King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 045

December 1982

Generalization of Higuchi's Conditions for Love Waves Propagating Through Two Welded Quarter Spaces with Two Surface Layers on Each

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GENERALIZATION OF HIGUCHI'S CONDITIONS FOR LOVE WAVES PROPAGATING THROUGH TWO WELDED QUARTER-SPACES WITH TWO SURFACE LAYERS ON EACH

by

M. H. KAZI and A. NIAZY

ABSTRACT

Higuchi (1932) considered a medium consisting of two quarter-spaces in welded contact, each having a single homogeneous layer of the same thickness overlying a homogeneous half-space, and gave conditions on the rigidities and densities of the two layers and two half-sapces in such a way that the plane Love waves normally incident on the vertical plane generate only reflected and transmitted Love waves without mode conversion or body wave scattering.

In this paper we generalize Higuchi's results to two layers overlying a half-space divided by a vertical plane. On one side of the vertical plane, the shear velocities and rigidities for the top and the bottom surface layers, and for the half-space are β_1, μ_1, $\beta_2(>\beta_1), \mu_2(>\mu_1)$ and $\beta_3(>\beta_2), \mu_3(>\mu_2)$ respectively. On the other side the corresponding shear velocities and rigidities for the two layers and the half-space are given by the primed quantities with $\beta_3' > \beta_2' > \beta_1'$ and $\mu_3' > \mu_2' > \mu_1'$. We show that if the following conditions are satisfied:

$$\frac{1}{\beta_1^2} - \frac{1}{\beta_1'^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2} = \frac{1}{\beta_3^2} - \frac{1}{\beta_3'^2}$$

and

$$\frac{\mu_1}{\mu_1'} = \frac{\mu_2}{\mu_2'} = \frac{\mu_3}{\mu_3'}$$

then plane Love waves normally incident on the vertical plane generate only reflected and transmitted Love waves without mode conversion or body wave scattering. These conditions for the special case under consideration provide a valuable check for various analytical and numerical approximations which ignore the body-wave contributions in similar diffraction problems.

INTRODUCTION

Let us suppose that a quarter-space consisting of a material of rigidity μ_2, shear velocity β_2, overlain by a layer of depth h, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$, is in welded contact with a similar quarter-space of material of rigidity μ_2' and shear velocity β_2', overlain by a layer of depth h, rigidity $\mu_1'(<\mu_2')$ and shear velocity $\beta_1'(<\beta_2')$. Higuchi (1932) showed (see Aki and Richards (1980) Chapter 13) that if

$$\frac{1}{\beta_1^2} - \frac{1}{\beta_1'^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2}, \tag{1a}$$

and

$$\frac{\mu_1}{\mu_2} = \frac{\mu_1'}{\mu_2'}$$

then plane Love waves normally incident on the vertical plane generate only reflected and transmitted Love waves without mode conversion or body-wave scattering. Higuchi's conditions have been used by Alsop (1966) to test his approximations based upon the assumption that the body wave contributions may be ignored. In testing their computer programs used to obtain the results published in Niazy and Kazi (1980 & 1982), the authors used Higuchi's condition as one of the tests.

In this paper we consider a structure consisting of two layers of uniform thickness overlying a half-space divided by a vertical plane (see Figure 1). On one side (medium I) of the vertical plane of welded contact, the shear velocities and rigidities for the top and the bottom

