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Symmetrals and X-Rays of Planar Convex Bodies

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1. Introduction.

In [2], it was shown that there are certain finite sets of directions such that the Steiner symmetrals of any planar convex body K in those directions determine the shape of K uniquely. In fact, all such sets of directions are characterized by Theorem 1 of [2]. In particular, no sets of three directions have this property, but some sets of four directions do.

In a related problem, the convex body K is fixed in advance, and the directions may be chosen to depend on K. Giering ([3]) shows that in this case certain sets of three directions suffice, and, surprisingly, this is the best one can do. The first part of this paper deals with this result, freely using the ideas of [3], together with new ones. The proof obtained here is, however, much shorter than that in [3], and avoids some inaccuracies in that paper; the theory of homeomorphisms of the circle is used.

Symmetrals of a convex body may be identified with (parallel) 'X-ray pictures'. The same types of problem may be tackled, where the X-ray pictures are taken from finite points. The main theorem of the last section shows that in some cases X-rays from two points are enough to determine shape.

In [1], K.J. Falconer has independently drawn conclusions similar to those of the last section of this paper. I thank him for sending a preprint of his work, which benefited mine. I also thank Y. Akyildiz for some very helpful discussions which led to the application

of the theory of homeomorphisms of the circle to the problems considered here.

2. Symmetrals of a convex body.

Let K be a convex body in the plane, and ℓ a line through the origin, perpendicular to the direction θ. The (Steiner) θ-symmetral of K is the convex body obtained by translating, in the direction θ, each chord of K parallel to θ, so that it is bisected by ℓ, and taking the union of these translates.

The next lemma is given in [2].

Lemma 2.1. The centroid of K lies on the same line in the direction θ as that of its θ-symmetral.

Proof. Consider moments of chords, parallel to θ, of K and its θ-symmetral. As the corresponding chords have equal lengths, the centroids must lie on the same line. |||

In this paper we shall henceforth identify any convex body with its translate which has centroid at the origin. The next lemma is then proved using Lemma 2.1 (see also [2]). (We write ∂K for the boundary of K.)

Lemma 2.2. Suppose $\{\theta_i\}$ is a set of directions, and H is a convex body with the same θ_i-symmetrals as K. Then H and K have common supporting lines in the directions θ_i, so that each line parallel to some θ_i meets H and K in chords of equal length. Also, $\mathrm{conv}(\partial H \cap \partial K)$ has a non-empty interior.

If θ_1 and θ_2 are two different directions, a (θ_1,θ_2)-*parallelogram* in K is a parallelogram whose sides are parallel to θ_1 or θ_2, and whose vertices lie in ∂K. We include the degenerate case of a chord of K, parallel to $\theta_1(\theta_2)$, which meets the supporting lines to K parallel to $\theta_2(\theta_1)$.

The following crucial lemma is Satz 9 of [3]. We give the proof for completeness.

Lemma 2.3. Suppose that the convex body H has the same θ_1- and θ_2-symmetrals as K. If $x \in \partial K$ is the vertex of a (θ_1,θ_2)-parallelogram P in K, then $x \in \partial H$.

Proof. The case where P is degenerate is clear, so suppose P is non-degenerate. Let $K_i (1 \leq i \leq 4)$ denote the (closed) subsets of K which lie outside the interior P^0 of P, between a line containing a side of P and a supporting line to K parallel to this side. Using m for planar Lebesgue measure, we have

$$m(K) = m(P) + \Sigma_i\, m(K_i).$$

Suppose H has the same θ_1- and θ_2-symmetrals as K, and H_i are the corresponding (closed) subsets of H, which lie outside P^0, and between a line containing a side of P and a supporting line of K parallel to this side. Then by Lemma 2.2, $m(H) = m(K)$, and $m(H_i) = m(K_i)$, $1 \leq i \leq 4$. The set $H - \cup_i H_i$ is contained in P^0, and

$$m(H - \cup_i H_i) \geq m(H) - \Sigma_i m(H_i) = m(K) - \Sigma_i m(K_i) = m(P).$$

It follows that $H - \cup_i H_i = P^0$, so the vertices of P all lie in ∂H. |||

Theorem 2.4. (Giering) If θ_1 and θ_2 can be chosen so that each point of ∂K is in the boundary of a (θ_1, θ_2)-parallelogram in K, then K is uniquely determined by its θ_1- and θ_2-symmetrals.

