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Modulational Instability of a Magneto Hydrodynamic Jet

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Abstract

The effect of a magnetic field on the nonlinear capillary instability of a fluid jet is examined using the method of multiple scales. It is a well-known result that a sufficiently strong magnetic field can, in the limit of zero viscosity and resistivity, completely suppress the linear capillary instability. It turns out that while the nonlinear (modulational) instability cannot be completely suppressed, the presence of a magnetic field does greatly increase the range of stable wave numbers.

1. INTRODUCTION

The capillary instability of a circular jet has been the subject matter of great attention since the pioneering work in the nineteenth century by Savart, Plateau and Rayleigh (see Rayleigh [1]). Neglecting the effect of the surrounding air, Rayleigh showed that only axisymmetrical surface disturbances with wave lengths larger than the circumference of the cylinder would grow. The surface waves grow as $e^{\omega t}$ where

$$\omega^2 = \frac{T}{\rho R^3} \frac{x I_1(x)}{I_0(x)} (1 - x^2) ,$$

$x = kR$, T is the surface tension, ρ is the density of the liquid, R is the undisturbed radius of the jet, k is the wave number and I_0, I_1 are modified Bessel functions. The dispersion curve shows a maximum growth rate at $x = 0.697$ and a cut-off wavelength at $x = 1$.

Weber[2] examined the stability of a viscous capillary jet. His linearized theory indicated that the viscosity would not change the stability criterion as predicted by the inviscid theory. However, the viscous effect would cause the wavelength of the most unstable state to become larger than that predicted by the inviscid theory.

Chandrasekhar[3] examined the effect of a uniform magnetic field along the axis of the jet on its capillary instability. He found in the limit of zero viscosity and resistivity, the magnetic field has the effect of increasing the wavelength at which capillary instability occurs. In fact the capillary instability can be suppressed completely by a magnetic field $H > H_c$, where

$$H_c = \sqrt{2\pi T/R}$$

The effect of nonlinearities on the capillary instability of a hydrodynamical jet was examined by Yuen[4)], Wang[5)], Nayfeh[6)] and Nayfeh and Hassan[7)] and a complete analysis was finally given by Kakutani, Inoue and Kan[8)]. These latter authors derived the nonlinear Schrödinger equation satisfied by the amplitude of a slowly modulated travelling harmonic wave and showed that there is modulational instability for x greater than approximately 1.3. In view of Chandrasekhar's result within the linearized theory, it is therefore of considerable interest to examine the nonlinear amplitude modulation of a travelling wave in the presence of a magnetic field. In order to describe the non-linear interactions of small but finite amplitude waves, we use the derivative expansion method with multiple scales (Kawahara[9)]) and assume that all the physical quantities have uniformly valid asymptotic expansions in powers of a small ordering parameter ε. By requiring that these formal expansions satisfy the exact governing equations and the boundary conditions for all values of ε, sets of linearized boundary value problems are obtained.

2. The basic equations

We consider a cylindrical jet of magnetic fluid, assumed to be incompressible, inviscid and perfectly conducting, on which is imposed a uniform magnetic field in the axial direction. We choose units such that the outer radius of the jet is $r = 1$, where r and z represent cylindrical polar coordinates with the z-axis along the axis of the jet. We consider an axially symmetrical motion of the jet in which the outer surface is distorted to $r = 1 + \eta(z,t)$. The equations holding in $r < 1 + \eta$ are then as follows:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}\Pi + (\underline{h}\cdot\underline{\nabla})\underline{h} \tag{1}$$

$$\frac{\partial \underline{h}}{\partial t} = (\underline{h}\cdot\underline{\nabla})\underline{u} - (\underline{u}\cdot\underline{\nabla})\underline{h} \tag{2}$$

$$\underline{\nabla}\cdot\underline{u} = 0 \ , \ \underline{\nabla}\cdot\underline{h} = 0 \ . \tag{3}$$

Here $\underline{u}(r,z,t)$ is the velocity field and $\sqrt{4\pi\rho}\,\underline{h}(r,z,t)$ is the magnetic field strength, ρ being the (constant) fluid density, and

$$\Pi = \frac{p}{\rho} + \frac{1}{2}\underline{h}^2$$

where p is the pressure.

Outside the jet, the magnetic field $\underline{h}^{(0)}$ is expressible in terms of a potential, and we write

$$\underline{h}^{(0)} = A_0\hat{\underline{e}}_z + \underline{\nabla}\phi \ , \ \nabla^2\phi = 0 \qquad (r > 1 + \eta). \tag{4}$$

Here $A_0\sqrt{4\pi\rho}$ is the magnitude of the imposed magnetic field, A_0 being the Alven velocity.

Finally, the boundary conditions on $r = 1 + \eta$ are:

$$u_r = \frac{\partial\eta}{\partial t} + u_z \frac{\partial\eta}{\partial z} \tag{5}$$

$$\Pi = \frac{T}{\rho}\left\{(1+\eta)^{-1}\left[1 + \left(\frac{\partial\eta}{\partial z}\right)^2\right]^{-1/2} - \frac{\partial^2\eta}{\partial z^2}\left[1 + \left(\frac{\partial\eta}{\partial z}\right)^2\right]^{-3/2}\right\} + \frac{1}{2}\underline{h}^{(0)2} \tag{6}$$

$$\underline{n}\cdot\underline{h} = \underline{n}\cdot\underline{h}^{(0)} \tag{7}$$

where T denotes the surface tension and $\underline{n}$ the unit normal to the surface. We shall choose units such that $T/\rho = 1$.

We wish to investigate motions which are small perturbations of the steady state. As usual we shall use the method of multiple scales (Kakutani, Inoue and Kan[8]; Kawahara[10]). If ϵ is a small parameter measuring the size of perturbation, we introduce slow distance and time scales

$$z_n = \epsilon^n z\ ,\ t_n = \epsilon^n t \qquad (n = 0,1,2,\ldots,N)$$

and represent the perturbed variables as power series in ϵ:

$$\eta(z,t) = \sum_{n=1}^{N+1} \epsilon^n \eta_n(z_0,z_1,\ldots,z_N,t_0,t_1,\ldots,t_N)$$

$$\underline{u}(r,z,t) = \sum_{n=1}^{N+1} \epsilon^n \underline{u}_n(r,z_0,z_1,\ldots,z_N,t_0,t_1,\ldots,t_N)$$

$$\Pi(r,z,t) = 1 + \frac{1}{2}A_0^2 + \sum_{n=1}^{N+1} \epsilon^n \Pi_n(r,z_0,z_1,\ldots,z_N,t_0,t_1,\ldots,t_N)$$

$$\underline{h}(r,z,t) = A_0\hat{\underline{e}}_z + \sum_{n=1}^{N+1} \epsilon^n \underline{h}_n(r,z_0,z_1,\ldots,z_N,t_0,t_1,\ldots,t_N)$$

$$\phi(r,z,t) = \sum_{n=1}^{N+1} \epsilon^n \phi_n(r,z_0,z_1,\ldots,z_N,t_0,t_1,\ldots,t_N).$$

Substituting these expansions into equations (1) - (3) and comparing the coefficients of $\epsilon^n (n = 1,2,3)$ we obtain the following systems of equations.

