King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 049

April 1983

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UNIQUE COMPOSITION OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The object of this article is to prove the uniqueness of the composition $f = P \circ \phi$, where P is a polynomial of degree at most p and ϕ is a normalized univalent function, whenever f is a close-to-convex function of order p, and possibly otherwise whenever f is a weakly close-to-convex function of order p.

AMS 1980 subject classifications. Primary 30C45

Key words and phrases: Multivalent functions, Close-to-convex functions, Covering surfaces.

UNIQUE COMPOSITION OF CLOSE-TO-CONVEX FUNCTIONS

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1. Introduction

In an article [4] which appeared recently, the author shows that every weakly close-to-convex function f of order p is a composition of a polynomial of degree at most p and a normalized univalent function in S. A natural question that arises is whether this composition is unique. In this paper, using the intrinsic properties of the image surface of a multivalent close-to-convex function, we show that such a composition is unique whenever f is a close-to-convex function of order p, as defined by Livingston [3], and possibly otherwise whenever f is a weakly close-to-convex function of order p, as defined by Styer [5]. This result completes the solution of the problem of writing a multivalent close-to-convex function as a polynomial of a univalent function.

Let $S_a(p)$ be the class of all functions f regular in the open unit disc $\mathbb{B}$, with p zeros there (counting multiplicity), such that $f(0) = 0$ and $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for all z in some annulus $A_r = \{z: r < |z| < 1\}$. Call $S_a(p)$ the class of <u>annular p-valent starlike functions</u> [2]. We consider the class $K(p)$ of <u>close-to-convex functions of order</u> $\underline{p}$ as introduced in [3]. A function f regular in $\mathbb{B}$, with $f(0) = 0$, belongs to $K(p)$ if there exists $g \in S_a(p)$ such that $\operatorname{Re}\{zf'(z)/g(z)\} > 0$ for all z in some annulus A_r. We say that a function f is <u>close-to-convex</u>

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* This paper was presented at the 88th annual meeting of the American Mathematical Society.

of order p on Cl($\mathbb{B}$) if f is regular on Cl($\mathbb{B}$), with $f(0) = 0$, such that there exists $g \in S_a(p)$ regular on Cl($\mathbb{B}$) for which $\mathrm{Re}\{zf'(z)/g(z)\} > 0$ on bd($\mathbb{B}$). Evidently, every such function belongs to $K(p)$ and has no zeros of its derivative on bd($\mathbb{B}$). The class of weakly close-to-convex functions of order p, $K_w(p)$, can be defined simply as the closure of $K(p)$ in the topology of uniform convergence on compact subsets of $\mathbb{B}$. For other definitions of $K_w(p)$, see [5].

Note that for every $f \in K(p)$, f is at most p-valent and f' has exactly $(p-1)$ zeros (counting multiplicity). A proof of this result may be found in [3]. It easily follows [5] that for every $f \in K_w(p)$, f is at most p-valent and f' has at most $(p-1)$ zeros (counting multiplicity).

2. Close-To-Convex Functions

A main purpose of this paper is to prove:

Theorem. Let $f \in K(p)$. Then there exists a unique pair of functions P, ϕ such that $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$.

For the proof we require the following lemmas.

Lemma 1. Let f be a nonconstant analytic function on the closure of $\mathbb{B}$, $Cl(\mathbb{B})$. Then f induces a bordered Riemann surface Y, the image surface of f, with an analytic projection $\Pi : Y \to \mathbb{C}$, and a conformal map $F : Cl(\mathbb{B}) \to Y$ such that $f = \Pi \circ F$.

The proof of this lemma may be found in [4]. For information on bordered surfaces see [1; 23, 117].

Suppose that X and Y are covering surfaces over $\mathbb{C}$. X is said to <u>extend</u> Y if Y is a proper subset of X and Y is endowed with the same conformal structure and projection map of X.

<u>Lemma</u> 2. Let f be close-to-convex of order p on $Cl(\mathbb{B})$, and let Y be its image surface. Then Y can be extended to the image surface X of a polynomial of degree p such that $X - Y$ is a disjoint union of rays (A ray in a covering surface of $\mathbb{C}$ is a lift of a euclidean ray) which start from $bd(Y)$.

For the proof of this lemma see [4].

