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# SYMPLECTIC GEOMETRY OF SHALLOW WATER WAVES

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Abstract

After touching on some answers to the comments made by Cavalcante and McKean in their article [1], we give a symplectic formulation of the shallow water equations on a sloping beach. We show that this is a completely integrable Hamiltonian system with infinitely many constants of motion which are in involution.

## I Introduction

Cavalcante and McKean in their paper [1] deal with the complete integrability of the classical shallow water equations on a horizontal bottom (homogeneous case) :

$$\begin{aligned} &u_t + uu_x + h_x = 0, \\ &h_t + (uh)_x = 0, \end{aligned} \tag{1}$$

where $-\infty < x < \infty$ is the horizontal coordinate, t is the time, $u = u(t,x)$ is the horizontal component of the velocity at the point x at time t, and $h = h(t,x) > 0$ is the height of the free surface above the point x at time t. It is assumed that the horizontal component of the velocity does not depend on height (irrotational case).

Before we came across [ 1 ] we knew that the system of nonlinear equations (1) possessed an infinite number of conservation laws [ 2 ], [ 3 ], [ 4 ]. What we did not know were :
1) This system can be written in Hamiltonian form relative to a symplectic structure introduced by Manin in 1978, [ 5 ], and the infinitely many conserved quantities are in involution relative to this symplectic form and 2) Cavalcante and McKean wish to know an explicit solution to the system (1), (see their comment number 5).

To us it was one of those instances in mathematics where not being a specialist in that particular field, and hence not being familiar with all the literature on the subject allowed one , unencumbered, to discover the solution : One day we were exploring the content of a colleague's preprint [ 6 ], where we met, for the first time, the system of equations (1). This evolved into our paper [ 2 ] in which, beside other things, we constructed all the non-simple solutions* of (1) by using the classical <u>hodograph transformation</u>, which is a well known method for quasi-linear homogeneous equations [ 7 ]. Thus, regarding comment number 5 in [ 1 ], we are happy to confirm that we have the answer in [ 2 ]. For example, here are two sets of explicit non-simple wave solutions found there

$$u = x/t, \quad h = 1/t \quad \text{and} \quad u = -t, \quad h = x + t^2/2,$$

for the homogeneous nonlinear system (1). We would also like to mention Nutku's simple-wave solution*

$$u = 2x/3t, \quad h = (x/3t)^2,$$

which is constructed in [ 6 ] by the scale-invariance argument.

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*A solution (u,h) of the system (1) is called non-simple if u and h are functionally independent, that is, $J = u_x h_t - u_t h_x \neq 0$. The singular solutions for which $J = 0$ are called simple-wave solutions. The transformation equation between [ 2 ] and this article is $h = c^2$.

In [ 2 ], for the system (1) we have also constructed a one-to-one correspondence between all the hodograph solutions ($J \neq 0$) and all the conservation laws (polynomial and non-polynomial). Basically, to a solution of the following linear equation

$$K_{uu} = hK_{hh} \tag{2}$$

we associate a hodograph solution as well as a conservation law for the system (1). (Perhaps we ought to call (2) the <u>moduli equation</u> for the system (1) because the solution space of (2) parametrizes the non-simple solutions as well as the conservation laws of the system (1) ). Furthermore, by using the linear nature of (2) we construct auto-Bäcklund transformations. This is, of course, nothing but a <u>superposition principle</u> for the system (1) : Given two solutions of (1), add the corresponding solutions of (2), and then construct the solution of (1) corresponding to this sum.

In the last section of [ 2 ] we consider the non-homogeneous system of nonlinear equations

$$u_t + uu_x + h_x = g\beta, \tag{3a}$$

$$h_t + (uh)_x = 0, \tag{3b}$$

representing shallow water waves on a sloping beach, where $g$ is the gravitational constant and $\beta$ represents the slope of the bottom. By using the polynomial conservation laws of the related homogeneous system (1), we were fortunate in finding ourselves able to construct an infinite family of polynomial conservation laws for the non-homogeneous system (3). Again, by using the solutions of (2), which is really the cylindrical wave equation, we constructed all the non-simple solutions, as well as an auto-Bäcklund transformation for (3), in addition to a Bäcklund transformation between the homogeneous and non-homogeneous systems.