$$\epsilon^1: \quad \left.\begin{aligned} &\frac{\partial \underline{u}_1}{\partial t_0} + \underline{\nabla}_0 \Pi_1 - A_0 \frac{\partial \underline{h}_1}{\partial z_0} = \underline{0} \\ &\frac{\partial \underline{h}_1}{\partial t_0} - A_0 \frac{\partial \underline{u}_1}{\partial z_0} = \underline{0} \\ &\frac{1}{r}\frac{\partial}{\partial r}(r u_{1r}) + \frac{\partial u_{1z}}{\partial z_0} = 0 \\ &\frac{1}{r}\frac{\partial}{\partial r}(r h_{1r}) + \frac{\partial h_{1z}}{\partial z_0} = 0 \end{aligned}\right\} \qquad (8)$$

$$\epsilon^2: \quad \left.\begin{aligned} &\frac{\partial \underline{u}_2}{\partial t_0} + \underline{\nabla}_0 \Pi_2 - A_0 \frac{\partial \underline{h}_2}{\partial z_0} = - \frac{\partial \underline{u}_1}{\partial t_1} - \frac{\partial \Pi_1}{\partial z_1} \hat{\underline{e}}_z + A_0 \frac{\partial \underline{h}_1}{\partial z_1} - (\underline{u}_1 \cdot \underline{\nabla}_0)\underline{u}_1 + (\underline{h}_1 \cdot \underline{\nabla}_0)\underline{h}_1 \\ &\frac{\partial \underline{h}_2}{\partial t_0} - A_0 \frac{\partial \underline{u}_2}{\partial z_0} = - \frac{\partial \underline{h}_1}{\partial t_1} + A_0 \frac{\partial \underline{u}_1}{\partial z_1} + (\underline{h}_1 \cdot \underline{\nabla}_0)\underline{u}_1 - (\underline{u}_1 \cdot \underline{\nabla}_0)\underline{h}_1 \\ &\frac{1}{r}\frac{\partial}{\partial r}(r u_{2r}) + \frac{\partial u_{2z}}{\partial z_0} = - \frac{\partial u_{1z}}{\partial z_1} \\ &\frac{1}{r}\frac{\partial}{\partial r}(r h_{2r}) + \frac{\partial h_{2z}}{\partial z_0} = - \frac{\partial h_{1z}}{\partial z_1} \end{aligned}\right\} \qquad (9)$$

$$\epsilon^3: \quad \frac{\partial \underline{u}_3}{\partial t_0} + \underline{\nabla}_0 \Pi_3 - A_0 \frac{\partial \underline{h}_3}{\partial z_0} = -\left(\frac{\partial \underline{u}_1}{\partial t_2} + \frac{\partial \underline{u}_2}{\partial t_1}\right) - \left(\frac{\partial \Pi_1}{\partial z_2} + \frac{\partial \Pi_2}{\partial z_1}\right) \hat{\underline{e}}_z + A_0 \left(\frac{\partial \underline{h}_1}{\partial z_2} + \frac{\partial \underline{h}_2}{\partial z_1}\right)$$

$$- u_{1z} \frac{\partial \underline{u}_1}{\partial z_1} + h_{1z} \frac{\partial \underline{h}_1}{\partial z_1} - (\underline{u}_1 \cdot \underline{\nabla}_0)\underline{u}_2 - (\underline{u}_2 \cdot \underline{\nabla}_0)\underline{u}_1$$

$$+ (\underline{h}_1 \cdot \underline{\nabla}_0)\underline{h}_2 + (\underline{h}_2 \cdot \underline{\nabla}_0)\underline{h}_1$$

$$\frac{\partial \underline{h}_3}{\partial t_0} - A_0 \frac{\partial \underline{u}_3}{\partial z_0} = -\left(\frac{\partial \underline{h}_1}{\partial t_2} + \frac{\partial \underline{h}_2}{\partial t_1}\right) + A_0 \left(\frac{\partial \underline{u}_1}{\partial z_2} + \frac{\partial \underline{u}_2}{\partial z_1}\right) + h_{1z} \frac{\partial \underline{u}_1}{\partial z_1} - u_{1z} \frac{\partial \underline{h}_1}{\partial z_1} \qquad (10)$$

$$+ (\underline{h}_1 \cdot \underline{\nabla}_0)\underline{u}_2 + (\underline{h}_2 \cdot \underline{\nabla}_0)\underline{u}_1 - (\underline{u}_1 \cdot \underline{\nabla}_0)\underline{h}_2 - (\underline{u}_2 \cdot \underline{\nabla}_0)\underline{h}_1$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_{3r}) + \frac{\partial u_{3z}}{\partial z_0} = -\frac{\partial u_{1z}}{\partial z_2} - \frac{\partial u_{2z}}{\partial z_1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r h_{3r}) + \frac{\partial h_{3z}}{\partial z_0} = -\frac{\partial h_{1z}}{\partial z_2} - \frac{\partial h_{2z}}{\partial z_1} .$$

In these equations, $\underline{\nabla}_0$ denotes the operator $\hat{\underline{e}}_r \frac{\partial}{\partial r} + \hat{\underline{e}}_z \frac{\partial}{\partial z_0}$. Equation (4) similarly leads to the following equations outside the jet:

$$\epsilon^1: \quad \nabla_0^2 \phi_1 = 0 \qquad (11)$$

$$\epsilon^2: \quad \nabla_0^2 \phi_2 = -2\frac{\partial^2 \phi_1}{\partial z_0 \partial z_1} \qquad (12)$$

$$\epsilon^3: \quad \nabla_0^2 \phi_3 = -2\frac{\partial^2 \phi_2}{\partial z_0 \partial z_1} - 2\frac{\partial^2 \phi_1}{\partial z_0 \partial z_2} - \frac{\partial^2 \phi_1}{\partial z_1^2} \qquad (13)$$

where $\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_0^2}$.

The boundary conditions (5) - (7) can be shifted to $r = 1$ and then become as follows.

$$\epsilon^1: \quad \left.\begin{aligned} & u_{1r} - \frac{\partial \eta_1}{\partial t_0} = 0 \\ & \Pi_1 + \eta_1 + \frac{\partial^2 \eta_1}{\partial z_0^2} - A_0 h_{1z}^{(0)} = 0 \\ & h_{1r} - h_{1r}^{(0)} = 0 \end{aligned}\right\} \qquad (14)$$

$$\epsilon^2: \quad \left.\begin{aligned} & u_{2r} - \frac{\partial \eta_2}{\partial t_0} = \frac{\partial \eta_1}{\partial t_1} - \eta_1 \frac{\partial u_{1r}}{\partial r} + \frac{\partial \eta_1}{\partial z_0} u_{1z} \\ & \Pi_2 + \eta_2 + \frac{\partial^2 \eta_2}{\partial z_0^2} - A_0 h_{2z}^{(0)} = \eta_1^2 - \frac{1}{2}\left(\frac{\partial \eta_1}{\partial z_0}\right)^2 - 2\frac{\partial^2 \eta_1}{\partial z_0 \partial z_1} - \eta_1 \frac{\partial \Pi_1}{\partial r} + A_0 \eta_1 \frac{\partial h_{1z}^{(0)}}{\partial r} + \frac{1}{2} \underline{h}_1^{(0)2} \\ & h_{2r} - h_{2r}^{(0)} = \eta_1 \frac{\partial}{\partial r}(h_{1r}^{(0)} - h_{1r}) - \frac{\partial \eta_1}{\partial z_0}(h_{1z}^{(0)} - h_{1z}) \end{aligned}\right\} \qquad (15)$$

$$\epsilon^3: \quad \left.\begin{aligned} & u_{3r} - \frac{\partial \eta_3}{\partial t_0} = -\eta_1 \frac{\partial u_{2r}}{\partial r} - \eta_2 \frac{\partial u_{1r}}{\partial r} - \frac{1}{2}\eta_1^2 \frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial \eta_1}{\partial t_2} + \frac{\partial \eta_2}{\partial t_1} + \frac{\partial \eta_1}{\partial z_0} u_{2z} \\ & \qquad + \left(\frac{\partial \eta_2}{\partial z_0} + \frac{\partial \eta_1}{\partial z_1}\right) u_{1z} \\ & \Pi_3 + \eta_3 + \frac{\partial^2 \eta_3}{\partial z_0^2} - A_0 h_{3z}^{(0)} = -\eta_1 \frac{\partial \Pi_2}{\partial r} - \eta_2 \frac{\partial \Pi_1}{\partial r} - \frac{1}{2}\eta_1^2 \frac{\partial^2 \Pi_1}{\partial r^2} + 2\eta_1 \eta_2 - \eta_1^3 + \frac{1}{2}\eta_1 \left(\frac{\partial \eta_1}{\partial z_0}\right)^2 \\ & \qquad - \frac{\partial \eta_1}{\partial z_0}\left(\frac{\partial \eta_2}{\partial z_0} + \frac{\partial \eta_1}{\partial z_1}\right) - 2\frac{\partial^2 \eta_2}{\partial z_0 \partial z_1} - 2\frac{\partial^2 \eta_1}{\partial z_0 \partial z_2} - \frac{\partial^2 \eta_1}{\partial z_1^2} + \frac{3}{2}\frac{\partial^2 \eta_1}{\partial z_0^2}\left(\frac{\partial \eta_1}{\partial z_0}\right)^2 \\ & \qquad + A_0\left(\eta_2 \frac{\partial h_{1z}^{(0)}}{\partial r} + \eta_1 \frac{\partial h_{2z}^{(0)}}{\partial r} + \frac{1}{2}\eta_1^2 \frac{\partial^2 h_{1z}^{(0)}}{\partial r^2}\right) + \underline{h}_1^{(0)} \underline{h}_2^{(0)} \end{aligned}\right\} \qquad (16)$$

$$+ \frac{1}{2} \eta_1 \underline{h}_1^{(0)} \cdot \frac{\partial \underline{h}_1^{(0)}}{\partial r}$$

$$h_{3r} - h_{3r}^{(0)} = \eta_1 \frac{\partial}{\partial r} (h_{2r}^{(0)} - h_{2r}) + \eta_2 \frac{\partial}{\partial r} (h_{1r}^{(0)} - h_{1r}) + \frac{1}{2} \eta_1^2 \frac{\partial^2}{\partial r^2} (h_{1r}^{(0)} - h_{1r})$$

$$- \frac{\partial \eta_1}{\partial z_0} (h_{2z}^{(0)} - h_{2z}) - \eta_1 \frac{\partial \eta_1}{\partial z_0} \frac{\partial}{\partial r} (h_{1z}^{(0)} - h_{1z}) - \left(\frac{\partial \eta_2}{\partial z_0} + \frac{\partial \eta_1}{\partial z_1} \right) (h_{1z}^{(0)} - h_{1z}) \Bigg]$$