This follows from Lemma 2.3.

3. Homeomorphisms of the circle.

In this section we briefly summarize some discoveries, which were initiated by Poincaré, from the theory of homeomorphisms of the circle, or more generally, those defined on a simple closed curve (see, for example, [5] or [6]).

Let C be a simple closed curve parametrized by x, $0 \leq x \leq 1$. Suppose that T is an orientation-preserving homeomorphism of C to itself. Then T can be represented by a monotonic increasing, continuous function $F : \mathbb{R} \to \mathbb{R}$ which satisfies $F(x + 1) = F(x) + 1$. This representation is not unique, for the function defined by $F(x) + n$, for an arbitrary integer n, may be used in place of $F(x)$. The number $\rho(F) = \lim_{|n| \to \infty} F^n(x)/n$ can be shown to exist for each $x \in \mathbb{R}$, and it is independent of x. Then $\rho(T)$ can be defined as the residue class of $\rho(F)$ modulo one, and this is called the rotation number of T; it is a topological invariant of T and the oriented curve C. Further, $\rho(T)$ is rational if and only if T has a periodic point, that is, a point x such that $T^m x = x$ for some integer m. In this case all periodic points have the same period, the smallest such $|m|$.

Suppose K is a convex body, and θ a direction. Define a map $\theta : \partial K \to \partial K$ as follows. If ℓ is any line parallel to θ which meets K, let θ on $\partial K \cap \ell$ be any orientation-reversing homeomorphism. Then θ is an orientation-reversing homeomorphism from ∂K to itself; also, the map θ interchanges the endpoints of the chord $K \cap \ell$. Now if θ_1, θ_2 are two distinct directions, the map $T = \theta_2 \circ \theta_1$ is an orientation-preserving homeomorphism of ∂K to itself, and the above remarks apply to T.

4. Determination of shape by symmetrals.

Theorem 4.1. (Giering) Given a convex body K, three directions may be chosen so that the symmetrals of K in those directions determine K uniquely.

We have seen in Theorem 2.4 that in special cases two suitable directions may be chosen. However, Example 4.2 shows that in general three directions are needed.

The bulk of Giering's long paper is dedicated to a proof of Theorem 4.1, but unfortunately some of his claims are false. For example, Satz 1 is not true. To see this, let (in the notation of [3]), A be a circle and s' and s'' correspond to two directions at an angle which is an irrational multiple of π. Then the union of the sequence $(s_1, s_2, \ldots)$ of chords of A is dense in the disc, and so cannot converge to a limit cycle composed of finitely many chords. In the language of the previous section, the map T defined by the two directions s' and s'' has an irrational rotation number; indeed, Satz 1 of [3] is true if and only if this rotation number is rational.

Before proving Theorem 4.1, we need a definition. A convex polygon Q is called a $\{\theta_i\}$-polygon (cf. 'S-polygon' in [2]) if whenever u is a vertex of Q and ℓ is a line through u parallel to some θ_i, then ℓ either supports Q in u alone, or contains another vertex v of Q.

Proof of Theorem 4.1. Let θ_1 be any direction such that ∂K does not contain a line segment parallel to θ_1 (we need only avoid countably many directions). Let k be the chord of K which meets the two supporting lines to K, which are parallel to θ_1, at a and b say. Suppose θ_2 is parallel to k and let the maps θ_1, θ_2 and $T = \theta_2 \circ \theta_1$ be defined as in Section 3. Note that $T^2a = a$ and $T^2b = b$.

