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A Geometric Approach for the Coefficient Estimates of Multivalent Close to convex Functions

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ABSTRACT

A geometric property associated with the Riemann surface spread over the complex plane of a multivalent close-to-convex function is used in order to obtain sharp coefficient estimates for two classes of functions: The class of close-to-convex functions of order p, due to A.E. Livingston, and the class of weakly close-to-convex functions of order p, due to D. Styer.

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1. Introduction

The object of this article is to present a surface theoretic method for obtaining uniform and sharp coefficient bounds for two classes of multivalent close-to-convex functions: The class of <u>close-to-convex functions of order p</u>, $K(p)$, due to Livingston [5], and the class of <u>weakly close-to-convex functions of order p</u>, $K_W(p)$, due to Styer [12].

Coefficient estimates for the class $K(p)$ were found by Hallenbeck and Livingston [3] via extreme point theory. Our approach, however, is direct and completely different. We merely use the intrinsic properties of the surface of a close-to-convex function as embedded into a polynomial surface spread over the complex plane.

We begin with defining the classes $K(p)$ and $K_W(p)$.

<u>Definition</u> 1.1. Let $K(p)$ be the class of all functions

$$f(z) = a_q z^q + a_{q+1} z^{q+1} + \cdot \cdot \cdot , \tag{1.1}$$

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where $1 \le q \le p$, which are regular in $\mathbb{B} = \{z: |z| < 1\}$ and satisfy

$$\int_0^{2\pi} \mathrm{Re}\{1 + re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta})\} d\theta = 2p\pi , \tag{1.2}$$

and

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\{1 + re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta})\} d\theta > -\pi, \tag{1.3}$$

for all r, with $0 < r_0 < r < 1$ for some r_0, and all $\theta_1 < \theta_2$.

We call $f \in K(p)$ which is regular on $\mathrm{Cl}(\mathbb{B})$ and satisfies (1.2) and (1.3) a <u>close-to-convex function of order p on</u> $\mathrm{bd}(\mathbb{B}) = \{z: |z| = 1\}$.

From (1.2), it follows directly that f' has $(p-1)$ zeros (counting multiplicity).

Note that $K(1)$ is the univalent class of close-to-convex functions due to Kaplan [4].

We will say that the sequence $\{f_n\}$ of functions regular in $\mathbb{B}$ converges almost uniformly to f if $\{f_n\}$ converges uniformly to f on every compact subset of $\mathbb{B}$.

There are six analytic definitions for the class $K_W(p)$ that are due to Styer [12]. The following two definitions are chosen only for relevance.

<u>Definition</u> 1.2. A function $f \neq 0$ belongs to the class $K_W(p)$ if there is a sequence $\{f_n\}$ of functions in $K(p)$ such that $f_n \to f$ almost uniformly in $\mathbb{B}$.

Definition 1.3. Let f be a nonconstant function regular in $\mathbb{B}$. f belongs to the class $K_W(p)$ if $f(0) = 0$ and there exists a univalent starlike function, h, with $h(0) = 0$ such that

$$\liminf_{r\to 1^-}[\min_{|z|=r} \operatorname{Re}\{zf'(z)/[h(z)]^p\}] \geq 0 \tag{1.4}$$

It is evident from the definitions that for every $f \in K_W(p)$, f' has at most $(p-1)$ zeros (counting multiplicity); and that $K(p)$ is a proper subset of $K_W(p)$.

2. Statements of Results

It will be useful to introduce the notation $g \ll h$, where g and h are functions regular in $\mathbb{B}$, to denote that the modulus of each coefficient of the Taylor's series expansion of g about $z = 0$ is at most the corresponding one of h.

Note that if $g \ll h$, then $g(z) \ll e^{i\psi}h(e^{i\theta}z)$ for any ψ and θ.

Our first result will be:

Theorem 2.1. Suppose that $f \in K(p)$ has form (1.1). Let $\beta_1, \beta_2, \cdots, \beta_{p-q}$ be the nonzero zeros of f' (counting multiplicity). Then

$$f \ll q\, a_q \int_0^z z^{q-1}(1+z)(1-z)^{-2p-1}\prod_{j=1}^{p-q}(1+|\beta_j|^{-1}z)(1+|\beta_j|z)dz, \tag{2.1}$$

and the indefinite integral belongs to $K(p)$.

