King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 050

May 1983

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Abstract

We prove that two classes of univalent functions are equal. This settles a conjecture of M.S. Robertson in the affirmative.

AMS 1980 subject classification. Primary 30C45

Key words and phrases: Univalent functions, Starlike functions.

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1. Introduction

Recently, M.S. Robertson [4] introduced two classes of univalent functions, G and G^*, and conjectured that they are equal. In this short note, we prove this conjecture in the affirmative.

First, let us define the classes G and G^*.

Definition 1. Let G be the class of all functions f, regular and nonvanishing in $\mathbb{B} = \{z: |z| < 1\}$, with $f(0) = 1$, such that

$$\mathrm{Re}\{2z \frac{f'(z)}{f(z)} + \frac{1+z}{1-z}\} > 0, \quad \text{for} \quad z \in \mathbb{B}.$$

Note that $1 \in G$.

Let D be a domain, and let a belong to the closure of D. We say that D is starlike with respect to a if for each $z \in D$, every point $tz + (1-t)a$, with $0 < t \leq 1$, belongs to D.

Definition 2. Let G^* be the class of functions f, regular and univalent in $\mathbb{B}$, with $f(0) = 1$ and $\lim_{r\to 1^-} f(r) = 0$, such that $f(\mathbb{B})$ is starlike with respect to zero, and $\mathrm{Re}(e^{ia}f) > 0$ for some real number a. Also, let $1 \in G^*$.

For the sake of clarity, we remind the reader of some familiar definitions which are needed.

Definition 3. Let S^* be the class of all functions f, regular in $\mathbb{B}$, with $f(0) = 0$, such that

$$\mathrm{Re}\{z \frac{f'(z)}{f(z)}\} > 0 , \quad \text{for} \quad z \in \mathbb{B}.$$

Note that there is no restriction on $f'(0)$ in this definition.

It is known that each $f \in S^*$ is univalent and maps $\mathbb{B}$ onto a domain starlike with respect to zero.

Definition 4. Let S_g be the class of functions f, regular and univalent in $\mathbb{B}$, which maps $\mathbb{B}$ onto a domain which contains the origin and is starlike with respect to the origin.

Definition 5. Let S_w be the class of functions f of the form

$$f(z) = h(z) \frac{(z - \zeta)(1 - \overline{\zeta}z)}{z} , \quad |\zeta| < 1 ,$$

where $h \in S^*$.

Our main tool for the proof will be the set relation $S_g = S_w$, which is due to J. Hummel, who developed the classes S_g and S_w in a more general setting [1].

3. Proof of Conjecture

Theorem. $G^* = G$.

Before we give the proof, we remark that the set-inclusion $G \subset G^*$ was established by M. Robertson [4]. Another proof of this fact will be given as a part of the proof of the theorem.

Proof of Theorem. (a) $G^* \subset G$. Suppose that $f \in G^*$ is not identical to 1. It follows directly from Definition 2 that f^2 is univalent, $\lim_{r \to 1^-} f^2(r) = 0$, and f^2 maps $\mathbb{B}$ onto a domain starlike with respect to zero. Let D_n be the domain obtained from the union of the range of f^2 and the open disc centered at the origin and of radius $\frac{1}{n}$. Evidently, each D_n is simply connected. Let f_n be a conformal map from $\mathbb{B}$ onto D_n that satisfies $f_n(0) = 1$ and $\arg f_n'(0) = \arg(f^2)'(0)$. By the Carathéodory Kernel Theorem [2, p.18], $f_n \to f^2$ uniformly on compact subsets of $\mathbb{B}$. From Definition 4, each $f_n \in S_g = S_w$. Hence we can write for each n

$$f_n(z) = h_n(z)\,\frac{(z - z_n)(1 - \bar{z}_n z)}{z}\,, \quad |z_n| < 1\,,$$

where $h_n \in S^*$. It can be verified that

$$f_n'(0) = \frac{1}{2}\,\frac{h_n''(0)}{h_n'(0)} + (1 + |z_n|^2)h_n'(0)\ .$$

Since $f_n'(0) \to (f^2)'(0) \neq 0$ and $h \in S^*$ gives $\left|\frac{h_n''(0)}{h_n'(0)}\right| \le 4$, for every n, $h_n'(0)$ is uniformly bounded. Hence, there exists a sequence of positive integers (n_k) so that (h_{n_k}) converges uniformly on compact subsets to either $h \in S^*$ or zero. The latter case is impossible, otherwize $zf_n \to 0$ uniformly on compact subsets of $\mathbb{B}$ and f will be identical to zero. Suppose that $z_{n_k} \to \zeta$, with $|\zeta| \le 1$ (otherwise we choose a subsequence of (z_{n_k}) that does so). Then we can write