Following this lemma, let Π_Y and Π_X be the projection maps of Y and X respectively. By Lemma 1, if F and P^* are the lifts to Y and X which are induced by f and P respectively, then $f = \Pi_Y \circ F$ and $P = \Pi_X \circ P^*$. Let $\phi = (P^*)^{-1} \circ F$. ϕ is a conformal map on $Cl(\mathbb{B})$. We can choose P so that $P^*(0) = F(0)$ and $P^{-1} \circ f$ is locally univalent at 0 with $(P^{-1} \circ f)'(0) = 1$. Then $\phi \in S$. Since $F = P^* \circ \phi$, $f = \Pi_Y \circ F = \Pi_X \circ (P^* \circ \phi) = (\Pi_X \circ P^*) \circ \phi = P \circ \phi$. We call a <u>ray</u> in the plane domain of P any curve whose image under P is a euclidean ray. Then $\mathbb{C} - \phi(\mathbb{B})$ is a disjoint union of rays ℓ which start from $bd(\phi(\mathbb{B}))$.

It can be easily shown that for any $a \in bd(\phi(\mathbb{B}))$, there exists a "sub-interval" $[b,c]$ of $bd(\phi(\mathbb{B}))$, with $a \in [b,c]$, such that P is univalent in the domain bounded by ℓ, ℓ', $[b,c]$, and which does not meet $\phi(\mathbb{B})$. This fact, together with a simple compactness argument, yields a partition of $C - \phi(\mathbb{B})$ which proves to be useful. The following lemma summarizes the preceding discussion and describes the desired partition.

Lemma 3. Let f be close-to-convex of order p on $Cl(\mathbb{B})$. Then $f = P \circ \phi$, where P is a polynomial of degree p and $\phi \in S$, with $\mathbb{C} - \phi(\mathbb{B})$ is a disjoint union of rays which start from $bd(\phi(\mathbb{B}))$. Of these rays, we can extract a finite collection which divides $\mathbb{C} - \phi(\mathbb{B})$ into disjoint simply connected domains H that satisfy:

1) P is univalent in every $Cl(H)$.

2) No $P(z)$, with $P'(z) = 0$, is covered by P in any H.

3) If two domains H are adjacent along some ray ℓ, then their images are adjacent along the euclidean ray $P(\ell)$; that is, they lie on the different sides of $P(\ell)$.

Now we return to the theorem.

Proof of Theorem. First we establish the proof for functions f which are close-to-convex of order p on $Cl(\mathbb{B})$.

Suppose that

$$f = P \circ \phi = Q \circ \psi ,$$

where P and ϕ are the same functions which appear in Lemma 3,

Q is a polynomial of degree p, and $\psi \in S$.

Call the domains of P and Q the z-plane and ζ-plane respectively, and call the image plane of f the w-plane. We shall mean by a ray in the z-plane or ζ-plane any simple curve whose image under P or Q is a euclidean ray respectively; whereas a ray in the w-plane is a euclidean ray. Let γ and Γ be the boundaries of $\phi(\mathbb{B})$ and $\psi(\mathbb{B})$ respectively. Note that γ and Γ are simple closed analytic curves. For any points, $A, B \in \Gamma$, let (A,B) be the subcurve of Γ obtained by traversing Γ positively from A to B.

Using Lemma 3, there exists a collection of disjoint rays ℓ_i, $1 \le i \le n$, which divides $\mathbb{C} - \phi(\mathbb{B})$ into disjoint simply connected domains H_i, $1 \le i \le n$, according to the lemma. Suppose that as we wind positively once around zero in $\mathbb{C} - \phi(\mathbb{B})$, the rays ℓ_i and domains H_i are laid out such that each H_i is bounded by ℓ_i and ℓ_{i+1}, with $\ell_{n+1} = \ell_1$, together with the positively oriented subcurve of γ which joins the initial points of ℓ_i and ℓ_{i+1}.

Let $h = \psi \circ \phi^{-1}$. Evidently, h is the single-valued analytic branch of $Q^{-1} \circ P$ which maps $Cl(\phi(\mathbb{B}))$ conformally onto $Cl(\psi(\mathbb{B}))$. In what follows we show that h can be extended analytically to an entire function which is locally univalent and has a pole of infinity. This implies that $\phi \circ \psi^{-1}(z) = az + b$, where $a, b \in \mathbb{C}$. Since $\phi, \psi \in S$, ϕ is identical to ψ and consequently P is identical to Q.

It is easily seen that the sets $\{f(z) : f'(z) = 0\}$, $\{P(z) : P'(z)=0\}$, and $\{Q(z) : Q'(z) = 0\}$ are equal. Thus by Lemma 3, none of the points of the latter set, in particular, belongs to any $P(H_i)$. Since the domains $P(H_i)$ are simply connected, the Monodromy Theorem yields p single-valued analytic branches of Q^{-1} in every $P(H_i)$, each of which is univalent. Given $P(H_i)$, $1 \le i \le n$, choose the branch of Q^{-1} so that $Q^{-1} \circ P$ and h agree on γ. We directly conclude that h has a single-valued analytic extension in $\mathbb{C} - \bigcup_{i=1}^{n} \ell_i$. Denote this extension by h itself.