The corresponding result for the class $K_W(p)$ yields the more general

Theorem 2.2. Suppose that $f \in K_W(p)$ has form (1.1). Let $\beta_1, \beta_2 \cdots \beta_s$ with $1 \le s \le p-q$, be the nonzero zeros of f' (counting multiplicity). Then

$$f \ll qa_q \int^z z^{q-1}(1+z)^{\alpha}(1-z)^{-2p-1}\prod_{j=1}^{s}(1+|\beta_j|^{-1}z)(1+|\beta_j|z)\,dz, \qquad (2.2)$$

where $\alpha = 2(p-q-s)+1$, and the indefinite integral belongs to $K_W(p)$.

3. Proofs of Theorems and Corollaries

In order to prove Theorem 2.1 we need a couple of lemmas, of which the first is most significant.

Lemma 3.1. Suppose that f is a close-to-convex function of order p on bd($\mathbb{B}$). Let Y be the image surface of f. Then Y can be embedded in the image surface, X, of a polynomial of degree exactly p, such that $X \setminus Y$ is a disjoint union of semi-lines, each of which starts from bd(Y).

The surfaces X and Y are taken over the complex plane, and a semi-line in X is a lift of a semi-line in the plane.

A proof of Lemma 3.1 as well as of a more general result is due to the author and can be found in [6].

Lemma 3.2. (1) Let f_i and g_i, with $1 \le i \le n$, be regular functions in $\mathbb{B}$. Suppose that for each i, $f_i \ll g_i$, and g_i has non-negative coefficients. Then

$$\prod_{i=1}^{n} f_i \ll \prod_{i=1}^{n} g_i$$

(2) Let $\{f_n\}$ and $\{g_n\}$ be sequences of regular functions that converge, respectively, to f and g almost uniformly. If for each n, $f_n \ll g_n$, then $f \ll g$.

This lemma is direct, and its proof is omitted.

Proof of Theorem 2.1. First, suppose that f is a close-to-convex function of order p on $bd(\mathbb{B})$. Let X and Y be as in Lemma 3.1, and let $X\setminus Y$ be ruled by disjoint semi-lines, ℓ, starting from $bd(Y)$.

We extract from the ruling $\{\ell\}$ a countable set of semi-lines $\{\ell_k\}$ such that $X\setminus Y = Cl(\bigcup_{k=1}^{\infty} \ell_k)$. Evidently, there exists a positive integer N so that each of the interior angles at infinity, included between two consecutive semi-lines of $\{\ell_k\}_{k=1}^{N}$ is at least $-\pi$.

Now we apply the Uniformization Theorem [12, pp.225]. For each $n > N$, there exists a function, f_n, regular in $\mathbb{B}$, with its first nonzero derivative at zero is $f_n^{(q)}(0)$, and $f_n^{(q)}(0)/a_q > 0$, such that the image surface of f_n is $X\setminus(\bigcup_{k=1}^{n} \ell_k)$. By virtue of the Carathéodary Kernel Theorem [8], $f_n \to f$ almost uniformly in $\mathbb{B}$, since $X\setminus(\bigcup_{k=1}^{n} \ell_k) \to Y$ in the general sense of convergence.

Making use of the Generalized Schwarz-Christoffel Transformation (see [1], [2], and [10]), we obtain for each function f_n the following integral representation:

$$f_n(z) = A_n\int_0^z z^{q-1}\prod_{j=1}^{n}(1-\zeta_j z)(1-z_j z)^{-\gamma_j}\prod_{j=1}^{p-q}(1-\beta_j^{-1}z)(1-\overline{\beta}_j z)dz\ , \tag{3.1}$$

where:

a) For each j, with $1 \le j \le n$, ζ_j and z_j are the inverse images under f_n of the finite and infinite ends of ℓ_j.

b) Each $\gamma_j\pi$, with $1 \le j \le n$, is the exterior angle at infinity of $X\setminus(\bigcup_{j=1}^{n}\ell_j)$, corresponding to the interior angle included by ℓ_j and ℓ_{j+1}, with $\ell_1 = \ell_{n+j}$.

c) Each β_j, $1 \le j \le p-1$, is a zero of f' in $\mathbb{B}$.

d) Each $A_n \neq 0$, and $A_n \to qa_q$.

