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MIXING LIMIT THEOREMS RELATED TO THE INVARIANCE PRINCIPLE AND THEIR CONVERGENCE RATES

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Abstract

Let $\{X_n\}$ be a sequence of i.i.d. r.v.'s defined on a probability space $(\Omega, \mathcal{S}, P)$ such that $EX_1 = 0$ and $\text{Var}\, X_1 = \sigma^2$. Let $S_n(t)$ denote the usual random function defined on $C[0,1]$ to be equal to $(S_k/\sigma\sqrt{n})$ if $t = (k/n)$ and $S_n(k/n) + n[S_n((k+1)/n) - S_n(k/n)](t-(k/n))$ for $(k/n) \le t \le (k+1)/n$, $k = 1,\cdots,n$, $0 \le t \le 1$.

The purpose of this investigation is to obtain conditioned, in the sense of Renyi (1958), L_p-limit theorem and its rate of convergence of $\sup_t |S_n(t)|$.

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1. Introduction

Let $\{X_n\}$ be a sequence of i.i.d. r.v.'s such that $EX_1 = 0$ and $\mathrm{Var}\, X_1 = \sigma^2$. Define $S_n = \Sigma_{j=1}^{n} X_j$ and consider the random function defined on $C[0,1]$, the space of continuous functions on $[0,1]$, defined by:

$$S_n(t) = \begin{cases} S_k/\sigma\sqrt{n}, & \text{if } t = k/n , \\ S_n(k/n) + n[S_n((k+1)/n) - S_n(k/n)](t-(k/n)), & \text{if } k/n \le t \le (k+1)/n, \end{cases} \quad (1.1)$$

$k = 1, \ldots, n$. Then, see Billingsley (1968), Theorem 10.1, $S_n(t)$ converges weakly to the standard Weiner process $W(t)$, $0 \le t \le 1$. Thus by the continuity theorem, Billingsley (1968), Theorem 5.1, $\sup_t |S_n(t)|$ converges in distribution to $\sup_{0\le t\le 1} |W(t)|$, where, as shown by Erdös and Kač (1964),

$$P[\sup_{0\le t\le 1} |W(t)| \le x] = G(x) = \frac{4}{\pi}\sum_{j=0}^{\infty} [(-1)^j/(2j+1)]\exp[-(2j+1)^2\pi^2/8x^2], \quad x > 0 . \quad (1.2)$$

It is also well-known, see Rozenkrantz (1967), that

$$\lim_{n\to\infty} P[\sup_{0\le t\le 1} |S_n(t)| \le x] = \lim_{n\to\infty} P[\max_{1\le k\le n} |S_k| \le x\sigma\sqrt{n}] = G(x). \quad (1.3)$$

It also follows from Theorem 5 of Rozenkrantz (1967) that the rate of convergence of the uniform deviation between the distribution functions of $\sup_{0\le t\le 1} |S_n(t)|$ and $\sup_{0\le t\le 1} |W(t)|$ is equivalent to that between the distribution functions of $\max_{1\le k\le n} |S_k|/\sigma\sqrt{n}$ and $\sup_{0\le t\le 1} |W(t)|$. Analogous result for the L_1-deviation is indicated by Pinelis (1980). Hence consider:

$$\Delta_{n(p)} = \begin{cases} \{\int |P[\max_{1\le k\le n} |S_k| \le x\sigma\sqrt{n}] - G(x)|^p \, dx\}^{1/p}, & 1 \le p < \infty \\ \\ \sup_x |P[\max_{1\le k\le n} |S_k| \le x\sigma\sqrt{n}] - G(x)|, & p = \infty. \end{cases} \tag{1.4}$$

For $p = \infty$, the rate in $\Delta_{n(\infty)}$ has long been subject to investigation. Chung (1948) showed that if $E|X_1|^3 < \infty$, then $\Delta_{n(\infty)} \le C(\ell n\ \ell n\ n^{\frac{1}{2}}/\ell n\ n^{\frac{1}{2}})^{\frac{1}{2}}$ while Prokhrov (1956) gave the upper bound $C(\ell n\ n)^2/n^{1/8}$ and in 1967, Rozenkrantz and independently Swayer (1967) showed that if $E|X_1^{2+p}| < \infty$, $p > 0$, then the upper bound is $C(\ell n\ n)^{\frac{1}{2}}/n^{-\frac{1}{2}(p-2)/(p+1)}$, and thus in their result if all moments are finite then the rate is $C(\ell n\ n)^{\frac{1}{2}}/n^{-\frac{1}{2}}$. In a major breakthrough, Nagaev (1970) showed that if $E|X_1|^3 < \infty$ then the rate is $C[(E|X_1|^3)^2/\sigma^3]n^{-\frac{1}{2}}$, finally the exact explicit rate was given by Sakhanenko (1974) to be: If $E|X_1|^3 < \infty$, then for all $n \ge 1$,

