we are unable to provide an answer for at this moment.

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