3. First order solution

We seek a solution of the first order equations (8), (11) and (14) in the form of a progressive harmonic wave. Assuming all field quantities proportional to $e^{i\psi}$ where $\psi = kz_0 - \omega t_0$ we easily find the following (real) solution

$$\left.\begin{aligned}
u_{1r} &= -i\alpha I_1(kr)(Ae^{i\psi} - \bar{A}e^{-i\psi}) \\
u_{1z} &= \alpha I_0(kr)(Ae^{i\psi} + \bar{A}e^{-i\psi}) \\
h_{1r} &= \frac{ikA_0\alpha}{\omega} I_1(kr)(Ae^{i\psi} - \bar{A}e^{-i\psi}) \\
h_{1z} &= -\frac{kA_0\alpha}{\omega} I_0(kr)(Ae^{i\psi} + \bar{A}e^{-i\psi}) \\
\phi_1 &= -iK_0(kr)(Be^{i\psi} - \bar{B}e^{-i\psi}) \\
\eta_1 &= \frac{\alpha}{\omega} I_1(k)(Ae^{i\psi} + \bar{A}e^{-i\psi})
\end{aligned}\right\} \qquad (17)$$

where $\qquad \alpha = \omega k(\omega^2 - A_0^2 k^2)^{-1} \qquad (18)$

and $\qquad B = \frac{\alpha A_0}{\omega} \frac{I_1(k)}{K_1(k)} A . \qquad (19)$

K_0 and K_1 being modified Bessel functions of the second kind. Furthermore the dispersion relation turns out to be

$$\omega^2 = k(k^2-1)\frac{I_1(k)}{I_0(k)} + \frac{A_0^2 k}{I_0(k)K_1(k)} . \qquad (20)$$

When the magnetic field is absent $(A_0 = 0)$, ω is real only for $k > 1$. The presence of the final term in (20) extends the region in which stable linear waves occur below $k = 1$ and in fact for $A_0^2 > \frac{1}{2}$, ω^2 is positive for all k (Chandrasekhar[3]).

4. Second order solution

Substituting the first order solutions (17) into equations (9) and (12) we obtain the following equations for the second order solution:

$$\left.\begin{aligned}
&\frac{\partial u_{2r}}{\partial t_0} + \frac{\partial \Pi'_2}{\partial r} - A_0 \frac{\partial h_{2r}}{\partial z_0} = i\alpha\, I_1(kr) \left(\frac{\partial A}{\partial t_1} + \frac{kA_0^2}{\omega}\frac{\partial A}{\partial z_1}\right) e^{i\psi} + CC \\
&\frac{\partial u_{2z}}{\partial t_0} + \frac{\partial \Pi'_2}{\partial z_0} - A_0 \frac{\partial h_{2z}}{\partial z_0} = -\alpha\, I_0(kr) \left(\frac{\partial A}{\partial t_1} + \frac{\omega}{k}\frac{\partial A}{\partial z_1}\right) e^{i\psi} + CC \\
&\frac{\partial h_{2r}}{\partial t_0} - A_0 \frac{\partial u_{2r}}{\partial z_0} = -i\alpha A_0 I_1(kr) \left(\frac{\partial A}{\partial z_1} + \frac{k}{\omega}\frac{\partial A}{\partial t_1}\right) e^{i\psi} + CC \\
&\frac{\partial h_{2z}}{\partial t_0} - A_0 \frac{\partial u_{2z}}{\partial z_0} = \alpha A_0 I_0(kr) \left(\frac{\partial A}{\partial z_1} + \frac{k}{\omega}\frac{\partial A}{\partial t_1}\right) e^{i\psi} + CC \\
&\frac{1}{r}\frac{\partial}{\partial r}(ru_{2r}) + \frac{\partial u_{2z}}{\partial z_0} = -\alpha\, I_0(kr) \frac{\partial A}{\partial z_1} e^{i\psi} + CC \\
&\frac{1}{r}\frac{\partial}{\partial r}(rh_{2r}) + \frac{\partial h_{2z}}{\partial z_0} = \frac{kA_0\alpha}{\omega} I_0(kr) \frac{\partial A}{\partial z_1} e^{i\psi} + CC
\end{aligned}\right\} \qquad (21)$$

$$\frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_2}{\partial r} + \frac{\partial^2 \phi_2}{\partial z_0^2} = -2k\, K_0(kr) \frac{\partial B}{\partial z_1} e^{i\psi} + CC \qquad (22)$$

where

$$\Pi'_2 = \Pi_2 + \frac{\alpha k}{2\omega}\left[I_0^2(kr)(Ae^{i\psi} + \bar{A}e^{i\psi})^2 - I_1^2(kr)(Ae^{i\psi} - \bar{A}e^{-i\psi})^2\right] \qquad (23)$$

The boundary conditions (15) on $r = 1$ similarly become

$$\left.\begin{aligned}
&u_{2r} - \frac{\partial \eta_2}{\partial t_0} = \frac{\alpha}{\omega} I_1(k) \frac{\partial A}{\partial t_1} e^{i\psi} + \frac{i\alpha^2 k}{\omega}\left[I_1(k)I_1'(k) + I_0(k)I_1(k)\right] A^2 e^{2i\psi} + CC \\
&\Pi_2 + \eta_2 + \frac{\partial^2 \eta_2}{\partial z_0^2} - A_0 \frac{\partial \phi_2}{\partial z_0} = -\frac{i\alpha}{\omega} I_1(k)\left[2k + A_0^2 \frac{K_0(k)}{K_1(k)}\right]\frac{\partial A}{\partial z_1} e^{i\psi} \\
&\qquad + \frac{\alpha^2}{2\omega^2} I_1^2(k)\left[2 - 2\omega^2 + k^2 - A_0^2 k^2 + \frac{A_0^2 k^2 K_0^2(k)}{K_1^2(k)}\right] A^2 e^{2i\psi} + CC \\
&\qquad + \frac{\alpha^2}{\omega^2} I_1^2(k)\left[2 - 2\omega^2 - k^2 + A_0^2 k^2 + \frac{A_0^2 k^2 K_0^2(k)}{K_1^2(k)}\right] A\bar{A} \\
&h_{2r} - \frac{\partial \phi_2}{\partial r} = -\frac{2ik\alpha^2 A_0}{\omega^2}\frac{I_1(k)}{K_1(k)} A^2 e^{2i\psi} + CC
\end{aligned}\right\} \quad (24)$$