Obviously, h is univalent in each H_i. For each i, $1 \le i \le n$, let $D_i = h(H_i)$. It can be easily seen that, like regions $Cl(H_i)$, each $Cl(D_i)$ is a simply connected region bounded by a Jordan curve that consists of two disjoint rays in the ζ-plane, together with the positively oriented subcurve of Γ which joins the initial points of these rays. In fact, the regions $Cl(H_i)$ and $Cl(D_i)$, taken in the geometries of the z-plane and ζ-plane respectively, are copies of each other.

Now we investigate the adjacency of any pair of domains D_i and D_{i+1}, $1 \le i \le n$, with $D_{n+1} = D_1$. For any ℓ_i, $P(\ell_i)$ is a euclidean ray. Consequently, $Q^{-1} \circ P(\ell_i)$ is a union of p rays L in the ζ-plane which does not enclose a bounded domain. A number m, with $m \le p$, of these rays start from the point ζ_i, where ζ_i is the image under h of the initial point of ℓ_i. If $m = 1$, it easily follows that D_i and D_{i+1} are adjacent along some ray L_i. However, if $m \ge 2$, it can be shown that there are exactly two rays L which start at ζ_i, and continue along a nondegenerate segment, also denoted by L_i, untill they split at some critical point of Q in such a manner that one bounds D_i and the other bounds D_{i+1}. In either case, the domains D_i and D_{i+1} are adjacent along a ray segment or ray L_i.

Next, we wish to show that the functional element $(h, \mathbb{C} - \bigcup_{i=1}^{n} \ell_i)$ can be continued analytically along each ℓ_i. Note that P has no critical points on any ℓ_i. Suppose to the contrary that there exists some ℓ_i, say ℓ_1, such that Q^{-1} can not be continued analytically along $P(\ell_1)$. This implies that L_1 passes through a critical point of Q. Since all the critical points of Q are in $\psi(\mathbb{B})$, L_1 must meet Γ. For each i, $1 \leq i \leq n$, let A_i be the initial points of L_i, and let B_i be the first point of intersection, if exists, of L_i with Γ. Then either L_1 separates the curve (A_1,B_1) from infinity or not. We assume the former possibility, since in either case the rest of the proof is essentially the same.

Since $bd(D_1)$ is a Jordan curve, A_2 must be interior to (A_1,B_1). Consequently, L_2 must intersect at B_2, where B_2 is interior to (A_1,B_1) and (A_2,B_2) is a subarc of (A_1,B_1). Repeating this argument, we conclude that A_n is interior to (A_1,B_1) and that L_n meets Γ at B_n, where B_n is interior to (A_1,B_1). Evidently, B_n is interior to (A_n,A_1). But this will contradict the fact that the domain D_n is bounded by a Jordan curve that contains (A_n,A_1) and the ray segments or rays L_n and L_1. Therefore, our assumption is false and the functional element $(h, \mathbb{C} - \bigcup_{i=1}^{n} \ell_i)$ continues analytically along each ℓ_i.

It can be easily verified that the resulting function from the continuation is an entire function with the desired properties. This proves the theorem for the case under concern.

Now suppose that $f \in K(p)$ and $f = P \circ \phi = Q \circ \psi$, where P, Q are distinct polynomials of degree p and ϕ, $\psi \in S$. Let $g(z) = f(rz)$ for some r close enough to 1. It is easy to show that g is close-to-convex of order p on $Cl(\mathbb{B})$. Write

$$g(z) = P \circ \phi(rz) = Q \circ \psi(rz)$$

as

$$g(z) = P_1 \circ \left(\frac{\phi(rz)}{r}\right) = Q_1 \circ \left(\frac{\psi(rz)}{r}\right),$$

where P_1, Q_1 are distinct polynomials of degree p, and $\frac{\phi(rz)}{r}$, $\frac{\psi(rz)}{r} \in S$. But this is impossible by the first part of the proof. This completes the proof.