Suppose E is the set of all vertices of (θ_1,θ_2)-parallelograms in K. Then E is a closed set, and the set of endpoints of components of $\partial K - E$ is a countable set D. Thus we may choose a direction θ_3 not parallel to any line joining two points of D.

Suppose H is a convex body with the same θ_i-symmetrals as K, $1 \leq i \leq 3$, and A is a non-empty component of $H^0 - K$. Let y and z be the common endpoints, in $\partial H \cap \partial K$, of the arcs in ∂H and ∂K which bound A.

Let ℓ be a line parallel to some θ_i, $1 \leq i \leq 3$. Then the line segments $(H - K) \cap \ell$ and $(K - H) \cap \ell$ have the same length, by Lemma 2.2. Define $\theta_i A = \cup\{(K^0 - H) \cap \ell \mid \ell \cap A \neq \phi\}$. Then $\theta_i A$ is a component of $K^0 - H$, with the same area $m(A)$ as A. Further, the endpoints y' and z' of the arcs of ∂H and ∂K which bound $\theta_i A$ belong to $\partial H \cap \partial K$.

Now the above process can be iterated through any sequence of directions from $\{\theta_i\}$, producing disjoint components of $K^0 - H$ or $H^0 - K$ with areas equal to $m(A)$. So, from A only a finite number of such components can be obtained, and the set of endpoints of these components clearly form the vertices of a $\{\theta_1,\theta_2,\theta_3\}$-polygon Q, which will be non-degenerate by Lemma 2.2.

The points $a, b \in E$, and by Lemma 2.3, $E \subset \partial H \cap \partial K$. Thus if u is any vertex of Q, $\theta_1 u$ is also a vertex. However, $\theta_2 u$ may not be a vertex of Q, in two exceptional cases. The first is if ∂K contains a line segment in the direction θ_2, when it is possible that u lies in the interior of this segment. In this case it is easy to see that u is a vertex of a (θ_1,θ_2)-parallelogram P in K. The second is if the line ℓ through u parallel to θ_2 supports Q in u alone; there are at most two vertices with this property, and we label them v and w.

Suppose u is any vertex of Q other than a vertex of P, or v, or w. Then Tu is also a vertex of Q different from these exceptional vertices, so u is periodic under T. As $T^2 a = a$, we have $T^2 u = u$ by the remarks of Section 3, so u is a vertex of a (θ_1,θ_2)-parallelogram in K, and $u \in E$. It also follows that $w = \theta_1 v$.

Thus each vertex u of Q except v and $\theta_1 v$ lies in E. Further, as u is an endpoint of some component of $H^0 - K$ or $K^0 - H$, $u \varepsilon D$.

Suppose neither v nor $\theta_1 v$ is among the vertices of Q. Then for some vertex u of Q, $\theta_3 u$ is also a vertex, and this contradicts our choice of θ_3.

Assume, then, that $y = v$, so v and z are endpoints of the component A. If z is not a or b, the points $\theta_1 z$, $\theta_2 z$ and Tz are distinct vertices of Q which belong to D. As Q is a $\{\theta_1, \theta_2, \theta_3\}$-polygon, we must have $\theta_3 v = \theta_1 z$ and $\theta_3 \theta_1 v = \theta_2 z$ (or $\theta_3 v = T z$ and $\theta_3 \theta_1 v = z$, which can be dealt with similarly), for otherwise two points of D are joined by a line in the direction θ_3. Then the line through z parallel to θ_3 does not meet another vertex of Q, so it supports Q in z alone. In this case, one endpoint, a say, of k must lie in the relative interior of the arc of ∂K bounding the component $\theta_3 A$ of $K^0 - H$, which is impossible, because $a \varepsilon \partial H \cap \partial K$.