Because of the boundary correspondence, the complex numbers $\zeta_1, \zeta_2, \cdots, \zeta_n$ and $z_1, z_2, \cdots, z_n$ are located alternately on $bd(\mathbb{B})$. As for each γ_j, $1 \le \gamma_j \le 2$. Since the exterior angle of $X\setminus(\bigcup_{j=1}^{n}\ell_j)$ at the finite ends of each ℓ_j is $-\pi$, $\sum_{j=1}^{n}\gamma_j = 2p + n$.

Rewrite (3.1) as

$$f_n(z) = A_n\int_0^z z^{q-1}\prod_{j=1}^{n}(1-\zeta_j z)(1-z_j z)^{-1}\prod_{j=1}^{n}(1-z_j z)^{1-\gamma_j}\prod_{j=1}^{p-q}(1-\beta_j^{-1}z)(1-\beta_j z)dz.$$

It is well known that there exists $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that

$$\mathrm{Re}\{e^{i\alpha}\prod_{j=1}^{n}(1-\zeta_j z)(1-z_j z)^{-1}\} > 0\ ,$$

in $\mathbb{B}$. This gives

$$\prod_{j=1}^{n}(1 - \zeta_j z)(1 - z_j z)^{-1} = e^{i\alpha}(P\cos\alpha + i\sin\alpha),$$

where P is a regular function with positive real part in $\mathbb{B}$, and $P(0) = 1$. This implies (see [9, pp.2])

$$\prod_{j=1}^{n}(1 - \zeta_j z)(1 - z_j z)^{-1} \ll \frac{1+z}{1-z}. \tag{3.2}$$

Next, for every j, $1 \le j \le n$, $1 \le \gamma_j \le 2$, or, $0 \le \gamma_j - 1 \le 1$, and $\sum_{j=1}^{n}(\gamma_j - 1) = 2p$. Apply a lemma due to Goodman [2, pp.213], we get

$$\prod_{j=1}^{n}(1 - z_j z)^{1-\gamma_j} \ll (1 - z)^{-2p}. \tag{3.3}$$

Using Lemma 3.2, we have

$$\prod_{j=1}^{p-q}(1 - \beta_j^{-1} z)(1 - \beta_j z) \ll \prod_{j=1}^{p-q}(1 + |\beta_j|^{-1} z)(1 + |\beta_j| z). \tag{3.4}$$

Again, in view of Lemma 3.2, and because of (3.2), (3.3), and (3.4), we conclude

$$f_n \ll A_n \int_0^z z^{q-1}(1+z)(1-z)^{-2p-1} \prod_{j=1}^{p-q}(1+|\beta_j|^{-1}z)(1+|\beta_j|z)\,dz.$$

Letting $n \to \infty$, we obtain (2.1).

Next, suppose that $f \in K(p)$. From Definition 1.1, it follows that for values ρ arbitrarily close to 1^-, $f(\rho z)$ is a close-to-convex function of order p on $bd(\mathbb{B})$. Therefore

$$f(\rho z) \ll \rho^q q a_q \int_0^z z^{q-1}(1+z)(1-z)^{-2p-1} \prod_{j=1}^{p-q}(1+|\beta_j|^{-1}z)(1+|\beta_j|z)\,dz.$$

By allowing $\rho \to 1^-$, the relation (2.1) follows.

Finally, for the fact that the indefinite integral in (2.1) belongs to $K(p)$ see [5]. This completes the proof.

Remark 3.1. Suppose that f is a regular function in $\mathbb{B}$ with a Schwarz-Christoffel representation. Then the product $\Pi(z - \beta_j)(1 - \overline{\beta}_j z)$ does not appear under the integral sign unless f' has nonzero zeros in $\mathbb{B}$.

The proof of Theorem 2.2 makes use of the following.

Lemma 3.3. Let $f(z) = a_q z^q + a_{q+1} z^{q+1} + \cdots \in K_W(p)$. Then there exist functions

$$f_n(z) = a_{q_n} z^q + a_{q_n+1} z^{q+1} + \cdots \in K(p)$$

for $n = 1, 2, \cdots$, such that $f_n \to f$ almost uniformly.