3. Weakly Close - To - Convex Functions

The theorem does not necessarily hold for weakly close-to-convex functions. Let $f = P \circ \phi \in K_w(p)$, where P is a polynomial of degree q, with $1 \le q \le p$, and $\phi \in S$. f may have other compositions with trivial distinctions as in the following cases:

1) If $q < p$, then there exists a polynomial Q of degree $q+1$ and $\psi \in S$ such that $f = Q \circ \psi$. Q can be obtained as follows. Let ℓ be a euclidean ray that avoids $\{f(z) : f'(z) = 0\}$. Cut a copy of the w-plane along ℓ, and the image surface of P along exactly one

lift of ℓ. Then join the surface with the w-plane crosswise along the corresponding edges to produce a parabolic covering over the w-plane with $(q+1)$ sheets. The resulting surface is the image surface of Q. Q can be chosen so that it yields $\psi \in S$, with $f = Q \circ \psi$.

2) The image surface of f may be embedded in $(q-1)$ sheets of the image surface of P. In this case, we undo the procedure in case (1) by cutting a complete sheet from the image surface of P and then pasting the remaining surface, in the usual manner. Again, the resulting surface is a parabolic covering over the w-plane with $(q-1)$ sheets. Consequently, $f = Q \circ \psi$, where Q is a polynomial of degree $(q-1)$ and $\psi \in S$.

3) The image surface of f may be embedded on the image surface of P so that a translation of the latter is conceivable. This will result in a polynomial Q distinct from P and $\psi \in S$ such that $f = Q \circ \psi$.

4) The image surface of f may be embedded on the image surface of P in more than one way. This will result in a function $\psi \in S$ distinct from ϕ such that $f = P \circ \psi$.

For obvious reasons, the cases of nonuniqueness discussed above are not satisfactory. In what follows, we use the Uniformization Theorem to construct a function $f \in K_w(9)$ such that $f = P \circ \phi = Q \circ \psi$, where $\phi, \psi \in S$, and P, Q are distinct polynomials of the same minimal degree, whose image surfaces have four branch points of order 1 that lie over four common finite complex numbers.

Example. Consider the polygonal domain D featured in Figure (1) and bounded by the polygon AB · · · U, where ABTU and GHIJ are squares centered at a and b respectively. Cut the domain D and a copy of the square ABTU along the segment aA, and join the resulting domain and square crosswise along the corresponding edges of the cuts. Then repeat the same construction over the square GHIJ. Now, adjoin the obtained surface with a symmetric copy of it about the real axis, by identifying every pair of points over a common point in VN. It can be easily seen that the resulting surface, denoted by Y, is a hyperbolic covering over $\mathbb{C}$, with four branch points of order 1 over a numbers a, b, $c = \overline{b}$, and $d = \overline{a}$ (see Figure (2)). Applying the Uniformization Theorem there exists a function f regular in $\mathbb{B}$, with $f(0) = 0$ and $f'(0) > 0$, such that its image surface is Y.

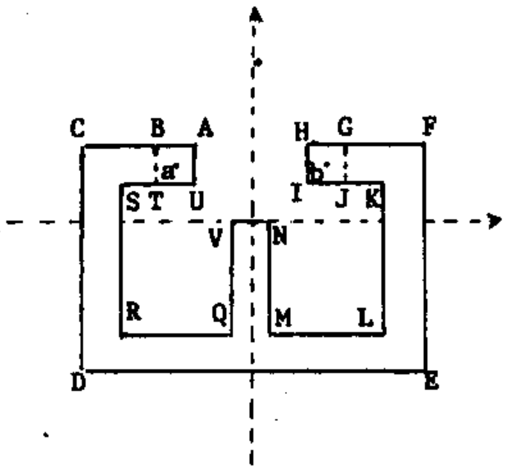

Figure (1)

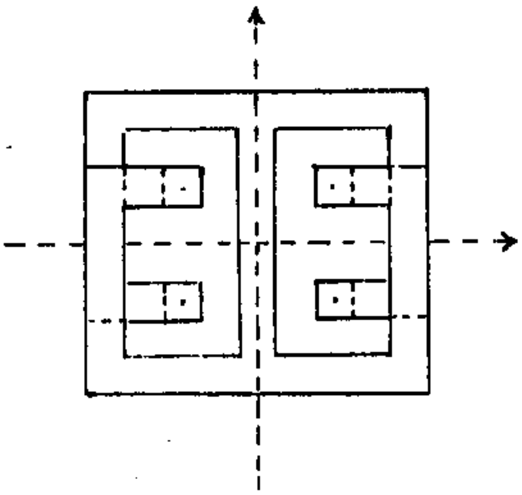

Figure (2)

Next we show that

1) $f \in K_w(9)$. To do this, we construct an image surface X of a polynomial of degree 9 such that X contains Y, and $X - Y$ is a disjoint union of rays in X. Making use of Theorem 4.1 in [4] the result follows.

