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ERROR ESTIMATES FOR GAUSS-JACOBI QUADRATURE FORMULA WITH SPECIAL WEIGHTS

by

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ABSTRACT

In [1] we gave an estimate on the error term when approximating the integral

$$\int_{\mathbb{R}} f(x)d\alpha(x)$$

by the Gauss-Jacobi quadrature formula $Q_n(d\alpha; f)$, assuming that f is an entire function satisfying a growth condition that depends naturally on the absolutely continuous m-distribution $d\alpha$. In [2] we estimated this error when $d\alpha$ was taken to be equal to $\exp(-x^2)dx$ ($\exp(-x^2)$ is the Hermite weight function).

In this paper and by using the same techniques used in [1] and [2], we investigate the growth of f and estimate the above error in the three special cases where the weight function is: (i) $\exp(-x^4)$, (ii) $\exp(-x^6)$ and (iii) the Pollaczek weight $|\Gamma(\lambda + ix)|^2$ respectively.

1. INTRODUCTION

For any m-distribution $d\alpha$ there exists a unique sequence of orthonormal polynomials $\{p_n(d\alpha;\ x)\}$ (see [5; Sect.I.1]) with the properties:

a) $p_n(d\alpha;\ x) = \gamma_n x^n + \dots$ is a polynomial of degree n and $\gamma_n > 0$;

b) $\int p_n(d\alpha) p_m(d\alpha) d\alpha = \delta_{mn}$, the Kronecker symbol.

It is well known that all zeros $x_{kn}(k = 1,2,\dots,n)$ of $p_n(d\alpha;\ x)$ are real, simple and are contained in the smallest interval overlapping the support of $d\alpha$. We shall assume, as usual, that $x_{1n} > x_{2n} > \dots > x_{nn}$.

The interpolatory quadrature formula

$$Q_n(d\alpha; f) \overset{\text{def}}{=} \sum_{k=1}^{n} \lambda_n(d\alpha; x_{kn}) f(x_{kn}) \qquad (\sim \int f d\alpha) \tag{1.1}$$

has the property that, for every polynomial π_{2n-1} of degree $\le 2n-1$,

$$Q_n(d\alpha;\ \pi_{2n-1}) = \int \pi_{2n-1}\ d\alpha.$$

The coefficients $\lambda_n(d\alpha;\ x_{kn})$ of this formula are called the Christoffel numbers and are given by

$$\lambda_n^{-1}(d\alpha;\ x) = \sum_{\nu=0}^{n-1} p_\nu^2(d\alpha;\ x)\ .$$

Equation (1.1) is the Gauss-Jacobi quadrature formula. The nodes x_{kn} are called the Gaussian abscissae with respect to $d\alpha$.

If, in addition, $d\alpha$ is an absolutely continuous m-distribution, then $d\alpha(x) = \alpha'(x)dx$ and $\alpha'(x)$ is a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $p_n(d\alpha)$ by $p_n(w)$.

2. MAIN RESULTS

Let f be an entire function. Let, for $x \in \mathbb{R}$, $w_1(x) = \exp(-x^4)$, $w_2(x) = \exp(-x^6)$, $w_3(x) = |\Gamma(\lambda + ix)|^2$, $\lambda > 0$, and $\Delta_n^{(j)} = \int_{\mathbb{R}} f(x)w_j(x)dx - Q_n(w_j; f)$ $(j = 1,2,3)$. Let us also denote the solution of

$$\frac{1 - x}{4} \exp\left(\frac{1 - x}{2x}\right) = 1$$

by a and $\max_{|z|=R} |f(z)|$, $z \in \mathbb{C}$, by $M(R)$. We will establish the following:

THEOREM 2.1. If

$$\beta = \limsup_{R \to \infty} \frac{\log M(R)}{R^4} < \frac{(1 - a)^3}{8a} = \rho \tag{2.1}$$

$(\rho \approx .243136729)$, then

$$\limsup_{n \to \infty} |\Delta_n^{(1)}|^{1/n} < 1 .$$

THEOREM 2.2. If

$$\limsup_{R \to \infty} \frac{\log M(R)}{R^6} < \frac{15(1 - a)^4}{128a} \quad (\approx .17495268), \text{ then}$$

$$\limsup_{n \to \infty} |\Delta_n^{(2)}|^{1/n} < 1 .$$

THEOREM 2.3. If

$$\limsup_{R\to\infty} \frac{\log M(R)}{\pi R} < \frac{2}{\pi}(1-a)^{\frac{1}{2}} \quad (\approx .557737056), \quad \text{then}$$

$$\limsup_{n\to\infty} |\Delta_n^{(3)}|^{1/n} < 1 .$$

THEOREM 2.4. The above three theorems remain valid if $\Delta_n^{(i)}$ is replaced by $\Delta_n^{(i,m)} = \int_{\mathbb{R}} f^{(m)}(x) w_i(x)dx - Q_n(w_i;\, f^{(m)})$, for $i = 1,2,3$ and $m = 1,2,3,\ldots\ldots$.