$$\Delta_{n(\infty)} \le C_1\ E|X_1|^3/\sigma^3 n^{\frac{1}{2}} . \tag{1.5}$$

Very recently, Pinelis (1980) proved that if $E|X_1|^3 < \infty$, then for all $n \ge 1$,

$$\Delta_{n(1)} \le C_2\ E|X_1|^3/\sigma^3 n^{\frac{1}{2}} . \tag{1.6}$$

From (1.5) and (1.6) it follows immediately that if $E|X_1|^3 < \infty$, then for all $n \ge 1$, and any $1 \le p \le \infty$,

$$\Delta_{n(p)} \le C_3\ E|X_1|^3/\sigma^3 n^{\frac{1}{2}} . \tag{1.7}$$

Note that if $\sigma^2 < \infty$, one can obtain from Pinelis (1980) that $\Delta_{n(1)} \to 0$ as $n \to \infty$.

In order to formulate the problem subject to investigation, assume that the sequence $\{X_n\}$ is defined on some probability space $(\Omega, \mathcal{S}, P)$ and let $B \in \mathcal{S}$ be such that $P(B) > 0$. Set

$$\Delta_{n(p)}(B) = \begin{cases} \{\int |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}|B] - G(x)|^p dx\}^{1/p}, & 1 \le p < \infty, \\ \sup_x |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}|B] - G(x)|, & p = \infty. \end{cases} \quad (1.8)$$

The first question to ask is whether $\Delta_{n(p)}(B) \to 0$, as $n \to \infty$, $1 \le p \le \infty$, for any $B \in \mathcal{S}$ with $P(B) > 0$? And if so, can one obtain a rate of this convergence?

In the classical central limit theorem for partial sums S_n, if one combines Corollary 3 of Rogge and Landers (1971) and Remark 2.9 of Ahmad (1981) it is easily seen that if

$$\delta_{n(p)}(B) = \begin{cases} \{\int |P[S_n \le \sigma x\sqrt{n}|B] - \Phi(x)|^p \, dx\}^{1/p}, & 1 \le p < \infty, \\ \sup_x |P[S_n \le \sigma x\sqrt{n}|B] - \Phi(x)|, & p = \infty, \end{cases} \quad (1.9)$$

then $\delta_{n(p)}(B) \to 0$, as $n \to \infty$ for any $B \in \mathcal{S}$ with $P(B) > 0$. As for the rate of convergence, Rogge and Landers (1977) showed that there does not exist a uniform positive constant C such that for any $B \in \mathcal{S}$ with $P(B) > 0$, $\delta_{n(p)}(B) \le Cn^{-\frac{1}{2}}$. A weaker result is possible, if $B \in \mathcal{F}_k$, the σ-field generated by $X_1, \cdots, X_k$, with $P(B) > 0$, then (see Ahmad (1981), Corollaries 2.2 and 2.7) $\delta_{n(p)}(B) \le [P(B)]^{1/r} C_r(k/n)^{\frac{1}{2}}$ for some $2 \le r \le 3$.

Thus the second question is reformulated to obtain rates of convergence of $\Delta_{n(p)}(B)$ for $B \in F_k$ with $P(B) > 0$.

2. Statement of main results

Theorem 2.1. Let $\{X_n\}$ be a sequence of iid rv's defined on the same probability space (Ω, S, P) such that $EX_1 = 0$ and $\text{Var } X_1 = \sigma^2 > 0$. Then for any $B \in S$ with $P(B) > 0$,

$$\Delta_{n(p)}(B) \to 0 \quad \text{as} \quad n \to \infty, \quad 1 \le p \le \infty \ . \tag{2.1}$$

The rate of convergence in $\Delta_{n(p)}(B)$ is given in the next result.