The solution of equations (21) and (22) will in general contain complementary solutions of the corresponding homogeneous equations analogous to the first order solution (17) with A and B replaced, say, by A_2 and B_2. However without any loss of generality we can take one of these coefficients equal to zero, which amounts to requiring that all of the basic harmonic is included in the first order solution in either $r < 1$ or $r > 1$. We take $A_2 = 0$. Then the solution of eqns (21) and (22) takes the form

$$u_{2r} = -\alpha r I_2(kr) \frac{\partial A}{\partial z_1} e^{i\psi} + ip_1 A^2 I_1(2kr) e^{2i\psi} + CC$$

$$u_{2z} = -\frac{i\alpha}{k}\left[krI_1(kr) - I_0(kr)\right]\frac{\partial A}{\partial z_1} e^{i\psi} - p_1 A^2 I_0(2kr) e^{2i\psi} + CC + C_{uz}(r, z_1, z_2, t_1, t_2)$$

$$h_{2r} = \frac{\alpha A_0}{\omega}\left[kr I_2(kr)\,\frac{\partial A}{\partial z_1} + I_1(kr)\left(\frac{\partial A}{\partial z_1} + \frac{k}{\omega}\frac{\partial A}{\partial t_1}\right)\right] e^{i\psi} - \frac{ikA_0}{\omega}\, p_1 A^2 I_1(2kr) e^{2i\psi} + CC$$

$$h_{2z} = \frac{i\alpha A_0}{\omega}\left[kr I_1(kr)\,\frac{\partial A}{\partial z_1} + \frac{k}{\omega} I_0(kr)\,\frac{\partial A}{\partial t_1}\right] e^{i\psi} + \frac{kA_0}{\omega}\, p_1 A^2 I_0(2kr) e^{2i\psi} + CC + c_{hz}(r,z_1,z_2,t_1,t_2)$$

$$\Pi_2' = \frac{i}{k}\left\{- kr I_1(kr)\;\frac{\partial A}{\partial z_1} + \left[\frac{2\alpha\omega}{k}\frac{\partial A}{\partial z_1} + \alpha\left(1 + \frac{A_0^2 k^2}{\omega^2}\right)\frac{\partial A}{\partial t_1}\right]\right\} I_0(kr)\; e^{i\psi}$$

$$- \frac{p_1}{\alpha} I_0(2kr) A^2 e^{2i\psi} + CC + c_\pi(z_1,z_2,t_1,t_2)$$

$$\phi_2 = \left[-iB_2 K_0(kr) + rK_1(kr)\,\frac{\partial B}{\partial z_1}\right] e^{i\psi} - iA_0 p_4 A^2 K_0(2kr) e^{2i\psi} + CC + c_\phi(z_1,z_2,t_1,t_2)\ .$$

The constants p_j are listed in the Appendix, eqns. (A.1) - (A.4).

The ψ-independent terms c_{uz}, c_{hz}, c_π and c_ϕ are retained in these second order solutions in order to avoid secularities at the third order. They will be determined in the following section. It is worth observing that in the approach used by Kawahara[10)] and Kakutani, Inoue and Kan[8)] such ψ-independent terms appear in the first order potentials whereas here they appear in the second order velocity and magnetic fields.

The boundary conditions (24) then give

$$\eta_2 = -\frac{i\alpha}{\omega}\left[I_2(k)\,\frac{\partial A}{\partial z_1} + \frac{1}{\omega} I_1(k)\,\frac{\partial A}{\partial t_1}\right] e^{i\psi} + p_2 A^2 e^{2i\psi} + CC + p_3 A\bar{A} - c_\pi(z_1,z_2,t_1,t_2)$$

$$B_2 = - \frac{i\alpha A_0}{\omega k K_1(k)} \left\{ \frac{\partial A}{\partial z_1} \left[k I_2(k) + I_1(k) + \frac{k K_0(k) I_1(k)}{K_1(k)} \right] + \frac{k}{\omega} I_1(k) \frac{\partial A}{\partial t_1} \right\} \tag{25}$$

and the secular equation

$$\frac{\partial A}{\partial t_1} + V_g \frac{\partial A}{\partial z_1} = 0$$

where

$$V_g = \frac{d\omega}{dk} = \frac{1}{\omega I_0(k)} \left\{ k^2 I_1(k) + A_0^2 k \frac{K_0(k) I_1(k)}{K_1(k)} + \frac{\omega k}{2\alpha} \frac{I_2(k) I_0(k)}{I_1(k)} + \frac{\omega^2}{2k} [2I_0(k) - kI_1(k)] + \frac{k^2 A_0^2}{2} \frac{K_0^2(k) I_1(k)}{K_1^2(k)} \right\}$$

is the group velocity of linearized wave theory. It follows as usual that the amplitude A depends on the slow variables z_1, t_1 through the combination $(z_1 - V_g t_1)$.

5. Third-order solution

We next substitute the first and second order solutions into the third order equations (10) and (13). Ignoring the terms proportional to $e^{\pm 2i\psi}$ and $e^{\pm 3i\psi}$ which are of no interest to us, we obtain

$$\left.\begin{aligned}
&\frac{\partial u_{3r}}{\partial t_0} + \frac{\partial \Pi_3''}{\partial r} - A_0 \frac{\partial h_{3r}}{\partial z_0} = L_1 e^{i\psi} + CC + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\
&\frac{\partial u_{3z}}{\partial t_0} + \frac{\partial \Pi_3''}{\partial z_0} - A_0 \frac{\partial h_{3z}}{\partial z_0} = L_2 e^{i\psi} + CC + M_2 + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\
&\frac{\partial h_{3r}}{\partial t_0} - A_0 \frac{\partial u_{3r}}{\partial z_0} = L_3 e^{i\psi} + CC + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\
&\frac{\partial h_{3z}}{\partial t_0} - A_0 \frac{\partial u_{3z}}{\partial z_0} = L_4 e^{i\psi} + CC + M_4 + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\
&\frac{1}{r}\frac{\partial}{\partial r}(ru_{3r}) + \frac{\partial u_{3z}}{\partial z_0} = L_5 e^{i\psi} + CC + M_5 + e^{\pm 2i\psi} \text{ terms} \\
&\frac{1}{r}\frac{\partial}{\partial r}(rh_{3r}) + \frac{\partial h_{3z}}{\partial z_0} = L_6 e^{i\psi} + CC + M_6 + e^{\pm 2i\psi} \text{ terms}\ .
\end{aligned}\right\} \quad (26)$$

$$\frac{\partial^3 \phi_3}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_3}{\partial r} + \frac{\partial^2 \phi_3}{\partial z_0^2} = L_7 e^{i\psi} + CC\ . \quad (27)$$

The coefficients L_j, M_j are listed in the Appendix, eqns. (A.5)-(A.14), while Π_3'' is defined by

$$\Pi_3'' = \Pi_3 - \frac{k}{\omega} p_1 A^2 \bar{A}\, e^{i\psi}\left[I_1(kr)I_1(2kr) + I_0(kr)I_0(2kr)\right]$$

$$+ \frac{i\alpha}{\omega}\left(A\frac{\partial \bar{A}}{\partial z_1} - \bar{A}\frac{\partial A}{\partial z_1}\right)\left\{ krI_1(kr)\left[I_2(kr) + I_0(kr)\right] - I_0^2(kr)\right. \tag{28}$$

$$\left. - \frac{\alpha k A_0^2}{\omega}\left(1 - \frac{k}{\omega}V_g\right)\left[I_1^2(kr) + I_0^2(kr)\right]\right\} .$$

In a similar way the boundary conditions (16) become

$$\left.\begin{aligned} u_{3r} - \frac{\partial \eta_3}{\partial t_0} &= L_8 e^{i\psi} + CC + M_8 + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\ \Pi_3 + \eta_3 + \frac{\partial^2 \eta_3}{\partial z_0^2} - A_0\frac{\partial \phi_2}{\partial z_0} &= L_9 e^{i\psi} + CC + M_9 + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \\ h_{3r} - \frac{\partial \phi_3}{\partial r} &= L_{10} e^{i\psi} + CC + M_{10} + e^{\pm 2i\psi},\ e^{\pm 3i\psi} \text{ terms} \end{aligned}\right\} \tag{29}$$

(see Appendix, eqns. (A.15) - (A.24)).