3. PRELIMINARY RESULTS

To prove the main results, we need the following:

LEMMA 3.1. Let $f(z)$ be an analytic function in a domain $\mathcal{D}$ containing the Gaussian abscissae x_{kn} $(k = 1,2,\ldots,n)$ and $x_{j,n+1}$ $(j = 1,2,\ldots,n+1)$, then we have

$$Q_{n+1}(d\alpha;f) - Q_n(d\alpha;f) = \frac{\gamma_{n+1}}{\gamma_n}\cdot\frac{1}{2\pi i}\oint_{C_n}\frac{f(z)dz}{p_n(d\alpha;z)\;p_{n+1}(d\alpha;z)} ,$$

where $C_n \subset \mathcal{D}$ is a simple closed curve containing the zeros of $p_n(d\alpha)$ and $p_{n+1}(d\alpha)$ in its interior. Consequently, the error term of the quadrature formula is expressible as

$$\int f d\alpha - Q_n(d\alpha;f) = \sum_{\nu=n}^{\infty}\frac{\gamma_{\nu+1}}{\gamma_\nu}\cdot\frac{1}{2\pi i}\oint_{C_\nu}\frac{f(z)dz}{p_\nu(d\alpha;z)\;p_{\nu+1}(d\alpha;\,z)} \tag{3.1}$$

LEMMA 3.2. For every even weight function $w(x)$, we have

$$\max_{1\le k\le n-1} \frac{\gamma_{k-1}}{\gamma_k} \le x_{1n} \le 2 \max_{1\le k\le n-1} \frac{\gamma_{k-1}}{\gamma_k} \tag{3.2}$$

LEMMA 3.3. We have for $m = 2,4,$ and 6

$$\lim_{n\to\infty} n^{-1/m} \frac{\gamma_{n-1}(w_m)}{\gamma_n(w_m)} = \left[m\binom{m-1}{\frac{m-2}{2}}\right]^{-1/m} = \left[\frac{\Gamma(m+1)}{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}+1)}\right]^{-1/m}, \tag{3.3}$$

where $w_m(x) = \exp(-x^m)$.

LEMMA 3.4. Let $Q(x)$, $x \in \mathbb{R}$, be a convex, even and differentiable function. Let $\{q_n\}$ be the unique positive solution of the equation

$$q_n Q'(q_n) = n ,$$

and

$$\omega_Q(x) = \exp\{-2Q(x)\}. \quad \text{Then}$$

$$\frac{1}{4} q_n \le x_{1n} \le 4q_{n-1}$$

and

$$\frac{1}{4} q_k \le \frac{\gamma_{k-1}}{\gamma_k} \le 2q_k \tag{3.4}$$

LEMMA 3.5. Let $w(x)$ be an even weight function. Then we have

$$\sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 \tag{3.5}$$

For the proofs of Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 see [6,7,8,3 & 2] respectively.

REMARK. Apart from the necessary changes due to the change in the weight function, the technique in proving Theorems 2.1, 2.2 and 2.3 is essentially the same. As to Theorem 2.4, it is just a direct consequence of [4; Theorem 2.4.1] and the previous three theorems. So, we are going to prove here Theorem 2.1 only.

4. PROOF OF THEOREM 2.1

The proof of this theorem is based essentially on estimating the error term (3.1) in the case when $d\alpha(x) = w_1(x)dx$. But first, we need to establish some inequalities.

Since $w_1(x) = \exp(-x^4)$, we conclude from (3.3) that

$$\lim_{n\to\infty} n^{-1/4} \frac{\gamma_{n-1}}{\gamma_n} = (12)^{-1/4} .$$

So, if we choose an arbitrarily small number $\eta > 0$, then there exists a positive number N_η such that

$$\frac{\gamma_{n-1}}{\gamma_n} \le ((12)^{-1/4} + \eta)n^{1/4} , \quad \text{for all} \quad n \ge N_\eta . \tag{4.1}$$

We also conclude from (3.4) that

$$\frac{\gamma_{n-1}}{\gamma_n} \le (8n)^{1/4} , \quad \text{for all} \quad n = 1,2,3, \ldots \tag{4.2}$$

and from (3.2) combined with (4.1) that

$$x_{1,n+1} \le 2((12)^{-1/4} + \eta)n^{1/4} , \quad \text{for all} \quad n \ge N_\eta . \tag{4.3}$$