At the present time, we do not know if (3) has any non-polynomial conservation laws at all. Its polynomial conservation laws are in one-to-one correspondence with the polynomial conservation laws of the homogeneous system (1) which has non-polynomial conservation laws as well. We know how to construct a solution to (3) from any solution (or conservation law) of (1), but we do not know, at present, how to construct a conservation law for (3) from a non-polynomial conservation law of (1). Perhaps such a construction is not possible, that is, (3) has no non-polynomial conservation laws. If so, we believe that this would be quite an interesting example to have in the theory of nonlinear differential equations as well as in the infinite dimensional symplectic geometry.

Coming across the following - which shows how great the differences between linear and nonlinear systems may be - was also an interesting experience for us : Carrier and Greenspan in 1958 [ 8 ] gave the following explicit formulae for the non-simple solutions of the non-homogeneous system (3) :

$$g\beta x = - \Phi_h + \Phi_v^2/2h^2 + h, \tag{4a}$$

$$g\beta t = \Phi_v/h - v, \tag{4b}$$

where $h=c^2$, $v = (u-g\beta t)$ and $\Phi$ satisfies the cylindrical wave equation (2). We would have expected that the homogeneous problem would have been solved before the non-homogeneous problem (because our academic life has been mostly along linear lines while living in a nonlinear physical world). We did not even know (until we read [ 1 ]) that the first part of our article [ 2 ] contained any new results. It is true that by setting $\beta = 0$ in the polynomial conservation laws of (3) we obtain conservation laws for the homogeneous system (1), but we get only one relation if we let $\beta = 0$ in the solutions (4) of the non-homogeneous system. This, in particular, shows that the Bäcklund transformation that we discovered in [ 2 ] between the solutions of the homogeneous and non-homogeneous systems is a highly non-trivial construction.

In this article we shall cast the shallow water equations (3) into Hamiltonian formalism relative to a symplectic form, and show that the infinitely many polynomial constants of motion found in I 2 I are in involution relative to this symplectic form. We shall also observe that the Bäcklund and auto-Bäcklund transformations constructed in [ 2 ] are symplectic maps.

At this point, one is tempted to conclude that the homogeneous and non-homogeneous problems are equivalent as two completely integrable Hamiltonian systems. But, let us not get carried away, because the non-homogeneous system may not have any polynomial conservation laws, whereas the homogeneous system has many.

The results of this article will be announced in Physics Letters A, 1983.

## II Shallow Water Waves: Symplectic Geometry

As explained in our previous paper[2], there are two interesting wave equations concerning the system (3):

$$4K_{uu} = K_{cc} \pm K_c/c \ . \qquad (5)$$

We shall call them the "plus wave equation" and the "minus wave equation", respectively. We discovered the following Bäcklund transformation between their solution spaces:

$$P_u = L_c/2c \quad \text{and} \quad P_c = 2L_u/c \ ; \qquad (6)$$

where P and L are solutions of the positive and negative wave equations, respectively,(easy to check). Thus, we can use either of these equations in our computations. However, later we realized that only the minus wave equation is suitable for the symplectic geometry which comes with the system (3), as will be seen in this article.

In 1958, Carrier and Greenspan applied Riemann's characteristic forms to the system (3) and obtained solutions for this system in terms of the solutions of the positive wave equation above. Below, we express their solutions in terms of the minus wave equation:

$$g\beta x = -\Phi_h + \Phi_v^2 /2h^2 + h, \tag{7a}$$

$$g\beta t = \Phi_v /h - v \tag{7b}$$

where $h=c^2$, $v=u-g\beta t$ and $\Phi$ satisfies the transformed minus wave equation in the new variables v and h:

$$\Phi_{vv} = h\,\Phi_{hh}. \tag{8}$$

These exhaust almost all the solutions of the system (3), (we miss the so-called <u>simple</u> <u>wave</u> solutions which form a set of measure zero in the space of all solutions. They correspond to the singular case in which v and h are functionally dependent ).

It is interesting to note that by letting $\beta=0$ in (7a) we obtain

$$\Phi = \tfrac{1}{2}(u^2h + h^2), \tag{9}$$

which is the Hamiltonian function of the associated homogeneous system,[1].According to the techniques developed in Akyildiz [2],there is only the following solution of the homogeneous problem corresponding to this Hamiltonian:

$$u=-t\ ,\ c=(x + t^2/2)^{1/2}. \tag{10}$$

That is, we do not get all the solutions of the homogeneous system by setting the nonhomogeneous term to zero in the solutions of the nonhomogeneous problem. This is quite contrary to what happens in the linear case.