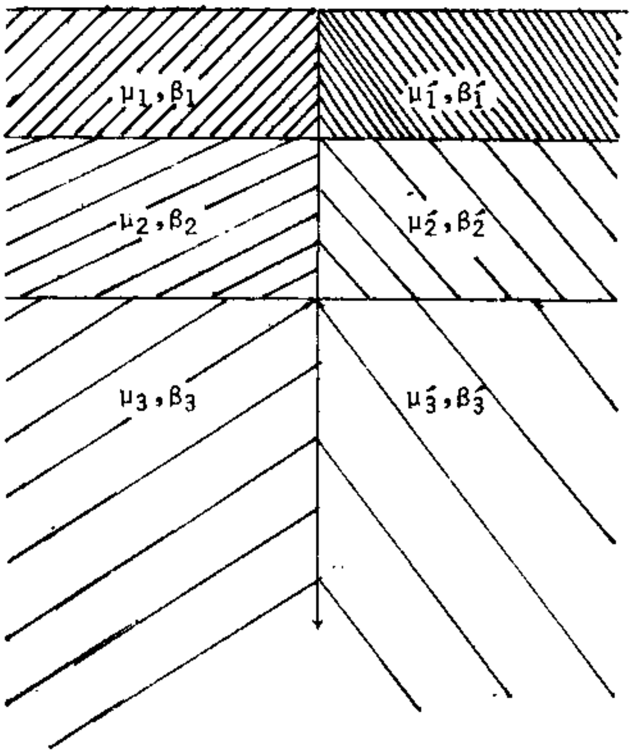

FIG. 1 GEOMETRY AND ELASTIC PARAMETERS OF THE PROBLEM

surface layers, and for the half-space, are β_1, μ_1, $\beta_2 > \beta_1, \mu_2 > \mu_1$ and $\beta_3 > \beta_2, \mu_3 > \mu_2$ respectively. On the other side (medium II) the corresponding shear velocities and rigidities for the two layers and the half-space are given by the primed quantities with $\beta_3' > \beta_2' > \beta_1'$ and $\mu_3' > \mu_2' > \mu_1'$ (See Figure 1). We prove that if

$$\frac{1}{\beta_1^2} - \frac{1}{\beta_1'^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2} = \frac{1}{\beta_3^2} - \frac{1}{\beta_3'^2} \tag{2a}$$

and

$$\frac{\mu_1}{\mu_2} = \frac{\mu_1'}{\mu_2'}, \quad \frac{\mu_2}{\mu_3} = \frac{\mu_2'}{\mu_3'}, \tag{2b}$$

then plane Love waves normally incident on the vertical plane generate only reflected and transmitted Love waves without mode conversion or body-wave scattering.

Our proof is based upon the spectral representation of the Love wave operator for two uniform and homogeneous layers over a half-space, which provides us with a complete set of proper and improper eigenfunctions, in terms of which the displacements on either side of the vertical plane of discontinuity may be expressed. We shall show that, under the conditions (2a) and (2b), plane Love waves normally incident on the vertical plane do not undergo mode conversion or body-wave scattering by proving that they do not couple with other relevant modes and body-waves.

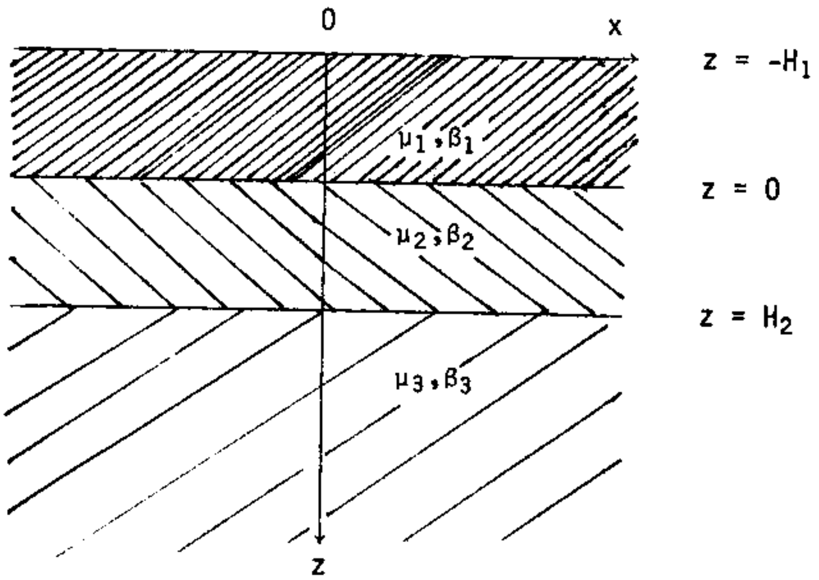

FIG. 2 TWO LAYERS OVER A HALF-SPACE

THE SPECTRAL REPRESENTATION OF THE LOVE WAVE OPERATOR FOR TWO LAYERS OVER A HALF-SPACE

Consider a uniform half-space of rigidity μ_3 and shear velocity β_3, overlaid by two infinite strips consisting of a layer of finite depth H_2, rigidity $\mu_2(<\mu_3)$, and shear velocity $\beta_2(<\beta_3)$, and another layer of depth H_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$ (See Figure 2). We suppose the density and rigidity of each layer to be constant and the top plane surface to be stress free. The coordinate system we choose is shown in Figure 2. Following our work on the spectral representation of the two-dimensional Love wave operator associated with the propagation of monochromatic SH waves in a structure consisting of a homogeneous layer overlying a homogeneous half-space (cf Kazi 1976), Kazi and Abu-Safiya (1982) have found that the spectrum of the corresponding two-dimensional Love wave operator for the structure shown in Figure 2 is the disjoint union of the discrete spectrum, which corresponds to the ordinary Love modes, and a continuous spectrum (corresponding to body waves) which is the interval $(-\infty\ , \omega^2/\beta_3^2)$ on the real axis of the complex λ-plane, where $\lambda = k^2$, k being the wave number and ω the angular frequency. Consequently, any function of z (such as the motion as a function of depth), which is square-integrable relative to the weight function