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Combining (3.5), (4.2) and (4.1) we get, for $n \geq N_\eta$

$$\sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 = \sum_{k=1}^{N_\eta -1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2 + \sum_{k=N_\eta}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2$$

$$\leq 2\sqrt{2} \sum_{k=1}^{N_\eta -1} k^{\frac{1}{2}} + ((12)^{-\frac{1}{4}} + \eta)^2 \sum_{k=N_\eta}^{n-1} k^{\frac{1}{2}}$$

$$\leq \frac{4\sqrt{2}}{3} (N_\eta^{3/2} - 1) + \frac{2}{3} ((12)^{-\frac{1}{4}} + \eta)^2 (n^{3/2} - N_\eta^{3/2})$$

$$\leq [\frac{4\sqrt{2}}{3} - \frac{2}{3} ((12)^{-\frac{1}{4}} + \eta)^2] N_\eta^{3/2} + \frac{2}{3} ((12)^{-\frac{1}{4}} + \eta)^2 n^{3/2}$$

Hence, putting $K_\eta = [\frac{4\sqrt{2}}{3} - \frac{2}{3}((12)^{-\frac{1}{4}} + \eta)^2] N_\eta^{3/2}$ and taking $n > N_\eta$, we get

$$\sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2 \leq K_\eta + \frac{2}{3} ((12)^{-\frac{1}{4}} + \eta)^2 n^{3/2} \tag{4.4}$$

From (4.1) and (4.2) we obtain

$$\frac{1}{\gamma_n} \leq ((12)^{-\frac{1}{4}} + \eta)^{n-N_\eta+1} \cdot [n(n-1)\dots(N_\eta+1)N_\eta]^{\frac{1}{4}} \cdot \frac{1}{\gamma_{N_\eta -1}} \quad \text{for} \quad n > N_\eta$$

and

$$\frac{1}{\gamma_{N_\eta -1}} \leq (8)^{N_\eta/4} \cdot [(N_\eta - 1)!]^{\frac{1}{4}} \cdot \frac{1}{\gamma_0} \quad \text{respectively.}$$

Hence, for $n > N_\eta$, we have

$$\frac{1}{\gamma_n^2} \leq \frac{1}{\gamma_0^2} (4\sqrt{6})^{N_\eta} \cdot ((12)^{-\frac{1}{4}} + \eta)^{2n} \cdot (n!)^{\frac{1}{2}} \tag{4.5}$$

Let us denote, throughout the rest of the proof, the nth orthogonal polynomial generated by w_1 by $p_n(x)$.

We now proceed to find a majorant for $|p_n(x)|^{-1}$. Since ω_1 is an even weight function, it follows that (see e.g. [9; Sect. 2.3(2)])

$$p_n(z) = \gamma_n z^{n-2[n/2]} \prod_{k=1}^{[\frac{n}{2}]} (z^2 - x_{kn}^2).$$

Hence,

$$\begin{aligned} |p_n(z)| &= \gamma_n |z|^{n-2[n/2]} \prod_{k=1}^{[\frac{n}{2}]} |z^2 - x_{kn}^2| \\ &= \gamma_n |z|^n \exp\Big(\sum_{k=1}^{[\frac{n}{2}]} \log\Big|1 - \frac{x_{kn}^2}{z^2}\Big|\Big) \\ &\geq \gamma_n |z|^n \exp\Big\{\sum_{k=1}^{[\frac{n}{2}]} \log\Big(1 - \frac{x_{kn}^2}{|z|^2}\Big)\Big\} \\ &\geq \gamma_n |z|^n \exp\Big(- \sum_{k=1}^{[\frac{n}{2}]} \frac{x_{kn}^2}{|z|^2 - x_{kn}^2}\Big) , \end{aligned}$$

for every $z \in \mathbb{C}$ such that $x_{1n} < |z|$. We also have

$$\frac{1}{|z|^2 - x_{kn}^2} \leq \frac{1}{|z|^2 - x_{1n}^2} , \quad \text{for } k = 1,2,\dots,n \text{ and } x_{1n} < |z|.$$

Therefore,

$$|p_n(z)|^{-1} \leq \frac{1}{\gamma_n |z|^n} \exp\Big(\frac{1}{|z|^2 - x_{1n}^2} \cdot \sum_{k=1}^{[\frac{n}{2}]} x_{kn}^2\Big).$$