We seek a solution of these equations in the form

$$\begin{aligned} u_{3r} &= a_{ur}e^{i\psi} + b_{ur} + \ldots & u_{3z} &= a_{uz}e^{i\psi} + b_{uz} + \ldots \\ h_{3r} &= a_{hr}e^{i\psi} + b_{hr} + \ldots & h_{3z} &= a_{hz}e^{i\psi} + b_{hz} + \ldots \\ \Pi_3'' &= a_\pi'' e^{i\psi} + b_\pi'' + \ldots & \phi_3 &= a_\phi e^{i\psi} + b_\phi + \ldots \\ \eta_3 &= a_\eta e^{i\psi} + b_\eta + \ldots \end{aligned} \tag{30}$$

where the dots indicate terms proportional to $e^{-i\psi}$, $e^{\pm 2i\psi}$ and $e^{\pm 3i\psi}$. The terms independent of ψ then lead to the equations

$$\frac{\partial b_\pi''}{\partial r} = 0 \ , \ M_2 = 0, \ M_4 = 0 \tag{31}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rb_{ur}) = M_5 \ , \ \frac{1}{r}\frac{\partial}{\partial r}(rb_{hr}) = M_6 \tag{32}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_\phi}{\partial r}\right) = 0 \qquad (r > 1) \tag{33}$$

and the following boundary conditions on $r = 1$:

$$b_{ur} = M_8 \ , \ b_\pi + b_\eta = M_9 \ , \ b_{hr} - \frac{\partial b_\phi}{\partial r} = M_{10} \ . \tag{34}$$

The conditions $M_2 = M_4 = 0$ allow c_{uz} and c_{hz} to be determined. If we assume that these two quantities depend on z_1 and t_1 only through the combination $x_1 - V_g t_1$, as A has already been shown to do, then eqns. (A.7) and (A.10) give

$$\begin{aligned} c_{uz} &= \frac{V_g}{\Delta} c_\pi + \frac{2\alpha^2 k A_0^2}{\omega\Delta}\left(1 - \frac{k}{\omega} V_g\right) A\bar{A}\left[I_1^2(kr) + I_0^2(kr)\right] \\ c_{hz} &= -\frac{A_0}{\Delta} c_\pi - \frac{2\alpha^2 k A_0 V_g}{\omega\Delta}\left(1 - \frac{k}{\omega} V_g\right) A\bar{A}\left[I_1^2(kr) + I_0^2(kr)\right] \end{aligned} \tag{35}$$

where $\Delta = V_g^2 - A_0^2$. There are also "constants" of integration, functions of z_2, t_2 , which should be included here for completeness, but since they have no effect on the final stability condition they have been neglected for simplicity.

Next we solve eqns. (32) for b_{ur} and b_{hr}. Using eqns. (A.12) and (A.14) with (35) we find that the solutions bounded at $r = 0$ are

$$b_{ur} = -\frac{V_g}{2\Delta}\frac{\partial C_\pi}{\partial z_1} r - \frac{2\alpha^2 A_0^2}{\omega\Delta}\left(1 - \frac{k}{\omega}V_g\right)\frac{\partial}{\partial z_1}(A\bar{A})\; I_0(kr)I_1(kr)$$

$$b_{hr} = \frac{A_0}{2\Delta}\frac{\partial C_\pi}{\partial z_1} r + \frac{2\alpha^2 A_0 V_g}{\omega\Delta}\left(1 - \frac{k}{\omega}V_g\right)\frac{\partial}{\partial z_1}(A\bar{A})\; I_0(kr)I_1(kr) \;. \quad (36)$$

The solution of (33) which produces a bounded magnetic field at infinity is

$$b_\phi = g_\phi(z_1,z_2,t_1,t_2)\;\ell n\, r + h_\phi(z_1,z_2,t_1,t_2) \;.$$

The first of eqns. (31) implies that $b''_\pi = b''_\pi(z_1,z_2,t_1,t_2)$.

Turning now to the boundary conditions (34), the second of these, after using eqn. (28), allows b_η to be expressed in terms of b''_π . This latter function cannot be determined at this order of approximation. The third boundary condition determines g_ϕ in terms of C_π while the first boundary condition determines C_π. Using eqns. (A.16) and (36) we obtain that

$$C_\pi = -\, qA\bar{A}$$

$$q = \left[V_g^2 - A_0^2 + \frac{1}{2}\right]^{-1}\left\{\frac{2\alpha^2}{\omega^2} I_0(k)I_1(k)(\omega V_g - kA_0^2) - \Lambda\left[p_3 + \frac{\alpha^2}{\omega^2} I_1^2(k)\right]\right\}$$

[It follows now from eqns (35) that on $r = 1$, $C_{uz} = q_1 A\bar{A}$ and $C_{hz} = q_2 A\bar{A}$ where q_1 and q_2 are constants which can readily be calculated - see eqns (A.21) - (A.24).]

After substituting (30) into the third order equations (26), the terms proportional to $e^{i\psi}$ lead to equations for a_{ur}, a_{uz} etc. The simplest procedure seems to be to eliminate all these variables except a_{ur}, obtaining an equation whose solution is

$$a_{ur} = N_1 r I_2(kr) + N_2 r^2 I_3(kr) + \frac{ik\alpha}{\omega} A(C_{uz}+V) I_1(kr) - i\alpha\, A_3 I_1(kr) \tag{38}$$

where the constants N_k are given in the Appendix, eqns. (A.25)-(A.32) and $V(r)$ is defined by the equation

$$r I_1^2(kr)\,\frac{\partial V}{\partial r} = -\,2\alpha\omega \int_0^r \frac{\partial}{\partial r}\left(C_{uz} + \frac{A_0 k}{\omega} C_{hz}\right) I_0(kr) I_1(kr)\, r\,dr\ .$$

After using (35) and (37) we find the result which will be needed later that

$$\left.\frac{\partial V}{\partial r}\right|_{r=1} = q_3 A\bar{A}$$

$$\text{where}\quad q_3 = \frac{4\alpha^3 A_0^2}{\Delta}\left(1 - \frac{k}{\omega} v_g\right)^2 \left\{ \frac{1}{k I_1^2(k)} \int_0^k \left[I_0^2(x) + I_1^2(x)\right]^2 x\,dx - \frac{I_0(k)}{I_1(k)}\left[I_0^2(k) + I_1^2(k)\right] \right\} \tag{39}$$

The constant A_3 in (38) arises from the solution of the homogeneous equations and may be set equal to zero on the same grounds as A_2 was. Using (26), the remaining a's can be found, and in particular we obtain

$$a_{hr} = N_3 I_1(kr) + N_4 r I_2(kr) + N_5 r^2 I_3(kr) + \frac{ik\alpha}{\omega} A \left(C_{hz} - \frac{A_0 k}{\omega} V\right) I_1(kr) \tag{40}$$

$$a''_\pi = N_6 I_0(kr) + N_7 r I_1(kr) + N_8 r^2 I_2(kr) \tag{41}$$

$$- \frac{1}{\omega} A\left[I_1(kr)\frac{\partial V}{\partial r} + kI_0(kr)V + 2\alpha\omega I_0(kr)\left(C_{uz} + \frac{kA_0}{\omega} C_{hz}\right)\right]$$

The $e^{i\psi}$ terms in eqn. (27) lead to the solution

$$a_\phi = \left(\frac{\partial B}{\partial z_2} + \frac{\partial B_2}{\partial z_1}\right) rK_1(kr) + \frac{i}{2k}\frac{\partial^2 B}{\partial z_1^2}\left\{kr^2 K_2(kr) - rK_1(kr)\right\} - iB_3 K_0(kr) \tag{42}$$

in the region $r > 1$. We must keep B_3 non-zero, while B_2 is given by (25).

The boundary conditions (29) lead to the following conditions on $r = 1$:

$$a_{ur} + i\omega a_\eta = L_8$$

$$a_\pi + (1-k^2) a_\eta - ikA_0 a_\phi = L_9$$

$$a_{hr} - \frac{\partial a_\phi}{\partial r} = L_{10}$$

where from (28)

$$a''_\pi = a_\pi - \frac{k}{\omega} p_1 A^2 \bar{A}\left[I_1(kr) I_1(2kr) + I_0(kr) I_0(2kr)\right] .$$

Substituting the solutions (38), (40) and (41) into these boundary conditions, eliminating a_η and B_3 and using (39) we are left with a nonlinear Schrödinger equation for A of the form

$$i\left(\frac{\partial A}{\partial t_2} + v_g \frac{\partial A}{\partial z_2}\right) + P \frac{\partial^2 A}{\partial z_1^2} = QA^2\bar{A} \tag{43}$$

where P and Q are given in the Appendix, eqns. (A.33) and (A.34).

6. Discussion

It is well-known (Hasimoto and Ono[11]) that the solutions of the nonlinear Schrödinger equation (43) are stable or unstable according as $PQ > 0$ or $PQ < 0$. Values of P, Q and PQ have been computed for various values of k and A_0^2 and typical graphs of these three quantities as functions of k for different values of A_0^2 are shown in Figures 1-3 respectively.

In the absence of a magnetic field $(A_0^2 = 0)$, P is negative for $k < 1.35$ and positive for $k > 1.35$. Since $Q < 0$ for all k we have modulational instability for $k > 1.35$ (in addition to the linear instability for $k < 1$). Apart from a small numerical discrepancy*, this result has already been given by Kakutani, Inoue and Kan[8].