Theorem 2.2. Let $\{X_n\}$ be as in Theorem 2.1 and let F_k denote the σ-field generated by $X_1, \cdots, X_k$. If $E|X_1|^3 < \infty$, and if r is some real number, such that $r \in [2,3]$, then there is a constant $C_r > 0$ such that for any $B \in F_k$ with $P(B) > 0$ and all $n \ge 2k$,

$$\Delta_{n(p)}(B) \le C_r[P(B)]^{-1/r} (k/n)^{\frac{1}{2}} \ . \tag{2.2}$$

In the proof of the above theorems we note that $\Delta_{n(p)}(B) \le [\Delta_{n(\infty)}(B)]^{(p-1)/p} [\Delta_{n(1)}(B)]^{1/p}$. Thus it suffices to obtain the results for $p = 1$ and $p = \infty$. This is precisely what we do in the next two sections.

3. L_∞-Results

The following inequality is instrumental in the proofs of Theorem 3.1 and 3.2.

Inequality 3.1. For any $x > 0$ and all $n \geq 2k$, there are positive constants C_1 and C_2 such that

$$|P[\max_{1\leq k\leq n} |S_k| \leq \sigma x\sqrt{n}|F_k] - G(x)|$$

$$\leq \Delta_{n-k} + C_1\frac{|S_k|}{\sqrt{n-k}} + C_2\frac{k}{n-k} + P[\max_{1\leq j\leq k} |S_j| \geq \sigma x\sqrt{n}|F_k]G(x\sqrt{\frac{n}{n-k}}) , \tag{3.1}$$

where $\Delta_{n-k} = \Delta_{(n-k)(\infty)}$, see (1.4), and $F_k = \sigma(X_1, \cdots, X_k)$ is the σ-field generated by $X_1, \cdots, X_k$.

Proof. For any $x > 0$,

$$|P[\max_{1\leq k\leq n} |S_k| \leq \sigma x\sqrt{n}|F_k] - G(x)|$$

$$\leq |P[\max_{1\leq k\leq n} |S_k| \leq \sigma x\sqrt{n}|F_k] - G(x\sqrt{\frac{n}{n-k}})| + |G(x\sqrt{\frac{n}{n-k}}) - G(x)| = I_1 + I_2, \tag{3.2}$$

say.

But since $G(x)$ is differentiable with bounded derivative $g(x)$ such that $\sup_x |x|g(x)$ is bounded, see Rozenkrantz (1967), p.547, then for any $x > 0$, all $n \geq 2k$.

$$I_2 \leq C|\sqrt{\frac{n}{n-k}} - 1| \leq C\frac{k}{n-k} \leq C(\frac{k}{n}), \tag{3.3}$$

where (here and else where) the constants referred to by C are all positive but not necessarily the same. Next, note that

$$\max_{1\leq k\leq n} |S_k| = \max\{\max_{1\leq j\leq k} |S_j|, \max_{k+1\leq j\leq n} |S_j|\}$$

$$= \max\{\max_{1\leq j\leq k} |S_j|, \max_{k+1\leq j\leq n} |S_k + T_{k,j}|\} , \tag{3.4}$$

where $T_{k,n} = \sum_{i=k+1}^{n} X_i$. Thus

$$\max\{\max_{1\le j\le k}|S_j|, \max_{k+1\le j\le n}|T_{k,j}| - |S_k|\} \le \max_{1\le k\le n}|S_k|$$

$$\le \max\{\max_{1\le j\le k}|S_j|, \max_{k+1\le j\le n}|T_{k,j}| + |S_k|\}. \quad (3.5)$$

Let $A_{n,k} = \{\max_{1\le j\le k}|S_j| < \sigma x\sqrt{n}\}$. Then it is not difficult to see that

$$P[\max_{k+1\le j\le n}|T_{k,j}| \le \sigma x\sqrt{n} - |S_k|\,|F_k, A_{n,k}]\, P[A_{n,k}|F_k]$$

$$\le P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_k] \le P[\max_{k+1\le j\le n}|T_{k,j}| \le \sigma x\sqrt{n} + |S_k|\,|F_k, A_{n,k}]P[A_{n,k}|F_k]. \quad (3.6)$$