Here we construct X. Denote by e and f the complex numbers located at Q and M respectively (see Figure (1)); and let $h = \overline{e}$ and $g = \overline{f}$. We put nine copies C_i, $1 \le i \le 9$, of $\mathbb{C}$ together, and let ℓ_a, ℓ_b, ℓ_c, ℓ_d, ℓ_e, ℓ_f, ℓ_g, and ℓ_h be disjoint simple curves in $\mathbb{C}$ from a, b, c, d, e, f, g, and h, respectively, to infinity. Make in C_1 four cuts along ℓ_e, ℓ_f, ℓ_g, and ℓ_h, and make in each of the copies C_2, C_3, C_4, and C_5 two cuts along ℓ_a and ℓ_e, ℓ_b and ℓ_h, ℓ_c and ℓ_g, and ℓ_d and ℓ_f, respectively. Also, make in each of the copies C_6, C_7, C_8, and C_9 one cut along ℓ_a, ℓ_b, ℓ_c, and ℓ_d, respectively. Now join the obtained copies as follows: Join the copy C_1 to the copies C_2, C_3, C_4, and C_5 crosswise along the corresponding edges of the cuts ℓ_e, ℓ_h, ℓ_g, and ℓ_f, respectively.

One way to visualize X is by the labeled simple graph in Figure (3). The vertices of the graph are the points a, b, c, d, e, f, g, and h; and two vertices are adjacent if and only if the branch points of X over these vertices can be joined directly on one sheet of X.

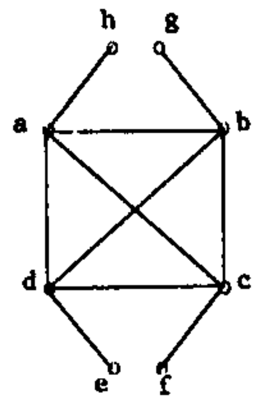

Figure (3)

2) The image surfaces of P and Q. We put five copies C_i, $1 \le i \le 5$, of $\mathbb{C}$ together, and let ℓ_a, ℓ_b, ℓ_c, and ℓ_d be disjoint simple curves in $\mathbb{C}$ from a, b, c, and d, respectively, to infinity. Make in each of the copies C_1, C_2, and C_3 two cuts along ℓ_a and ℓ_b, ℓ_b and ℓ_c, and ℓ_a and ℓ_d, respectively. Also, make in each of the copies C_4 and C_5 one cut along ℓ_c and ℓ_d, respectively. Now, join the copies as follows: Join the copy C_1 to the copies C_2 and C_3 crosswise along the corresponding edges of the cuts ℓ_b and ℓ_c, respectively. Also, join the copies C_2 and C_3 to C_4 and C_5 crosswise along the corresponding edges of the cuts ℓ_c and ℓ_d, respectively. Denote the resulting surface by X_1. Note that X_1 is not symmetric about the real axis. Let X_2 be the symmetric surface of X_1 about the real axis (see Figures (4.a) and (4.b) for the corresponding graphs of X_1 and X_2, where the adjacency of verticies is as defined previously). Both surfaces X_1, X_2 are 5-sheeted elliptic coverings over $\mathbb{C}$, with four common base points a, b, c, and d for their four branch points of order 1. Applying the Uni-

formization Theorem, there exists a unique pair of distinct polynomials P and Q whose image surfaces are X_1 and X_2, respectively, and such that $P(0) = Q(0) = 0$ and $P'(0)$, $Q'(0) > 0$.

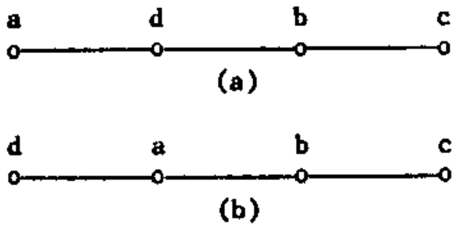

Figure (4)

It can be easily verified that X_1 extends Y, and because of symmetry, X_2 extends Y also. This directly implies that there exists a unique pair of distinct functions $\phi, \psi \in S$ such that $f = P \circ \phi = Q \circ \psi$. Since, by the construction of f, f' has four zeros of order 1, the degree 5 of P and Q is the minimum of all the degrees of polynomials which appear in such a composition.

4. Conclusion

It would be of interest to characterize the class of functions $\phi \in S$ such that $f = P \circ \phi$, where $f \in K(P)$ and P is a polynomial of degree p.

The author would like to thank Professor David Styer for his helpful comments during the preparation of this work.

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