An interesting feature of the numerical results is that when $A_0^2 \neq 0$, P becomes negative for sufficiently large k. Since Q remains negative for large k this means that the presence of the magnetic field produces modulational stability for all sufficiently large k. For example when $A_0^2 = 0.5$, which is the critical value above which the linear instability disappears, there is modulational stability for all $k > 3.05$ (i.e. for wavelengths smaller than about 2.06 radii of the jet). When $A_0^2 = 1.0$ there is stability for all $k > 2.18$ (i.e. for wavelength < 2.88 radii).

A second significant feature is the presence, for all $A_0^2 \neq 0$, of a second harmonic resonance, characterized by the vanishing of the denominator in p_1 (see eqn. (A.1)). The graphs of Q and PQ have

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* The analytical results have been compared in great detail in the case $A_0^2 = 0$ with those given by Kakutani et al[8]. There is complete agreement except that in eqn. (3.14b) in ref. 8) for Q the term $(3k^4 - 2k^2 - 9)/2(k^2 - 1)$ should be replaced by $(k^4 - 9)/2(k^2 - 1)$. This causes the small numerical discrepancy.

vertical asymptotes at the resonant k-value. The analysis given in this paper is not valid in the vicinity of such a resonance. To investigate the solution near resonance the method used by Malik and Singh[12)] can be employed.

The stability regions in the k-A_0^2 plane are shown in Figure 4. The solid curves labeled P or Q are stability boundaries across each of which P or Q changes sign. The shaded regions are those of modulational instability $(PQ < 0)$. The solid curve labeled k_c gives the linear cut-off wave number and the doubly-hatched region beneath this curve is the region of linear instability. The dashed curve labeled R2 gives the second harmonic resonance wave number: in some neighbourhood of this curve the analysis does not apply. In particular this means that the precise location of the lowest of the three stability boundaries cannot be assumed to be accurately given by the curve shown in the figure.

It is clear from Figure 4 that for any A_0^2 there are two bands of k-values which give instability. When $A_0^2 = 1.14$, the upper instability band shrinks to the single value $k = 2.05$. This result may be of some technological value since it means that the wavelength of the unstable disturbance (and hence presumably the size of the droplets into which the jet breaks up) can be accurately controlled by the magnetic field.

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Appendix

The constants $p_1,\ldots,p_4$ appearing in the second order solution are given in the following formulas. Unless otherwise indicated, any Bessel function I_j or K_j has the argument k.

$$p_1 = \left[\frac{kA_0^2}{\omega} I_1(2k)K_0(2k) - \frac{1}{\alpha} I_0(2k)K_1(2k) + (4k^2-1)\frac{1}{2\omega} I_1(2k)K_1(2k)\right]^{-1}$$

$$\cdot\left\{\frac{\alpha^2}{2\omega^2}(4k^2-1)K_1(2k)I_1(2kI_0-I_1) - \frac{\alpha k}{2\omega}K_1(2k)(I_1^2-I_0^2)\right. \tag{A.1}$$

$$\left. + \frac{\alpha^2}{2\omega^2} I_1^2K_1(2k)\left[2 - 2\omega^2 + k^2(1-A_0^2) + \frac{A_0^2k^2K_0^2}{K_1^2} + \frac{2\alpha^2kA_0^2}{\omega^2}\frac{I_1K_0(2k)}{K_1}\right]\right\}$$

$$p_2 = \frac{\alpha^2}{2\omega^2} I_1(2kI_0-I_1) - \frac{I_1(2k)}{2\omega} p_1 \tag{A.2}$$

$$p_3 = \frac{\alpha k}{\omega} I_0^2 + \frac{\alpha^2}{\omega^2} I_1^2\left[2 - \omega^2 - k^2 + \frac{A_0^2k^2K_0^2}{K_1^2}\right] \tag{A.3}$$

$$p_4 = \frac{\alpha^2 I_1}{\omega^2 K_1K_1(2k)} - \frac{I_1(2k)}{2\omega K_1(2k)} p_1 \tag{A.4}$$

The coefficients L_j, M_j appearing in the third order equations are:

$$L_1 = i\alpha\, I_1(kr)\left(\frac{\partial A}{\partial t_2} + \frac{kA_0^2}{\omega}\frac{\partial A}{\partial z_2}\right) - \frac{k^2}{\omega} I_1(kr)(C_{uz} + \alpha A_0 C_z)A$$

$$+ \frac{\alpha A_0^2}{\omega}\frac{\partial^2 A}{\partial z_1^2}\left[\left(1 - \frac{k}{\omega}V_g\right)I_1(kr) + \left(1 - \frac{\omega V_g}{kA_0^2}\right)krI_2(kr)\right] \tag{A.5}$$

$$L_2 = -\alpha\, I_0(kr)\left(\frac{\partial A}{\partial t_2} + \frac{\omega}{k}\frac{\partial A}{\partial z_2}\right) + \frac{i\alpha\omega}{k^2}\left(1 - \frac{k}{\omega} v_g\right)\frac{\partial^2 A}{\partial z_1^2}\,[krI_1(kr) - 2I_0(kr)]$$

$$+ \frac{ik}{\omega} I_1(kr)\; A\,\frac{\partial}{\partial r}\,(C_{uz} + \alpha A_0 C_z) - \frac{ik^2}{\omega} I_0(kr)\; A(C_{uz} + \alpha A_0 C_z) \qquad (A.6)$$

$$M_2 = -\frac{\partial C_{uz}}{\partial t_1} + A_0\,\frac{\partial C_{hz}}{\partial z_1} - \frac{\partial C_\pi}{\partial z_1} \qquad (A.7)$$

$$L_3 = -\,iA_0\alpha\, I_1(kr)\left(\frac{\partial A}{\partial z_2} + \frac{k}{\omega}\frac{\partial A}{\partial t_2}\right) + \alpha k I_1(kr) C_z A$$

$$+ \alpha A_0\left(1 - \frac{k}{\omega} v_g\right)\frac{\partial^2 A}{\partial z_1^2}\left[\frac{v_g}{\omega} I_1(kr) - rI_2(kr)\right] \qquad (A.8)$$

$$L_4 = A_0\alpha I_0(kr)\left(\frac{\partial A}{\partial z_2} + \frac{k}{\omega}\frac{\partial A}{\partial t_2}\right) + ik\alpha I_0(kr) C_z A + i\alpha I_1(kr)\,\frac{\partial C_z}{\partial r}\,A$$

$$+ \frac{i\alpha A_0}{k}\left(1 - \frac{k}{\omega} v_g\right)\frac{\partial^2 A}{\partial z_1^2}\left[-\,krI_1(kr) + \left(1 + \frac{k}{\omega} v_g\right) I_0(kr)\right] \qquad (A.9)$$

$$M_4 = -\frac{\partial C_{hz}}{\partial t_1} + A_0\,\frac{\partial C_{uz}}{\partial z_1} + \frac{2\alpha^2 k A_0}{\omega}\left(1 - \frac{k}{\omega} v_g\right)[I_1^2(kr) + I_0^2(kr)]\,\frac{\partial}{\partial z_1}\,(A\bar{A}) \qquad (A.10)$$

$$L_5 = -\,\alpha\,\frac{\partial A}{\partial z_2}\, I_0(kr) + \frac{i\alpha}{k}\frac{\partial^2 A}{\partial z_1^2}\,[krI_1(kr) - I_0(kr)] \qquad (A.11)$$

$$M_5 = -\frac{\partial C_{uz}}{\partial z_1} \qquad (A.12)$$

$$L_6 = \frac{A_0 k\alpha}{\omega}\frac{\partial A}{\partial z_2}\, I_0(kr) - \frac{iA_0\alpha}{\omega}\frac{\partial^2 A}{\partial z_1^2}\left[krI_1(kr) - \frac{k}{\omega} v_g I_0(kr)\right] \qquad (A.13)$$

$$M_6 = -\frac{\partial C_{hz}}{\partial z_1} \tag{A.14}$$

Here $C_z = (A_0 k/\omega) C_{uz} + C_{hz}$. The coefficients in the third order boundary conditions are as follows

$$L_8 = \frac{\alpha I_1}{\omega}\frac{\partial A}{\partial t_2} + \frac{i\alpha V_g}{\omega}\left(I_2 - \frac{V_g}{\omega} I_1\right)\frac{\partial^2 A}{\partial z_1^2} + i(\Gamma_1 + \frac{k\alpha}{\omega} q_1 I_1) A^2\bar{A} - i\alpha k I_1' C_\pi A \tag{A.15}$$