Hence for all $x > 0$ we easily see that

$$|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_k] - G(x\sqrt{\tfrac{n}{n-k}})|$$

$$\le \max\{\sup_x|P[\max_{k+1\le j\le n}|T_{k,j}| \le \sigma x\sqrt{n} + |S_k|\,|F_k] - G(x\sqrt{\tfrac{n}{n-k}})|\,I(A_{n,k})$$

$$+ (1 - I(A_{n,k}))G(x\sqrt{\tfrac{n}{n-k}}), \sup_x|P[\max_{k+1\le j\le n}|T_{k,j}| \le \sigma x\sqrt{n} - |S_k|\,|F_k]$$

$$- G(x\sqrt{\tfrac{n}{n-k}})|\,I(A_{n,k}) + (1 - I(A_{n,k}))\, G(x\sqrt{\tfrac{n}{n-k}})\}.$$

$$= \max\{J_{1n}\cdot I(A_{n,k}) + (1-I(A_{n,k}))G(x\sqrt{\tfrac{n}{n-k}}), J_{2n}\cdot I(A_{n,k}) + (1-I(A_{n,k}))G(x\sqrt{\tfrac{n}{n-k}})\},$$

say. (3.7)

Now,

$$J_{1n} \le \sup_x \left| P\left[\frac{\max_{k+1\le j\le n} |T_{k,j}|}{\sigma\sqrt{n-k}} \le \frac{\sigma x\sqrt{n} - |S_k|}{\sigma\sqrt{n-k}} \middle| F_k\right] - G\left(\frac{x\sqrt{n} - |S_k|/\sigma}{\sqrt{n-k}}\right)\right|$$

$$+ \sup_x \left| G\left(\frac{x\sqrt{n} - |S_k|/\sigma}{\sqrt{n-k}}\right) - G\left(x\sqrt{\frac{n}{n-k}}\right)\right| .$$

$$= \sup_u \left| P[\max_{1\le j\le n-k} |S_j| \le \sigma u\sqrt{n-k}] - G(u)\right| + \sup_x \left| G\left(x\sqrt{\frac{n}{n-k}} - \frac{|S_k|}{\sigma\sqrt{n-k}}\right) - G\left(x\sqrt{\frac{n}{n-k}}\right)\right|$$

$$\le \Delta_{n-k} + C\frac{|S_k|}{\sqrt{n-k}} , \qquad (3.8)$$

where in the last inequality we used the fact that $|G(x+\epsilon) - G(x)| \le C|\epsilon|$, while the first term of the last inequality follows from the fact that S_k and $T_{m,j}$, $k+1\le j\le n$, are independent and that $T_{k,j}$ has the same distribution as S_{j-k}, $j = k+1, \cdots, n$. Next

$$(1 - I(A_{n,k}))G\left(x\sqrt{\frac{n}{n-k}}\right) = P[\max_{1\le j\le k} |S_j| \ge \sigma x\sqrt{n} | F_k]\ G\left(x\sqrt{\frac{n}{n-k}}\right) . \qquad (3.9)$$

Hence the first term in the extreme right-hand-side of (3.7) is bounded above by:

$$\Delta_{n-k} + \frac{C|S_k|}{\sqrt{n-k}} + P[\max_{1\le j\le k} |S_j| \ge \sigma x\sqrt{n} | F_k]\ G\left(x\sqrt{\frac{n}{n-k}}\right) . \qquad (3.10)$$

It is easy to see that the upper bound in (3.10) also holds for the second term in the extreme right-hand-side of (3.7). Hence it follows from (3.2), (3.3), (3.7), and (3.10), that for any $x > 0$, $n \ge 2k$

$$|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_k] - G(x)|$$

$$\le \Delta_{n-k} + C_1\frac{|S_k|}{\sqrt{n-k}} + C_2\frac{k}{n-k} + P[\max_{1\le j\le k}|S_j| \ge \sigma x\sqrt{n}|F_k]G(x\sqrt{\tfrac{n}{n-k}}). \qquad (3.11)$$

The proof of (3.1) is now complete.||

The next two theorems are L_∞-versions of Theorems 2.1 and 2.2 above and would be used in their proofs; but also they are of independent interest of their own.