$$\begin{aligned} \mu(z) &= \mu_1 \qquad -H_1 < z < 0 \\ &= \mu_2 \qquad 0 < z < H_2 \\ &= \mu_3 \qquad H_2 < z \end{aligned} \tag{3}$$

may be expressed in terms of normalized eigenfunction $\phi^n(z)$ (ordinary Love modes) corresponding to the eigenvalues belonging to the discrete spectrum and the normalized improper eigenfunctions $\psi(z,\lambda)$ (body waves) corresponding to the continuous spectrum. The expression for $\phi^n(z)$ and $\psi(z,\lambda)$ are found to be:

$$\begin{aligned} \phi^n(z) &= \phi_1^n(z) \qquad -H_1 \leq z \leq 0 \\ &= \phi_2^n(z) \qquad 0 \leq z \leq H_2 \\ &= \phi_3^n(z) \qquad H_2 \leq z \end{aligned} \tag{4}$$

where

$$\phi_1^n(z) = F_n \frac{\cos\{\sigma_1^n(z+H_1)\}}{\cos(\sigma_1^n H_1)} \quad , \quad -H_1 \leq z \leq 0$$

$$\phi_2^n(z) = G_n \frac{\mu_2\sigma_2^n \cos\{\sigma_2^n(z-H_2)\} - \mu_3\sigma_3^n \sin\{\sigma_2^n(z-H_2)\}}{\cos(\sigma_2^n H_2)}, \quad 0 \leq z \leq H_2$$

$$\phi_3^n(z) = G_n \frac{\mu_2\sigma_2^n \, e^{-\sigma_3^n(z-H_2)}}{\cos(\sigma_2^n H_2)} \quad , \quad z \geq H_2$$

$$F_n = [\{ \frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)_{\lambda=\lambda_n}} \}]^{\frac{1}{2}}$$

$$G_n = \frac{1}{M} F_n$$

$$M = \mu_2\sigma_2 + \mu_3\sigma_3\tan(\sigma_2 H_2),$$

$$\Delta = \mu_1\sigma_1\mu_2\sigma_2 \tan(\sigma_1 H_1) + \mu_1\sigma_1\mu_3\sigma_3\tan(\sigma_2 H_2)\tan(\sigma_1 H_1) - \mu_3\sigma_3\mu_2\sigma_2 + (\mu_2\sigma_2)^2 \tan(\sigma_2 H_2)$$

$$\sigma_1(\lambda) = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right)^{\frac{1}{2}}, \quad \sigma_2(\lambda) = \left(\frac{\omega^2}{\beta_2^2} - \lambda\right)^{\frac{1}{2}}, \quad \sigma_3(\lambda) = \left(\lambda - \frac{\omega^2}{\beta_3^2}\right)^{\frac{1}{2}}$$

$$\sigma_i(\lambda_n) = \sigma_i^n \,, \quad i = 1,2,3$$

and $\lambda_n = k_n^2$, $k_n > 0$ are the roots of

$$\Delta = 0 \tag{5}$$

which is the <u>dispersion equation</u> for Love wave propagation in two layers over a half space (see Ewing et al. 1957 p 229), and where

$$\psi(z,\lambda) = \psi_1(z,\lambda) = G^k \mu_2 \sigma_2^k \frac{\cos\{\sigma_1^k(z+H_1)\}}{\cos(\sigma_1^k H_1)\cos(\sigma_2^k H_2)} \,, \quad -H_1 \leq z \leq 0$$

$$= \psi_2(z,\lambda) = \frac{G^k}{\cos(\sigma_2^k H_2)}\{\mu_2\sigma_2^k\cos(\sigma_2^k z) - \mu_1\sigma_1^k\sin(\sigma_2^k z)\tan(\sigma_1^k H_1)\} \,, \quad 0 \leq z \leq H_2 \tag{6}$$

$$= \psi_3(z,\lambda) = -\frac{\sin\{\theta^k + s_3^k(z-H_2)\}}{\sqrt{\pi\mu_3 s_3^k}} \,, \quad H_2 \leq z$$

where

$$G^k = \frac{\mu_3 s_3^k \cos\theta^k}{p\sqrt{\pi\mu_3 s_3^k}} \,,$$

$$s_3^k = \left(\frac{\omega^2}{\beta_3^2} - \lambda\right)^{\frac{1}{2}} \quad \text{real and positive} \,,$$