$$f^2(z) = h(z) \frac{(z - \zeta)(1 - \bar{\zeta} z)}{z}, \quad |\zeta| \le 1.$$

Since f does not admit zero in $\mathbb{B}$, $|\zeta| = 1$. Furthermore, since $\lim_{r \to 1^-} f(r) = 0$ and h is bounded away from zero for values of z close to $\partial\mathbb{B}$, $\zeta = 1$. Therefore,

$$f^2(z) = -h(z) \frac{(1 - z)^2}{z},$$

which yields

$$\mathrm{Re}\{2z \frac{f'(z)}{f(z)} + \frac{1 + z}{1 - z}\} = \mathrm{Re}\{z \frac{h'(z)}{h(z)}\} > 0,$$

and $f \in G$.

(b). $G \subset G^*$. Let $f \in G$, with f not identical to 1, and let

$$h(z) = f^2(z) \frac{z}{(1 - z)^2}.$$

Then by simple calculation we have

$$\mathrm{Re}\{z \frac{h'(z)}{h(z)}\} = \mathrm{Re}\{2z \frac{f'(z)}{f(z)} + \frac{1 + z}{1 - z}\} > 0.$$

So we have

$$f^2(z) = h(z) \frac{(1 - z)^2}{z}, \tag{1}$$

where $h \in S^*$, with $h'(0) = 1$.

For every positive integer n, let $r_n = 1 - \frac{1}{n}$, and let

$$g_n(z) = -\frac{h(z)}{r_n}\frac{(z - r_n)(1 - r_n z)}{z} \quad .$$

From Definition 5, each $g_n \in S_w = S_g$. Note that each $g_n(0) = 1$, and $g_n'(0) = \frac{h''(0)}{2} - \frac{r_n^2 + 1}{r_n} \to \frac{h''(0)}{2} - 2 = (f^2)'(0) \neq 0,\infty$, since $h(z) \neq \frac{z}{(1-z)^2}$ otherwise f is identically 1. Also, note that $g_n \to f^2$ uniformly on compact subsets of $\mathbb{B}$. By Hurwitz's Theorem, this implies the univalence of f^2 in $\mathbb{B}$. Let $\Delta = f^2(\mathbb{B})$, and let $\Delta_n = g_n(\mathbb{B})$. Then by the Carathéodory Kernel Theorem [2, p.18], $\Delta_n \to \Delta$ as $n \to \infty$. Now we show that Δ is a domain starlike with respect to zero. Let $w \in \Delta$. From the definition of the kernel, there exists a domain U and a positive integer N such that $1,w \in \Delta$ and U is contained in Δ_n for all $n > N$. Let H be the domain consisting of all open-closed segments starting from zero and ending in U. Since each $g_n \in S_g$, each Δ_n is starlike with respect to zero. This implies that H is contained in Δ_n for all $n > N$. Hence H is contained in Δ, and consequently Δ is starlike with respect to zero. Since $0 \notin \Delta$, it is not hard to show that there exists a radial slit from 0 to ∞ which does not meet Δ. Hence, the univalence of f^2, and the starlikeness of Δ about zero lead to the univalence of f, and to the starlikeness of $f(\mathbb{B})$ about zero; moreover, there exists a real number a such that $\mathrm{Re}(e^{ia}f) > 0$ in $\mathbb{B}$. Since zero is an accessible boundary point of $f(\mathbb{B})$, there exists ζ, with $|\zeta| = 1$, so that $\lim_{r\to 1^-} f(r\zeta) = 0$ (see [3; p.277]). Since h is bounded away from zero for values of z near ∂B, (1) implies that $\zeta = 1$. Therefore, $f \in G^*$ and the proof is complete.

The author can give an alternative shorter proof to the theorem based on D. Styer [5]. However, this proof is quite involved, and was avoided for the sake of clarity.

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