<u>Theorem 3.1</u>. <u>Under the conditions of Theorem 2.1,</u>

$$\Delta_{n(\infty)}(B) = \sup_x |P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|B] - G(x)| \to 0, \quad \underline{as} \quad n\to\infty. \qquad (3.12)$$

<u>Proof</u>: Let $F_\infty = \sigma(X_1, X_2, \cdots)$ denote the σ-field generated by $\{X_n\}$. Then there exists a sequence $\{\eta_m\}$ of r.v.'s such that $0 \le \eta_m \le 1$, $k \ge 1$ and

$$E|\eta_m - P(B|F_\infty)| \to 0, \quad \text{as} \quad k\to\infty. \qquad (3.13)$$

Now, for any $x > 0$,

$$|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}, B] - G(x)P(B)|$$

$$\le E|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}, P(B|F_\infty)] - P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}, \eta_m]|$$

$$+ E|P\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}, \eta_m] - G(x)E\eta_m| + G(x)|E\eta_m - P(B)|$$

$$= L_{1n} + L_{2n} + L_{3n}, \quad \text{say.} \qquad (3.14)$$

But, it easily follows that

$$L_{1n} \le E|P(B|F_\infty) - \eta_m| < \varepsilon/3, \quad \text{for sufficiently large } m. \tag{3.15}$$

Similarly, for this large m,

$$L_{3n} \le |E\eta_m - P(B)|G(x) \le E\{E|P(B|F_\infty) - \eta_m|\} < \varepsilon/3 . \tag{3.16}$$

It remains to show that for large m and as $n \to \infty$, $L_{2n} < \varepsilon/3$.

$$L_{2n} \le E\{\eta_m|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_m] - G(x)|\}$$

$$\le E^{\frac{1}{2}}\eta^2_m E^{\frac{1}{2}}\{|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_m] - G(x)|\}^2$$

$$\le E^{\frac{1}{2}}\{\Delta_{n-m} + C_1\frac{|S_m|}{\sqrt{n-m}} + C_2\frac{m}{n-m} + P[\max_{1\le j\le m}|S_j| \ge \sigma x\sqrt{n}|F_m]G(x\sqrt{\tfrac{n}{n-m}})\}^2, \tag{3.17}$$

by using Inequality 3.1. Using Minkowisky's inequality we easily see that the last upper bound in (3.17) is less than or equal to

$$\Delta_{n-m} + C_1(\tfrac{m}{n-m})^{\frac{1}{2}} E^{\frac{1}{2}}(|S_m|^2/m) + C_2(\tfrac{m}{n-m}) + \{P[\max_{1\le j\le m}|S_j| \ge \sigma x\sqrt{n}]\}^{\frac{1}{2}}G(x\sqrt{\tfrac{n}{n-m}}). \tag{3.18}$$

But from the result of Erdös and Kač (1946), $\Delta_{n-m} < \varepsilon/12$ for sufficiently large n. Also since, $\sup_m \frac{1}{\sqrt{m}} E^{\frac{1}{2}}|S_m|^2 < \infty$, the second term in (3.18) is less than $\varepsilon/12$ for large n, while the third is obviously less than $\varepsilon/12$ for large n. Finally we need to show that the final term is less than $\varepsilon/12$ for large n. To see this first consider the case $x > n^{-1/3}$. Then using Kolmogorov's inequality,

$$\{P[\max_{1\le j\le m}|S_j| \ge \sigma x\sqrt{n}]\}^{\frac{1}{2}} G(x\sqrt{\tfrac{n}{n-m}}) \le \{P[\max_{1\le j\le m}|S_j| \le \sigma n^{-\frac{1}{6}}]\}^{\frac{1}{2}} \le C n^{-\frac{1}{6}} < \varepsilon/12, \quad (3.19)$$

for large n. While if $0 < x \le n^{-\frac{1}{3}}$ we have for n large such that $n > 2m$

$$\{P[\max_{1\le j\le m}|S_j| \ge \sigma x\sqrt{n}]\}^{\frac{1}{2}} G(x\sqrt{\tfrac{n}{n-m}}) \le G(n^{-\frac{1}{3}}\sqrt{\tfrac{n}{n-m}}) \le G(\sqrt{2}\, n^{-\frac{1}{3}}) < \varepsilon/12 , \quad (3.20)$$

Hence for all $x > 0$ and large m and n, the last term in (3.18) is $< \varepsilon/3$. The proof is now complete.||

Next, we give the order of approximation of L_∞-version.

Theorem 3.2. Under the conditions of Theorem 2.2, there exists a positive constant C_r, $2 \le r \le 3$ such that if $B \in F_k$, with $P(B) > 0$, then

$$\Delta_{n(\infty)}(B) \le C_r\, [P(B)]^{-\frac{1}{r}} (k/n)^{\frac{1}{2}} . \quad (3.21)$$

Remark 3.1. Note that if $E|X_1|^3 < \infty$, then $\Delta_{n(\infty)} < C n^{-\frac{1}{2}}$, as proved by Nagaev (1970), this is the instrumental fact used in the proof of Theorem 3.2 to follow.