$$\theta^k = \tan^{-1}\left(\frac{q}{p}\right) \,, \tag{7}$$

$$p = \mu_1\sigma_1^k\mu_2\sigma_2^k\tan(\sigma_1^k H_1) + \mu_2^2(\sigma_2^k)^2\tan(\sigma_2^k H_2),$$

$$q = \mu_1\sigma_1^k\mu_3 s_3^k\ \tan(\sigma_2^k H_2)\tan(\sigma_1^k H_1) - \mu_2\sigma_2^k\mu_3 s_3^k\ ,$$

$$\sigma_1^k = \left(\frac{\omega^2}{\beta_1^2} - k^2\right)^{\frac{1}{2}}, \quad \sigma_2^k = \left(\frac{\omega^2}{\beta_2^2} - k^2\right)^{\frac{1}{2}} .$$

The functions $\phi^n(z)$, $\psi(z,\lambda)$ satisfy the following orthonormality relations:

$$\int_{-H_1}^{\infty} \mu(z)\phi^m(z)\phi^n(z)dz = \delta_{mn}\ , \quad 1 \leq m,\ n \leq N$$

$$\int_{-H_1}^{\infty} \mu(z)\psi(z,\lambda)\ (z,\lambda')dz = \delta(\lambda-\lambda')\ ,\ -\infty \leq \lambda\ ,\ \lambda' < \frac{\omega^2}{\beta_3^2}$$

and

$$\int_{-H_1}^{\infty} \mu(z)\phi^m(z)\psi(z,\lambda)dz = 0\ , \quad 1 \leq m \leq N\ , \quad -\infty \leq \lambda < \frac{\omega^2}{\beta_3^2}$$

THE COUPLING INTEGRALS AND THE CONCLUSIONS

We may write the complete solution for the displacement in domain I $(x<0)$ of Figure 1 in terms of the eigenfunctions $\phi^n(z)$ given by equation (4) and the improper eigenfunction $\psi(z,\lambda)$ given by equation (6). Similarly, we may write the complete solution for the displacement in domain II $(x>0)$ of Figure 1 in terms of eigenfunctions $\phi^{n\prime}(z)$ and improper eigenfunctions $\psi'(z,\lambda)$ appropriate to the medium if it were a half-space instead of a quarter-space. The expressions for $\phi^{n'}(z)$ and $\psi'(z,\lambda)$ are the same as for $\phi^{n}(z)$ and $\psi(z,\lambda)$ but in primed notation.

In order to show that, for plane Love waves normally incident on the vertical plane of discontinuity in Figure 1, there is no mode conversion or body-wave scattering under conditions (2a) and (2b), we have to prove that

$$P_{im} = \int_{-H_1}^{\infty} \mu(z)\phi'^{i}(z)\phi^{m}(z)dz = 0$$

and $$I(m,\lambda') = \int_{-H_1}^{\infty} \mu(z)\phi^{m}(z)\psi'(z,\lambda')dz = 0.$$

Omitting details, we give only the final expressions for P_{im} and $I(m,\lambda')$.

$$P_{im} = \frac{\mu_1 F_i' F_m}{(k_m^2 - k_i'^2) + \omega^2\left(\frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2}\right)}[\sigma_1^{i'}\tan(\sigma_1^{i'}H_1) - \sigma_1^{m}\tan(\sigma_1^{m}H_1)]$$

$$+ \frac{\mu_2 G_i' G_m}{(k_m^2 - k_i'^2) + \omega^2\left(\frac{1}{\beta_2'^2} - \frac{1}{\beta_2^2}\right)}[\sigma_2^{m}\{\mu_2\mu_2'(\sigma_2^{i'})^2 + \mu_3\mu_3'\sigma_3^{m}\sigma_3^{i'}\}\tan(\sigma_2^{i'}H_2)$$

$$- \sigma_2^{i'}\{\mu_2\mu_2'(\sigma_2^{m})^2 + \mu_3\mu_3'\sigma_3^{m}\sigma_3^{i'}\}\tan(\sigma_2^{m}H_2)$$

$$+ \{\mu_3\mu_2'\sigma_3^{m}(\sigma_2^{i'})^2 - \mu_2\mu_3'\sigma_3^{i'}(\sigma_2^{m})^2\}\tan(\sigma_2^{m}H_2)\tan(\sigma_2^{i'}H_2) +$$

$$+\frac{\sigma_2^{m}\sigma_2^{i'}(\mu_3'\mu_2\sigma_3^{i'}-\mu_3\mu_2'\sigma_3^{m})}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2^{i'}H_2)}-\sigma_2^{m}\sigma_2^{i'}\{\mu_2\mu_3'\sigma_3^{i'}-\mu_2'\mu_3\sigma_3^{m}\}\ ]$$