And by using (4.4), we find that

$$|p_n(z)|^{-1} \le \frac{1}{\gamma_n |z|^n} \exp[\frac{1}{|z|^2 - x_{1n}^2} \{K_\eta + \frac{2}{3}((12)^{-1/4} + \eta)^2 n^{3/2}\}],$$

which implies that

$$|p_n(z)p_{n+1}(z)|^{-1} \le \frac{1}{\gamma_n\gamma_{n+1}} \cdot \frac{1}{|z|^{2n+1}} \cdot \exp\left\{ \frac{2K_\eta + \frac{4}{3}((12)^{-1/4} + \eta)^2 (n+1)^{3/2}}{|z|^2 - x_{1,n+1}^2} \right\}.$$

Let $I_n = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)dz}{p_n(z)p_{n+1}(z)}$ and choose the path of integration C_n to be the circle $|z| = R_n$ such that

$$R_n^2 \ge \frac{x_{1,n+1}^2}{1-\varepsilon}, \quad \text{for} \quad a < \varepsilon < 1 . \tag{4.6}$$

For $|z| = R_n$, we will have

$$|p_n(z)p_{n+1}(z)|^{-1} \le \frac{1}{\gamma_n\gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{2K_\eta + \frac{4}{3}((12)^{-1/4} + \eta)^2 (n+1)^{3/2}}{\varepsilon R_n^2}\right\} \tag{4.7}$$

From (2.1) it follows that for every $\delta > 0$, we can find an N_δ such that

$$|f(z)| \le \exp\{(\beta + \delta)R_n^4\}, \qquad \text{for all} \quad R_n \ge N_\delta \tag{4.8}$$

Using (4.7), (4.5) and (4.8), we conclude, for large enough R_n, that

$$|I_n| \le \frac{1}{\gamma_0^2} (4\sqrt{6})^{N_\eta}((12)^{-1/4} + \eta)^{2n} \cdot (n!)^{1/2} \cdot \frac{1}{R_n^{2n}} \cdot$$

$$\cdot \exp\left\{(\beta + \delta)R_n^4 + \frac{2K_\eta + \frac{4}{3}((12)^{-1/4} + \eta)^2 (n+1)^{3/2}}{\varepsilon R_n^2}\right\}.$$

Next, we are going to choose R_n so that it will minimize the right-hand side of this last inequality and, at the same time, satisfies (4.6) for $\alpha < \varepsilon < 1$. To do so, we consider the function

$$h(R) = \frac{1}{R^{2n}} \cdot \exp\left\{(\beta + \delta)R^4 + \frac{2K_\eta + \frac{4}{3}((12)^{-\frac{1}{4}} + \eta)^2(n+1)^{3/2}}{\varepsilon R^2}\right\}.$$

By differentiating $h(R)$ and setting $h'(R) = 0$, we get

$$2(\beta + \delta)R^6 - \frac{2}{\varepsilon}\{K_\eta + \frac{2}{3}((12)^{-\frac{1}{4}} + \eta)^2(n+1)^{3/2}\} - nR^2 = 0 \qquad (4.9)$$

Hence, we choose R_n to satisfy (4.9). From this choice of R_n and (4.3) we can see that

$$R_n^2 \geq \frac{x_{1,n+1}^2}{2\varepsilon^{1/3}(\beta + \delta)^{1/3}} .$$

Consequently, (4.6) will be satisfied if

$$\beta + \delta = \frac{(1 - \varepsilon)^3}{8\varepsilon} < \frac{(1 - a)^3}{8a} = \rho . \qquad (4.10)$$

Since R_n satisfies (4.9), we have

$$(\beta + \delta)R_n^4 = \frac{n}{2} + \frac{K_\eta + \frac{2}{3}((12)^{-\frac{1}{4}} + \eta)^2(n+1)^{3/2}}{\varepsilon R_n^2} ,$$

and it follows that

$$|I_n| \leq \frac{1}{\gamma_0^2}(4\sqrt{6})^{N_\eta} \cdot ((12)^{-\frac{1}{4}} + \eta)^{2n} \cdot (n!)^{\frac{1}{2}} .$$