Proof. For any $x > 0$, and using Holder's inequality we can see that

$$|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}, B] - G(x)P(B)|$$

$$= E\{|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n} \mid F_k] - G(x) \mid I(B)\}$$

$$\le [P(B)]^{1-1/r} E^{1/r}\{|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n} \mid F_k] - P(B)|\}^r, \quad (3.22)$$

$2 \le r \le 3$.

But using Inequality 3.1 and Minkowisky's inequality we get that

$$E^{\frac{1}{r}}\{|P[\max_{1\le k\le n}|S_k| \le \sigma x\sqrt{n}|F_k] - G(x)|^r\}$$

$$\le \Delta_{n-k} + C_1\sqrt{\frac{k}{n-k}}(E^{\frac{1}{r}}|S_k|^r/\sqrt{k}) + C_2\frac{k}{n-k} + E^{\frac{1}{r}}\{P[\max_{1\le j\le k}|S_j| \ge \sigma x\sqrt{n}|F_k]\}^r G(x\sqrt{\frac{n}{n-k}})$$

$$\le C_3(\frac{k}{n})^{\frac{1}{2}} + C_4(\frac{k}{n})^{\frac{1}{2}} + E^{\frac{1}{r}}\{P[\max_{1\le j\le k}|S_j| \ge \sigma x\sqrt{n}|F_k]\}^r G(x\sqrt{\frac{n}{n-k}}) , \qquad (3.23)$$

where the last upper bound is achieved using the facts that $\Delta_{n-k} \le C(n-k)^{-\frac{1}{2}}$, $2k\le n$, and $\sup_k \frac{1}{\sqrt{k}} E^{\frac{1}{r}}|S_k|^r < \infty$ for all $2 \le r \le 3$. Thus to conclude the proof we need only to establish that the last term in the upper bound of (3.23) is less than or equal to $C_r(k/n)^{\frac{1}{2}}$. To see this, since $x > 0$, first assume that $x \ge n^{-1/3}$, they by Kolmogorov's inequality,

$$E^{1/r}\{P[\max_{1\le j\le k}|S_j| \ge \sigma x\sqrt{n}|F_k]\}^r G(x\sqrt{\frac{n}{n-k}})$$

$$\le E^{1/3}\{P[\max_{1\le j\le k}|S_j| \ge \sigma x\sqrt{n}|F_k]\}^3$$

$$\le E^{1/3}[\frac{E(|S_k|^3|F_k)}{\sigma^3 n^2}]^3 \le C(k/n)^{\frac{1}{2}}. \qquad (3.24)$$

While, if $x < n^{-1/3}$, then the left-hand-side of (3.24) is less than or equal to

$$G(x\sqrt{\frac{n}{n-k}}) \le \frac{4}{\pi}\exp\{-\pi^2/8(\sqrt{\frac{n}{n-k}}\, n^{-1/3})^2\}$$

$$= C \exp\{-C\, n^{2/3}(\frac{n}{n-k})\}$$

$$\le C[1 + Cn^{2/3}(\frac{n}{n-k}) + \frac{C^2}{2!}\, n^{4/3}(\frac{n}{n-k})^2 + \cdots]^{-1}$$

$$\le C\, n^{-2/3}(\frac{n}{n-k})^{-1} \le C(k/n)^{\frac{1}{2}} \,. \tag{3.25}$$

Hence from (3.24) and (3.25) we see that the upper bound of (3.23) is less than or equal to $C(k/n)^{\frac{1}{2}}$ and this completes the proof.||

Remark 3.2. It is an interesting and open question to demonstrate that if $E|X_1|^{2+\delta} < \infty$, for some $0 < \delta \le 1$ then $\Delta_{n(\infty)} \le C\, n^{-\delta/2}$ (this would extend the work of Nagaev (1970) and Sakhanenko (1974)) and if this is true it is easy to see that the bound (in n) of Theorem 3.2 becomes $(k/n)^{\delta/2}$.

4. L_1-Results

First we obtain an inequality analogous to Inequality 3.1 and use it in obtaining the main results of this section. Since some parts of the proofs are analogous to those in Section 3 we shall be brief without distorting from clarity.