Proof. We apply Definition 1.4. There exists a univalent starlike function h such that (1.4) is satisfied. Following Styer [12, pp.107], let

$$f_t(z) = \int_0^z \left[\frac{h(tz)}{h(z)}\right]^p \left(f'(z) + (1 - t)\frac{[h(z)]^p}{z}\right) dz$$

where $0 < t < 1$. It follows from the proof of Proposition 4 in the latter reference that for each t, $f_t \in K(p)$, and $f_t \to f$ almost uniformly as $t \to 1$. It can be easily verified that each f_t has a zero of order q at zero. This ends the proof.

Proof of Theorem 2.2. According to the previous lemma, there are functions $f_n \in K(p)$, with $n = 1, 2, \cdots$, such that

$$f_n(z) = a_{q_n} z^q + a_{q_n+1} z^{q+1} + \cdots ,$$

and $f_n \to f$ almost uniformly in $\mathbb{B}$.

Suppose that the nonzero zeros of each f_n' are $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{p-q}^{(n)}$. With an appropriately chosen notation, the argument principle leads us to write:

$$\beta_j^{(n)} \to \beta_j, \quad \text{for all } j, \text{ with } 1 \le j \le s , \tag{3.5}$$

and

$$|\beta_j^{(n)}| \to 1, \quad \text{otherwise.} \tag{3.6}$$

An application of Theorem 2.1 gives:

$$f_n \ll q a_{q_n} \int_0^z z^{q-1}(1+z)(1-z)^{-2p-1} \prod_{j=1}^{p-q} (1+|\beta_j^{(n)}|^{-1}z)(1+|\beta_j^{(n)}|z)dz. \tag{3.7}$$

For convenience, let g_n be the function on the right-hand side of $\ll$ in (3.7), and let

$$C_n(z) = \prod_{j=1}^{s} (1 + |\beta_j^{(n)}|^{-1}z)(1 + |\beta^{(n)}|z) ,$$

and

$$D_n(z) = \prod_{j=s+1}^{p-q} (1 + |\beta_j^{(n)}|^{-1}z)(1 + |\beta_j^{(n)}|z) .$$

Because of (3.5) and (3.6), we easily conclude that

$$C_n(z) \to \prod_{j=1}^{s} (1 + |\beta_j|^{-1}z)(1 + |\beta_j|z) ,$$

and

$$D_n(z) \to (1 + z)^{2(p-q-s)}$$

almost uniformly in $\mathbb{B}$. Consequently, it is not hard to conclude that g_n converges to the function on the right-hand side of $\ll$ in (2.2) almost uniformly in $\mathbb{B}$. Using Lemma 3.2, the relation (2.2) follows. Since each $g_n \in K(p)$, the indefinite integral in (2.2) belongs to $K_W(p)$. This ends the proof.

Corollary 3.1. Theorem 2.2 implies Theorem 2.1.

The proof follows directly by simply letting $s = p-q$ in (2.2)

Corollary 3.2. Suppose that

$$f(z) = a_1 z + a_2 z^2 + \cdots \in K_W(p) ,$$

such that f' has no zeros in $\mathbb{B}$. Then

$$f \ll (4p)^{-1} a_1 [\{(1 + z)(1 - z)^{-1}\}^{2p} - 1] \in K_W(p)$$

The proof follows directly from Remark 3.1 and Theorem (2.2).

Corollary 3.3. Suppose that

$$f(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \in K_W(p) ,$$

then

$$f \ll a_p z^p (1 - z)^{-p} \in K(p)$$

Proof. Using Remark 3.1, by letting $p = q$ in Theorem 2.2 we get

$$f << pa_p\int_0^z [z(1-z)^{-2}]^{p-1}[(1+z)(1-z)^{-3}]dz = a_p z^p (1-z)^{-2p}.$$

For $p=1$ in the above corollary, we obtain the Bieberbach's conjecture for $K(1)$ which was done by Reade [8].

Our extremal functions are geometrically interesting. Every extremal function of $K_W(p)$, resulting from Theorem 2.2, has an image surface of a polynomial minus a radial slit. The polynomial surface must be symmetric about the slit, and its branch points must lie over the straight line upon which the slit projects. Moreover, if the derivative of the extremal function has s, with $0 \le s \le p-1$, zeros (counting multiplicity), then the finite end of the slit is a branch point of order $p-s-1$.

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