$$M_8 = \left[\frac{\alpha^2}{\omega} I_1\left(2I_0 - \frac{V_g}{\omega} I_1\right) - p_3 V_g\right]\frac{\partial}{\partial z_1}(A\bar{A}) - \frac{\partial C_\pi}{\partial t_1} \tag{A.16}$$

$$L_9 = -\frac{i\alpha I_1}{\omega}\left(2k + \frac{A_0^2 K_0}{K_1}\right)\frac{\partial A}{\partial z_2} + \frac{\alpha}{\omega}\left[\left(A_0^2 - 1 + \frac{2k}{\omega} V_g\right) I_1 - 2kI_2\right]\frac{\partial^2 A}{\partial z_1^2}$$

$$+ \Gamma_2 A^2 \bar{A} - \frac{\alpha}{\omega}(2 - \omega^2) I_1 C_\pi A \tag{A.17}$$

$$M_9 = \frac{i\alpha}{\omega} I_1\left(\bar{A}\frac{\partial A}{\partial z_1} - A\frac{\partial \bar{A}}{\partial z_1}\right)\left\{2kI_2 + \frac{\alpha}{\omega}\left[\frac{2V_g}{\omega} I_1 + k\left(1 - \frac{k}{\omega} V_g\right) I_1 + (k^2-2) I_2\right]\right.$$

$$\left. - \frac{\alpha A_0^2 k}{\omega}\left[3\left(1 - \frac{k}{\omega} V_g\right) I_1 + \frac{1}{K_1} + \frac{K_0^2}{K_1^3} - \frac{k}{\omega} V_g \frac{K_0^2 I_1}{K_1^2}\right]\right\} + A_0\frac{\partial C_\phi}{\partial z_1} \tag{A.18}$$

$$L_{10} = i\left(\Gamma_3 + \frac{k\alpha}{\omega} q_2 I_1\right) A^2\bar{A} + \frac{ik\alpha A_0}{\omega K_1} C_\pi A \tag{A.19}$$

$$M_{10} = -\frac{2\alpha^2 A_0 I_1}{\omega^2 K_1}\frac{\partial}{\partial z_1}(A\bar{A}) \tag{A.20}$$

where

$$\Gamma_1 = \frac{\alpha}{\omega} p_1 I_1 [I_1(2k) - kI_0(2k)] + \alpha p_2 (kI_0 + I_1) + k\alpha p_3 I_1' + \frac{\alpha^3}{2\omega^2} I_1^2 [(k^2+2)I_1 - kI_0] \tag{A.21}$$

$$\Gamma_2 = I_1 \Bigg\{ \frac{2k}{\omega} p_1 I_1(2k) + \frac{\alpha^2 k}{2\omega^2} I_1 (5kI_0 + I_1) - k(p_2+p_3) + \frac{\alpha}{\omega}\left[2(1-k^2)p_2 + 2p_3 + \frac{\alpha^2}{2\omega^2} I_1^2 (-6+k^2-3k^4)\right] - \frac{\alpha A_0^2 k^2}{\omega}\left[p_2 + p_3 + 2p_4\left(K_1(2k) - \frac{K_0 K_0(2k)}{K_1} + \frac{\alpha^2 I_1^2}{2\omega^2}\right)\left(\frac{5kK_0}{K_1} - 1\right)\right]\Bigg\} \tag{A.22}$$

$$\Gamma_3 = -\frac{2\alpha k A_0}{\omega} p_4 I_1 [K_1(2k) + kK_0(2k)] - \frac{\alpha A_0 k}{\omega K_1}(p_2+p_3) + \frac{\alpha k A_0}{\omega^2} p_1 I_1 [kI_0(2k) - I_1(2k)] + \frac{\alpha^3 k A_0}{2\omega^3}\frac{I_1^2}{K_1} \tag{A.23}$$

and q_1 and q_2 are defined by the conditions that on $r = 1$

$$C_{uz} = q_1 A\bar{A} \, , \; C_{hz} = q_2 A\bar{A} \; . \tag{A.24}$$

The following coefficients occur in the third order solution

$$N_1 = -\alpha\frac{\partial A}{\partial z_2} + \frac{i\alpha}{2k}\frac{\partial^2 A}{\partial z_1^2} \tag{A.25}$$

$$N_2 = \frac{i\alpha}{2}\frac{\partial^2 A}{\partial z_1^2} \tag{A.26}$$

$$N_3 = \frac{A_0\alpha}{\omega}\left(\frac{\partial A}{\partial z_2} + \frac{k}{\omega}\frac{\partial A}{\partial t_2}\right) + \frac{i\alpha A_0 V_g}{\omega^2}\left(1 - \frac{k}{\omega}V_g\right)\frac{\partial^2 A}{\partial z_1^2} \tag{A.27}$$

$$N_4 = \frac{A_0\alpha k}{\omega}\frac{\partial A}{\partial z_2} - \frac{i\alpha A_0}{2\omega}\left(3 - \frac{2k}{\omega}V_g\right)\frac{\partial^2 A}{\partial z_1^2} \tag{A.28}$$

$$N_5 = -\frac{iA_0 k\alpha}{2\omega}\frac{\partial^2 A}{\partial z_1^2} \tag{A.29}$$

$$N_6 = \frac{i\alpha}{k}\left[\left(1 + \frac{k^2 A_0^2}{\omega^2}\right)\frac{\partial A}{\partial t_2} + \frac{2\omega}{k}\frac{\partial A}{\partial z_2}\right] - \frac{\alpha}{k^2}\left[\frac{\omega}{k}\left(3 - \frac{2k}{\omega}V_g\right) - \frac{\alpha A_0^2 k^3 V_g^2}{\omega^3}\right]\frac{\partial^2 A}{\partial z_1^2} \tag{A.30}$$

$$N_7 = -i\frac{\partial A}{\partial z_2} + \frac{1}{k}\left[\frac{\alpha\omega}{k}\left(1 + \frac{A_0^2 k^2}{\omega^2}\right)\left(1 - \frac{k}{\omega}V_g\right) + \frac{1}{2}\right]\frac{\partial^2 A}{\partial z_1^2} \tag{A.31}$$

$$N_8 = -\frac{1}{2}\frac{\partial^2 A}{\partial z_1^2} \tag{A.32}$$

The coefficients in the nonlinear Schrödinger equation (43) are:

$$P = -\frac{1}{2\alpha\omega I_0}\left\{\frac{1}{2}\omega\ (kI_0-2I_1) + \frac{\alpha(1-k^2)}{2k}\ [(k^2+6)I_1 - 3kI_0]\right.$$

$$- \omega I_1\left[\frac{\alpha\omega}{k}\left(1+\frac{A_0^2k^2}{\omega^2}\right)\left(1-\frac{k}{\omega}V_g\right)+\frac{1}{2}\right] + \frac{\alpha k A_0^2 I_1}{2K_1}(K_1+kK_0)$$

$$+ \frac{\alpha\omega^2 I_0}{k}\left[\frac{3\omega}{k} - 2V_g - \frac{A_0^2k^3V_g^2}{\omega^3}\right] + \alpha k\left[\left(3+\frac{2k}{\omega}V_g\right)I_1 - 2kI_0\right]$$

$$+ \frac{\alpha V_g}{\omega}(k^2-1)\left[kI_0 - \left(2+\frac{k}{\omega}V_g\right)I_1\right] + \alpha k A_0^2 I_1$$

$$+ \frac{\alpha A_0^2}{K_1}(kK_1-K_0)\left[kI_0 - \left(1+\frac{k}{\omega}V_g\right)I_1 + \frac{kK_0I_1}{K_1}\right]$$

$$- \frac{\alpha A_0^2K_0}{K_1}\left[(k^2+3)I_1 - \left(2+\frac{k}{\omega}V_g\right)\left(1-\frac{k}{\omega}V_g\right)I_1 - \frac{1}{2}\left(1+\frac{2k}{\omega}V_g\right)kI_0\right.$$

$$\left.\left. + \frac{kK_0I_1}{2K_1}\left(1+\frac{2k}{\omega}V_g\right) - \frac{k^2K_0I_0}{K_1} - \frac{k^2K_0^2I_1}{K_1^2}\right]\right\} \qquad (A.33)$$

$$Q = \frac{k}{2\alpha\omega I_0}\left\{\omega\Gamma_2 + (k^2-1)\Gamma_1 - \frac{\omega A_0K_0}{K_1}\Gamma_3 - kp_1\left[I_1I_1(2k)+I_0I_0(2k)\right]\right.$$