Inequality 4.1. For all n such that $n \ge 2k$, there are positive constants C_1 and C_2 such that

$$\int |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}\,|F_k] - G(x)|dx$$
$$\le \Delta_{(n-k)(1)} + C_1\frac{|S_k|}{\sqrt{n-k}} + C_2\frac{k}{n-k} + \int P[\max_{1\le j\le k} |S_j| \ge \sigma x\sqrt{n}\,|F_k]G(x\sqrt{\frac{n}{n-k}})dx. \tag{4.1}$$

Proof. First note that

$$\int |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}|F_k] - G(x)|dx$$

$$\le \int |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}|F_k] - G(x\sqrt{\tfrac{n}{n-k}})|dx + \int |G(x\sqrt{\tfrac{n}{n-k}}) - G(x)|dx. \quad (4.2)$$

But since $G(\cdot)$ is twice differentiable with bounded derivatives we have that

$$\int |G(x\sqrt{\frac{n}{n-k}}) - G(x)|dx \le C|\sqrt{\frac{n}{n-k}} - 1| \int |x|\, d\,G(x) \le C\frac{k}{n-k}, \quad (4.3)$$

where here and else where C's are positive (not necessarily the same) constants. Next, following argument similar to that of Inequality 3.1 we see that

$$\int |P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n}|F_k] - G(x\sqrt{\frac{n}{n-k}})|dx$$

$$\le \max\{\int |P[\max_{k+1\le j\le n} |T_{k,j}| \le \sigma x\sqrt{n} + |S_k||F_k] - G(x\sqrt{\frac{n}{n-k}})|\, I(A_{n,k})dx$$

$$+ \int (1 - I(A_{n,k}))G(x\sqrt{\tfrac{n}{n-k}})dx,\ \int |P[\max_{k+1\le j\le n} |T_{k,j}| \le \sigma x\sqrt{n} - |S_k||F_k]$$

$$- G(x\sqrt{\frac{n}{n-k}})|\, I(A_{n,k})dx + \int (1 - I(A_{n,k}))\, G(x\sqrt{\frac{n}{n-k}})\, dx\}. \quad (4.4)$$

Now

$$\int |P[\max_{k+1\le j\le n} |T_{k,j}| \le \sigma x\sqrt{n} + |S_k||F_k] - G(x\sqrt{\tfrac{n}{n-k}})|I(A_{n,k})dx$$

$$\le \int |P[\max_{k+1\le j\le n} |T_{k,j}| \le \sigma x\sqrt{n} + |S_k||F_k] - G(\frac{\sigma x\sqrt{n} + |S_k|}{\sqrt{n-k}})|dx$$

$$+ \int |G(\frac{\sigma x\sqrt{n} + |S_k|}{\sqrt{n-k}}) - G(x\sqrt{\frac{n}{n-k}})|dx$$

$$\le \int |P[\max_{1\le \ell\le n-k} |S_\ell| \le \sigma x\sqrt{n-k}] - G(x)|dx + C\frac{|S_k|}{\sqrt{n-k}}. \quad (4.5)$$

Using similar argument we can also show that

$$\int \left| P[\max_{k+1\le j\le n} |T_{k,j}| \le \sigma x\sqrt{n} - |S_k| \,|\, F_k] - G(x\sqrt{\tfrac{n}{n-k}}) \right| I(A_{n,k})dx$$

$$\le \int \left| P[\max_{1\le \ell\le n-k} |S_\ell| \le \sigma x\sqrt{n-k}] - G(x) \right| dx + C\,\frac{|S_k|}{\sqrt{n-k}} \qquad (4.6)$$

The inequality follows directly from (4.5) and (4.6).||

Using the above inequality we can now establish mixing L_1-central limit theorem for $\max_{1\le k\le n} |S_k|$ and discuss its rate of convergence.

<u>Theorem 4.1.</u> <u>Under the conditions of Theorem 2.1,</u>

$$\Delta_{n(1)}(B) = \int \left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n} | B] - G(x) \right| dx \to 0 \ \underline{\text{as}}\ n \to \infty. \qquad (4.7)$$