We conclude that $z = a$ or b, so that $\partial H \cap \partial K$ consists of the four points a, b, v and $\theta_1 v$. Again, we have a contradiction, for H and K cannot then have common supporting lines in the direction θ_3. |||

<u>Example 4.2</u>. A general construction procedure for examples is due to Giering ([3], p. 241). Let K be a convex body, and θ_1 a direction. Suppose ℓ is the line midway between and parallel to the two supporting lines to K in this direction. Further, suppose the θ_1-symmetral of K

is symmetric about ℓ. If θ_2 is any other direction, and k is a chord of K parallel to θ_2 and divided by ℓ in the ratio $\alpha : \beta$, translate k parallel to θ_2 so that it is divided by ℓ in the ratio $\beta : \alpha$. Let H be the union of all such translates of the chords of K parallel to θ_2. It is not difficult to see that H and K have the same θ_1- and θ_2-symmetrals.

If at least one chord of K parallel to θ_2 is not bisected by ℓ, H is different from K. Applying this process to the circle, we see that the circle is determined by its symmetrals in two directions if and only if these directions are at right angles.

Now take K to be a centrally symmetric hexagon, such that none of the diagonals of K are parallel to the edges. Then the above procedure may be applied for arbitrary distinct θ_1 and θ_2, showing that in general three directions are necessary to determine shape.

5. Point X-rays.

An X-ray picture of a convex hole in an otherwise homogeneous solid would, theoretically at least, give the length of each chord of the hole, parallel to the direction of the X-ray. In this case, where the X-rays are taken from infinity, we may identify the X-ray picture with the Steiner symmetral in the same direction, for the information content is the same. Thus the results of the preceding sections may be regarded as theorems on X-rays of convex bodies, where one knows the shape of one convex body, K, in advance. This viewpoint stems from problems posed by P.C. Hammer in [4]. Hammer also mentioned the possibility that the X-ray pictures are taken from finite points, and we consider this here.

Suppose K is a convex body in the plane, and p is any point. Let $k(\theta)$ $(0 \leq \theta < \pi)$ be the length of the chord of K contained in the line through p, parallel to the direction θ. Then $k(\theta)$ is called the chord function of K at p, and it gives the same information as an X-ray picture of K taken from p.

Hammer's question was, in part: how many points are needed so that the corresponding chord functions determine the shape of any convex body uniquely? We could also ask, in the spirit of Giering's paper: given K, how many points do we have to choose so that the chord functions of K at these points determine K? (Here, we exclude the trivial case where the points lie in ∂K).

Theorem 5.1 bears on both these questions. It shows that the answer to the second question, and in certain cases the first question, is two. It was found by the author in 1978, but since then K.J. Falconer ([1]) has independently proved the same result. Actually, Falconer's result is stronger in that it gives a procedure by which ∂K can be reconstructed from the chord functions. Our proof is shorter, however. The author acknowledges the influence of [1] in correcting an error in his original proof.

Theorem 5.1. Suppose p_1 and p_2 are two points in the plane, and H and K are convex bodies with the same chord functions at p_1 and p_2. If the interiors of H and K meet the line segment joining p_1 and p_2, and p_1 and p_2 are exterior to H and K, then $H = K$.

Proof. We may suppose that p_1 and p_2 lie on the x-axis. Then for $i = 1, 2$, $h_i(\theta)$, $k_i(\theta)$ denote the lengths of the chords of H, K respectively, on the lines $\ell_i(\theta)$ through p_i in the direction θ.

We make the initial assumption that the boundaries ∂H and ∂K intersect the x-axis in common points x and y, such that p_1, x, y, p_2 lie on the x-axis in that order. Suppose that $H \neq K$. We define a sequence of line segments of lengths r_n, inductively.

For some angle θ_1 sufficiently close to π, we may assume that the line $\ell_2(\theta_1)$ contains distinct points $k_{21}(\theta_1)$, $h_{21}(\theta_1)$, $k_{22}(\theta_1)$, $h_{22}(\theta_1)$ and p_2, in that order, with $h_{2i}(\theta_1) \in \partial H$ and $k_{2i}(\theta_1) \in \partial K$ for $i = 1, 2$. Then r_1 is the length of the line segment $[k_{21}(\theta_1), h_{21}(\theta_1)]$.