$$+\frac{\mu_2\mu_2'\sigma_2^{m}\sigma_2^{i'}\mu_3G_mG_i'(\sigma_3^{i'}-\sigma_3^{m})}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2^{i'}H_2)[(k_i'^2-k_m^2)+\omega^2(\frac{1}{\beta_3^2}-\frac{1}{\beta_3'^2})]}\ , \qquad (8)$$

$$I(m,\lambda')=\frac{\mu_1F_mG^{k'}}{(k_m^2-k'^2)+\omega^2(\frac{1}{\beta_1'^2}-\frac{1}{\beta_1^2})}[\sigma_1^{k'}\tan(\sigma_1^{k'}H_1)-\sigma_1^{m}\tan(\sigma_1^{m}H_1)]$$

$$+\frac{\mu_2G^{k'}G_m}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2^{k'}H_2)}\cdot\frac{1}{(k'^2-k_m^2)+\omega^2(\frac{1}{\beta_2^2}-\frac{1}{\beta_2'^2})}[\{\mu_2\mu_2'(\sigma_2^{m})^2\sigma_2^{k'}$$

$$+\mu_3\sigma_3^{m}\mu_1'\sigma_2^{k'}\sigma_1^{k'}-\tan(\sigma_1^{k'}H_1)\}\sin(\sigma_2^{m}H_2)-\{\mu_2\mu_2'\sigma_2^{m}(\sigma_2^{k'})^2+\mu_3\sigma_3^{m}\mu_1'\sigma_1^{k'}\sigma_2^{m}\}\sin(\sigma_2^{k'}H_2)$$

$$+\{\mu_3\sigma_3^{m}\mu_2'\sigma_3^{m}\sigma_2^{k'}-\mu_1'\mu_2\sigma_2^{m}\sigma_1^{k'}\sigma_2^{k'}\tan(\sigma_1^{k'}H_1)\}\{\cos(\sigma_2^{k'}H_2)-\cos(\sigma_2^{m}H_2)\}]$$

$$-\frac{\mu_3\sigma_2^{m}}{\cos(\sigma_2^{m}H_2)}\ \frac{p'}{\mu_3s_3'}\cdot\frac{G_mG^{k'}\{s^{k'}-\frac{\mu_3's_3^{k'}\sigma_3^{m}}{\mu_1'\sigma_1^{k'}}\cot(\sigma_1^{m}H_1)\}}{(k_m^2-k'^2)+\omega^2(\frac{1}{\beta_3'^2}-\frac{1}{\beta_3^2})} \qquad (9)$$

If $$\frac{1}{\beta_1^2}-\frac{1}{\beta_1'^2}=\frac{1}{\beta_2^2}-\frac{1}{\beta_2'^2}=\frac{1}{\beta_3^2}-\frac{1}{\beta_3'^2} \qquad (2a)$$

and $$\frac{\mu_1}{\mu_2}=\frac{\mu_1'}{\mu_2'}\ ,\quad \frac{\mu_2}{\mu_3}=\frac{\mu_2'}{\mu_3'} \qquad (2b)$$

then using the dispersion equation (5) and equation (7) the expressions on the right hand sides of equations (8) and (9) can be shown to be zero after some effort. We conclude, therefore, that under the afore-mentioned conditions, there is no mode conversion or body-wave scattering.

We conjecture the following generalisation of Higuchi's conditions for Love waves propagating through two welded quarter-spaces with n surface layers on each, geometrically aligned across the vertical plane of welded contact: If

$$\frac{1}{\beta_1^2} - \frac{1}{\beta_1'^2} = \frac{1}{\beta_2^2} - \frac{1}{\beta_2'^2} = \dots\dots\dots = \frac{1}{\beta_n^2} - \frac{1}{\beta_n'^2}$$

and

$$\frac{\mu_1}{\mu_1'} = \frac{\mu_2}{\mu_2'} = \dots\dots\dots\dots\dots\dots = \frac{\mu_n}{\mu_n'}$$

then plane Love waves normally incident on the vertical plane will generate only reflected and transmitted Love waves without mode conversion or body wave scattering. The proof of the conjecture does not appear to be easy.

Finally, we remark that if the layers on either side of the vertical plane are not of the same geometrical alignment so that step-discontinuities may exist, then no constraints on the material constants or shear wave velocities can eliminate body wave scattering.

ACKNOWLEDGEMENTS

This work has been done as a part of research project No.(AR-3-032) sanctioned by Saudi Arabian National Centre of Science and Technology(SANCST). The authors acknowledge the SANCST support with thanks.

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