<u>Proof.</u> As in Theorem 3.1 we have , using Fubini Theorem,

$$\int \left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n},\ B] - G(x)P(B) \right| dx$$

$$\le \int E\left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n},\ P(B|F_\infty)] - P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n},\ \eta_m] \right| dx$$

$$+ \int E\left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n},\ \eta_m] - G(x)E\eta_m \right| dx + \int G(x)dx \left| E\,\eta_m - P(B) \right|$$

$$= L_{1n} + L_{2n} + L_{3n}, \quad \text{say.} \qquad (4.8)$$

Now, for sufficiently large m,

$$L_{1n} \leq \int P[\max_{1\leq k\leq n} |S_k| \leq \sigma x\sqrt{n}]dx\ E|P(B|F_\infty) - \eta_m| \leq CE|P(B|F_k) - \eta_m| < \varepsilon/3. \quad (4.9)$$

Next,

$$L_{3n} \leq C|E\eta_n - P(B)|(\varepsilon/3) \quad \text{for sufficiently large } m.$$

Finally,

$$L_{2n} \leq E^{\frac{1}{2}}\eta_m^2\ E^{\frac{1}{2}}\{\int|P[\max_{1\leq k\leq n}|S_k| \leq \sigma x\sqrt{n}|F_k] - G(x)|dx\}^2$$

$$\leq E^{\frac{1}{2}}\{\int|P[\max_{1\leq l\leq n-k}|S_\ell| \leq \sigma x\sqrt{n-k}] - G(x)|dx + C_1\frac{|S_k|}{\sqrt{n-k}} + C_2\frac{k}{m-k}$$

$$+ \int P[\max_{1\leq j\leq k}|S_j| > \sigma x\sqrt{n}|F_k]\ G(x\sqrt{\tfrac{n}{n-k}})\ dx\}^2$$

$$\leq \Delta_{(n-k)(1)} + C_1\frac{E^{\frac{1}{2}}|S_k|^2}{\sqrt{n-k}} + C_2\frac{k}{n-k} + \{\int P^2[\max_{1\leq j\leq k}|S_j| \geq \sigma x\sqrt{n}]G^2(x\sqrt{\tfrac{n}{n-k}})dx\}^{\frac{1}{2}}$$

$$\leq \varepsilon/12 + \varepsilon/12 + \varepsilon/12 + \{\frac{1}{\sigma\sqrt{n}}\int P[\max_{1\leq j\leq k}|S_j| \geq u]du\}^{\frac{1}{2}} < \varepsilon/3\ , \quad (4.10)$$

for m and n sufficiently large. This concludes the proof.||

Finally, we present the rate of convergence in the L_1-version of the central limit theorem.

Theorem 4.2. <u>Under the conditions of Theorem 2.2, there is a positive constant</u> C_r, $2 \leq r \leq 3$ <u>such that for any</u> $B \in F_k$ <u>with</u> $P(B) > 0$,

$$\Delta_{n(1)}(B) \leq C_r[P(B)]^{-1/r}\ (k/n)^{\frac{1}{2}}. \quad (4.11)$$

<u>Proof</u>. Similar to Theorem 3.2, we get, using Holder's inequality that

$$\int \left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n},\ B] - G(x)P(B) \right| dx$$

$$\le [P(B)]^{1-1/r}\, E^{1/r}\left\{ \int \left| P[\max_{1\le k\le n} |S_k| \le \sigma x\sqrt{n} \,|\, \mathcal{F}_k] - G(x) \right| dx \right\}^r$$

$$\le [P(B)]^{1-1/r}\left\{ \Delta_{(n-k)(1)} + C_1 \frac{E^{1/r}|S_k|^r}{\sqrt{n-k}} + C_2 \frac{k}{n-k} \right.$$

$$\left. + E^{1/r}\left[\int P[\max_{1\le j\le k} |S_j| \ge \sigma x\sqrt{n} \,|\, \mathcal{F}_k]\, G\!\left(x\sqrt{\tfrac{n}{n-k}}\right) dx\right]^r \right\}. \tag{4.12}$$

But we easily see that the last term is less than or equal to

$$\left\{ \int P^r[\max_{1\le j\le k} |S_j| \ge \sigma x\sqrt{n}]\; G^r\!\left(x\sqrt{\tfrac{n}{n-k}}\right) dx \right\}^{1/r}$$

$$\le \frac{1}{\sigma\sqrt{n}} \left\{ \int P^r[\max_{1\le j\le k} |S_j| \quad u]\, du \right\}^{1/r} \le C\, n^{-1/2} \le C(k/n)^{1/2}\ .$$

Finally since $E|X_1|^3 < \infty$ implies that $\Delta_{(n-k)(1)} \le C(n-k)^{-1/2}$, see Pinelis (1980). The proof is now complete.||

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