In our previous article[2], by using the same wave equations (5), we were able to construct almost all the solutions of the homogeneous system as well. Thus, after the necessary relabelling of the variables, a solution of (8) can be used to generate a solution to either of the problems: homogeneous and nonhomogeneous. In this way, we find a correspondence between the non-simple wave solutions of the two systems. This correspondence can be thought of as a Bäcklund transformation between the homogeneous and nonhomogeneous problems. By using the linear nature of the same wave equation (8), we can also construct auto-Bäcklund transformations for each system. These are, of course, nothing but superposition principles for the nonlinear systems under consideration: Given two solutions of (3), add the corresponding solutions of (8), and then construct the solution of (3) corresponding to this sum.

In Akyildiz [2], we were also able to construct polynomial conserved quantities for the system (3) in the form $\int_0^\infty \Psi\, dx$, where $\Psi$ is a solution of the same wave equation (8). In this way, via the solutions of (8), we establish correspondences between conservation laws, non-simple wave solutions and Bäcklund transformations of both the homogeneous and nonhomogeneous systems.

Finally, we shall cast the system (3) into Hamiltonian form and show that the solution space of the negative wave equation (8) is isotropic relative to the accompanying symplectic form (Poisson bracket); hence, proving that all the constructions carried out above are canonical. After absorbing the nonhomogeneous term $g\beta t$ in (3b) as

$$(u-g\beta t)_t + uu_x + 2cc_x = 0, \tag{11}$$

we can introduce the following Hamiltonian formalism for the system (3):

$$\begin{pmatrix} u-g\beta t \\ c^2 \end{pmatrix}_t = J \nabla H, \quad J=-\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}, \quad H=\frac{1}{2}(u^2c^2 + c^4), \qquad (12)$$

where $J$ is the Hamiltonian operator, $H$ the Hamiltonian, $D$ differentiation with respect to x, and $\nabla$ the gradient operator in (v,h)-space with $v=u-g\beta t$ and $h=c^2$. Since $u_x=v_x$, we equivalently have

$$\begin{pmatrix} v \\ h \end{pmatrix}_t = J \nabla H, \quad \text{with } H=\frac{1}{2}(v^2h + h^2). \qquad (13)$$

The associated <u>Poisson</u> <u>bracket</u> is defined to be

$$\left[A , B\right] = \int_0^{\infty} \nabla A J \nabla B \, dx \qquad (14)$$

for two functions A and B of the variables v and h. It is easy to verify that (14) satisfies the Jacobi identity. We assume that the boundary conditions $u=0$, $c=0$ are satisfied at $x=0$ and $\infty$. Since $v=u-g\beta t$, in order to have meaningful space integrals in (14) we must not have v's appearing on their own in our polynomial expressions (because t may become arbitrarily large). v's must always come multiplied with u's or c's. This was an essential point in constructing conserved quantities for the system (3) in our earlier work,[2],p.1727.

Now, we shall show that the Poisson bracket (14) vanishes on the solution space of the wave equation (8), which parametrizes both the non-simple wave solutions and conservation laws of the system (3): Let A and B be two solutions of (8). The integrant in (14) is

$$(A_vDB_h + A_hDB_v)dx = A_v dB_h + A_h dB_v . \qquad (15)$$

Since

$$d(A_v dB_h + A_h dB_v) = dA_v \wedge dB_h + dA_h \wedge dB_v$$
$$= (A_{vv}dv + A_{vh}dh) \wedge (B_{hv}dv + B_{hh}dh)$$
$$+(A_{hv}dv + A_{hh}dh) \wedge (B_{vv}dv + B_{vh}dh)$$
$$= (A_{vv}B_{hh} - A_{hh}B_{vv})dv \wedge dh$$
$$= 0 \qquad \text{by (8)},$$

the integrant in (14) is closed and is, therefore, exact on simply connected regions; that is,

$$[A , B] = \int_0^\infty dC = 0,$$

for a function C of the arguments v and h. This finishes the proof that the Poisson bracket (14) vanishes on the solution space of the wave equation (8), from which we obtain solutions, Bäcklund transformations and conservation laws for the system (3). Thus, we have simultaneously proved that the Bäcklund transformations above are symplectic and that conserved quantities found in Akyildiz [2] for the system (3) are in involution relative to the symplectic structure introduced in this article.

Whether the isotropic space of the solutions of the above-mentioned wave equation is Lagrangian is an important question. This is, in fact, the problem of complete integrability of an infinite dimensional Hamiltonian system. See pp. 7 and 9 of Cavalcante & McKean [1] on this important issue.

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