Suppose we have defined θ_m, ψ_m, $h_{2i}(\theta_m)$, $k_{2i}(\theta_m)$, $h_{1i}(\psi_m)$, $k_{1i}(\psi_m)$ and r_m, for $m < n$ and $i = 1, 2$.

Choose θ_n so that the line $\ell_2(\theta_n)$ meets the point $k_{12}(\psi_{n-1})$. Define $k_{21}(\theta_n) = k_{12}(\psi_{n-1})$, and suppose $k_{21}(\theta_n)$, $h_{21}(\theta_n)$, $k_{22}(\theta_n)$, $h_{22}(\theta_n)$ and p_2 are distinct points, in that order, on the line $\ell_2(\theta_n)$, with $h_{2i}(\theta_n) \in \partial H$ and $k_{2i}(\theta_n) \in \partial K$, for $i = 1, 2$. Define r_n to be the length of the line segment $[k_{21}(\theta_n), h_{21}(\theta_n)]$. To complete the induction, choose ψ_n so that the line $\ell_1(\psi_n)$ meets the point $h_{22}(\theta_n)$. Define $h_{11}(\psi_n) = h_{22}(\theta_n)$, and suppose $h_{11}(\psi_n)$, $k_{11}(\psi_n)$, $h_{12}(\psi_n)$, $k_{12}(\psi_n)$ and p_1 are the distinct points on $\ell_1(\psi_n)$, in that order, with $h_{1i}(\psi_n) \in \partial H$ and $k_{1i}(\psi_n) \in \partial K$, for $i = 1, 2$.

Our aim is to show that $r_n \not\to 0$; this will contradict the assumption that ∂H and ∂K coincide on the x-axis.

Suppose the line through $k_{22}(\theta_n)$ and $k_{11}(\psi_n)$ meets the x-axis at y_n, and α_n = angle $(p_2, y_n, k_{22}(\theta_n))$. Similarly, let the line through $h_{12}(\psi_n)$ and $h_{21}(\theta_{n+1})$ meet the x-axis at x_n, and set β_n = angle $(p_2, x_n, h_{12}(\psi_n))$.

As H and K have the same chord functions at p_2, the length of the line segment $[k_{22}(\theta_n), h_{22}(\theta_n)]$ is also r_n. With this and some trigonometry, we see that the length of the line segment $[h_{22}(\theta_n), k_{11}(\psi_n)]$ is $\dfrac{\sin(\theta_n - \alpha_n)}{\sin(\alpha_n - \psi_n)} \cdot r_n$.

Now a similar argument, using the chord functions at p_1, gives

$$r_{n+1} = \frac{\sin(\theta_n - \alpha_n)\cdot\sin(\beta_n - \psi_n)}{\sin(\alpha_n - \psi_n)\ \sin(\theta_{n+1} - \beta_n)} \cdot r_n \,,$$

for each n. So, to show $r_n \not\to 0$ it will suffice to show that the products $\prod_n \dfrac{\sin(\alpha_n - \psi_n)}{\sin(\theta_n - \alpha_n)}$ and $\prod_n \dfrac{\sin(\theta_{n+1} - \beta_n)}{\sin(\beta_n - \psi_n)}$ are convergent.

By convexity, $x_n \to x$ and $y_n \to y$. Also, as the interiors of H and K meet the x-axis, we have $\alpha_n \to \alpha$ and $\beta_n \to \beta$, for some α, β with $0 < \alpha, \beta < \pi$. Now

$$\left|\frac{\sin(\alpha_n - \psi_n)}{\sin(\theta_n - \alpha_n)}\right| = \left|-\cos(\theta_n - \psi_n) + \cot(\theta_n - \alpha_n)\ \sin(\theta_n - \psi_n)\right|$$

$$\leq 1 + (1 + \varepsilon)\left|\cot \alpha\right|\ \sin(\theta_n - \psi_n),$$

for some $\varepsilon > 0$ and sufficiently large n; for $\theta_n \to \pi$. By a standard result in the theory of infinite products, the convergence of the first product above will follow from that of the sum $\Sigma_n \sin(\theta_n - \psi_n)$.