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$$\cdot \frac{\varepsilon^{n/3}\ (\beta+\delta)^{n/3}}{[K_\eta + \frac{2}{3}\ ((12)^{-1/4}+\eta)^2 (n+1)^{3/2}]^{n/3}} \cdot$$

$$\cdot \exp\left\{ \frac{n}{2} + \frac{2((12)^{-1/4}+\eta)^2 (n+1)^{3/2}}{\varepsilon R_n^2} \right\}$$

$$\le \frac{1}{\gamma_0^2}\ (4\sqrt{6})^{N_\eta} ((12)^{-1/4}+\eta)^{2n} \cdot \frac{\varepsilon^{n/3}(\beta+\delta)^{n/3}\ (n!)^{1/2}}{[K_\eta + \frac{2}{3}((12)^{-1/4}+\eta)^2 (n+1)^{3/2}]^{n/3}}$$

$$\cdot \exp\left[\frac{\varepsilon^{1/3}(\beta+\delta)^{1/3}\{3K_\eta + 2((12)^{-1/4}+\eta)^2 (n+1)^{3/2}\}}{\varepsilon\{K_\eta + \frac{2}{3}\ ((12)^{-1/4}+\eta)^2 (n+1)^{3/2}\}}\right]$$

$$\le \frac{1}{\gamma_0^2}(4\sqrt{6})^{N_\eta}((12)^{-1/4}+\eta)^{4n/3}(\tfrac{3}{2})^{n/3} \cdot \frac{\varepsilon^{n/3}(\beta+\delta)^{n/3}(n!)^{1/2}}{(n+1)^{n/2}} \cdot$$

$$\cdot \exp\left[\frac{n}{2} + \frac{(\beta+\delta)^{1/3}\{3K_\eta + 2((12)^{-1/4}+\eta)^2 (n+1)^{3/2}\}}{\varepsilon^{2/3}(\frac{2}{3})^{1/3}((12)^{-1/4}+\eta)^{2/3}(n+1)^{1/2}}\right].$$

By using (4.10) and the Stirling formula, we can write this inequality, for sufficiently large n as

$$|I_n| \le K_\eta^*((12)^{-1/4}+\eta)^{4n/3}(\tfrac{3}{2})^{n/3} \cdot \frac{n^{1/4}(1-\varepsilon)^n}{2^n} \cdot$$

$$\cdot \exp\left\{\frac{1-\varepsilon}{2\varepsilon}\ (\tfrac{3}{2})^{1/3} \cdot \frac{3K_\eta + 2((12)^{-1/4}+\eta)^2 (n+1)^{3/2}}{((12)^{-1/4}+\eta)^{2/3}(n+1)^{1/2}}\right\}$$

$$\le K_\eta^*\ n^{1/4} \cdot [(\frac{1-\varepsilon}{4} + \varepsilon_1(\eta))\exp\{\frac{1-\varepsilon}{2\varepsilon}(1+\varepsilon_2(\eta)+\varepsilon_3(n))\}]^n \ ,$$

where K_η^* is a constant that depends on η, $\varepsilon_i(\eta) \to 0$ as $\eta \to 0$ $(i = 1,2)$ and $\varepsilon_3(n) \to 0$ as $n \to \infty$.

Since $g(\varepsilon) = \frac{1-\varepsilon}{4}\exp(\frac{1-\varepsilon}{2\varepsilon})$ is a continuous decreasing function on $(0,1)$ and $g(a) = 1$, it follows that $0 < g(\varepsilon) < 1$ for $a < \varepsilon < 1$ and consequently, we can find, for small enough η and large enough n, a number $\tau < 1$ such that

$$(\frac{1-\varepsilon}{4} + \varepsilon_1(\eta))\exp\left\{\frac{1-\varepsilon}{2\varepsilon}(1 + \varepsilon_2(\eta) + \varepsilon_3(n))\right\} < \tau < 1 ,$$

which establishes that $\sum_k |I_k|$ is a convergent series.

Therefore, for a sufficiently large n, we have

$$|\Delta_n^{(1)}| \le \sum_{k=n}^{\infty} |I_k| \le K_{\eta,\varepsilon} n\cdot[(\frac{1-\varepsilon}{4} + \varepsilon_1(\eta))\exp\left\{\frac{1-\varepsilon}{2\varepsilon}(1+\varepsilon_2(\eta)+\varepsilon_3(n))\right\}]^n,$$

where $K_{\eta,\varepsilon}$ is a constant that depends on ε and η only. Hence,

$$\limsup_{n\to\infty}|\Delta_n^{(1)}|^{1/n} \le (\frac{1-\varepsilon}{4} + \varepsilon_1(\eta))\exp\left\{\frac{1-\varepsilon}{2\varepsilon}(1 + \varepsilon_2(\eta))\right\}.$$

Since η can be chosen as small as we like and $\varepsilon_i(\eta) \to 0$ as $\eta \to 0$ $(i = 1, 2)$, it follows that

$$\limsup_{n\to\infty}|\Delta_n^{(1)}|^{1/n} \le g(\varepsilon) < 1 . \qquad \text{Q.E.D.}$$

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