By applying the law of sines to the triangles $(p_1, p_2, h_{22}(\theta_n))$, we see that

$$\frac{\sin(\theta_n - \psi_n)}{\sin(\pi - \theta_n)} \longrightarrow \frac{|p_1 - p_2|}{|p_1 - y|} . \tag{1}$$

The last step involves a little more trigonometry. Let x_n^*, y_n^* denote respectively the projections of the points $k_{12}(\psi_n)$ and $h_{22}(\theta_n)$ onto the x-axis. Considering four triangles, namely $(p_1, x^*, k_{12}(\psi_n))$, $(p_2, x_n^*, k_{12}(\psi_n))$, $(p_1, y_n^*, h_{22}(\theta_n))$, and $(p_2, y_n^*, h_{22}(\theta_n))$, and noting that $x_n^* \to x$, $y_n^* \to y$, we obtain

$$\frac{\sin(\pi - \theta_{n+1})}{\sin(\pi - \theta_n)} \to \frac{|p_1 - x|\cdot|p_2 - y|}{|p_1 - y|\cdot|p_2 - x|} < 1. \tag{2}$$

Now the convergence of $\Sigma_n \sin(\theta_n - \psi_n)$ follows from (1), (2), and the ratio test. The convergence of the product $\Pi_n \frac{\sin(\alpha_n - \psi_n)}{\sin(\theta_n - \alpha_n)}$ is established, and that of the other product is obtained similarly.

It follows that the boundaries ∂H and ∂K meet the x-axis in distinct points. Consequently, there are non-empty components A_1 and A_2, of $H^0 - K$ and $K^0 - H$ respectively, which meet the x-axis. Suppose p_1, A_1, A_2, p_2, meet the x-axis in that order. Let $\ell_1(\theta)$ be any line through p_1 which meets A_1. As $h_1(\theta) = k_1(\theta)$, $\ell_1(\theta)$ also meets A_2, and in fact meets A_1 and A_2 in chords of equal length. It follows that $m(A_1) < m(A_2)$. However, the same argument applied to lines $\ell_2(\theta)$ through p_2 gives $m(A_2) < m(A_1)$. This contradiction proves the theorem. ||

The theorem can be proved in exactly the same way to cover the cases where the interiors of H and K meet the line through p_1 and p_2, and (i) p_1 and p_2 are exterior to H and K, and on the same side of H and K, or (ii) p_1 and p_2 are both interior to H and K.

K.J. Falconer suggested that one of the arguments used in the proof of Theorem 5.1 gives the following more 'practical' result.

Theorem 5.2. Suppose H and K are convex bodies with the same chord functions at points p_i, $1 \leq i \leq 3$. If H and K lie in the interior of the triangle (p_1, p_2, p_3), then $H = K$.

Proof. If $H \neq K$, then either $H^0 - K$ or $K^0 - H$ (say the former) contains a component A of maximal area in $(H^0 - K) \cup (K^0 - H)$. Suppose y and z are the endpoints of A lying in $\partial H \cap \partial K$, m is the line through y and z, and E is the open half-space bounded by m and containing A. Let $p_i \in E$, let ℓ be a line through p_i, and note that $(H - K) \cap \ell$ and $(K - H) \cap \ell$ have the same length. Define $p_i A = \cup\{(K^0 - H) \cap \ell | p_i \in \ell \text{ and } \ell \cap A \neq \phi\}$. Then $p_i A$ is a component of $K^0 - H$ which has greater area than that of A, contradicting the choice of A. Thus $p_i \notin E$, $1 \leq i \leq 3$, and H and K cannot lie in the interior of the triangle (p_1, p_2, p_3).

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