$$+ \alpha q\left[\frac{2(A_0^2k - \omega V_g)}{\Delta}I_0 + (3-\omega^2-k^2)I_1 + k(k^2-1)I_0 + \frac{kA_0^2K_0}{K_1}\right]$$

$$\left.+ \frac{4\alpha^3A_0^2}{\Delta I_1}\left(1-\frac{k}{\omega}V_g\right)^2 L(k)\right\} \qquad (A.34)$$

where

$$L(k) = \int_0^k [I_1^2(x) + I_0^2(x)]^2 \, x dx \tag{A.35}$$

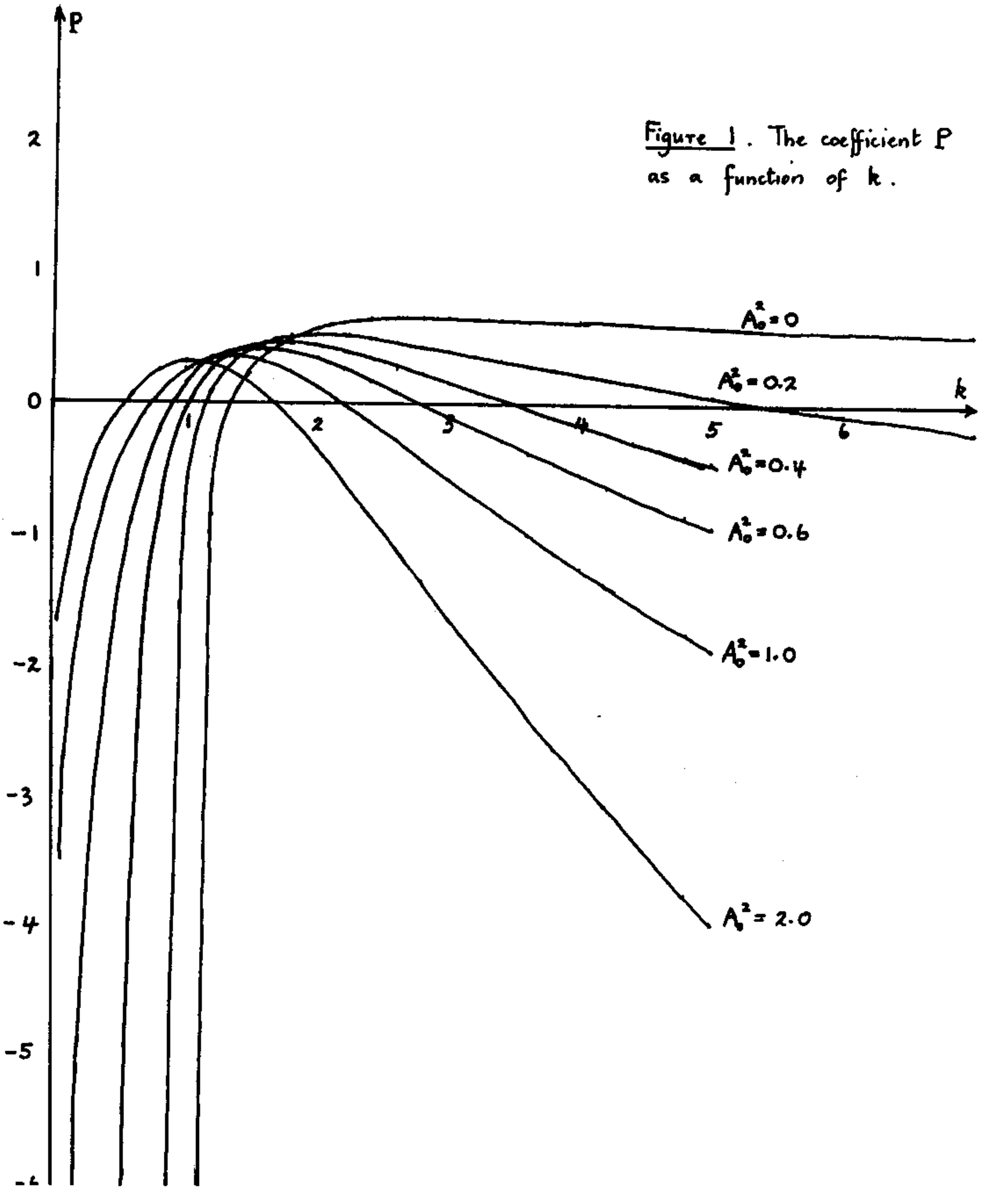


Figure 1. The coefficient P as a function of k.

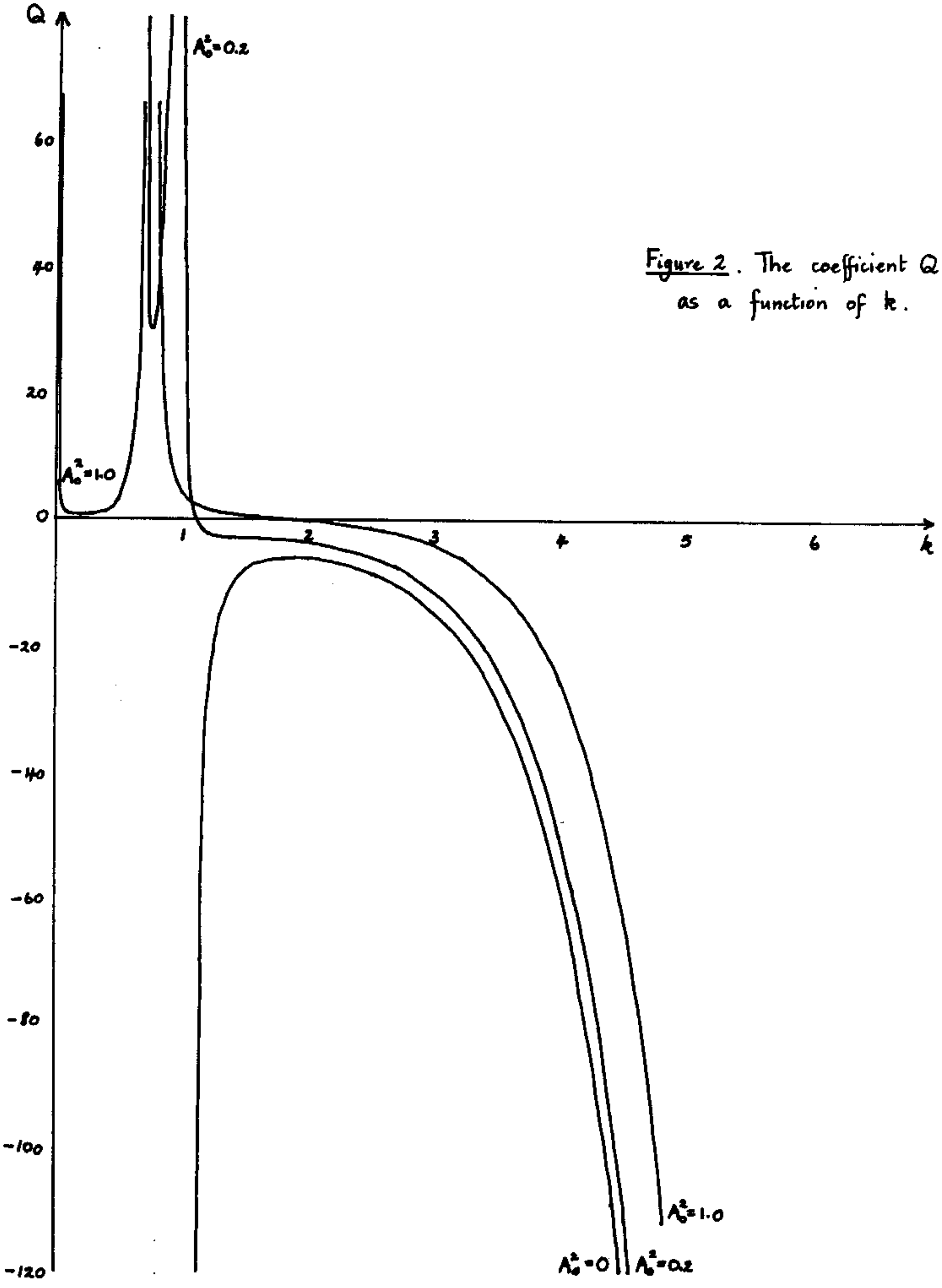
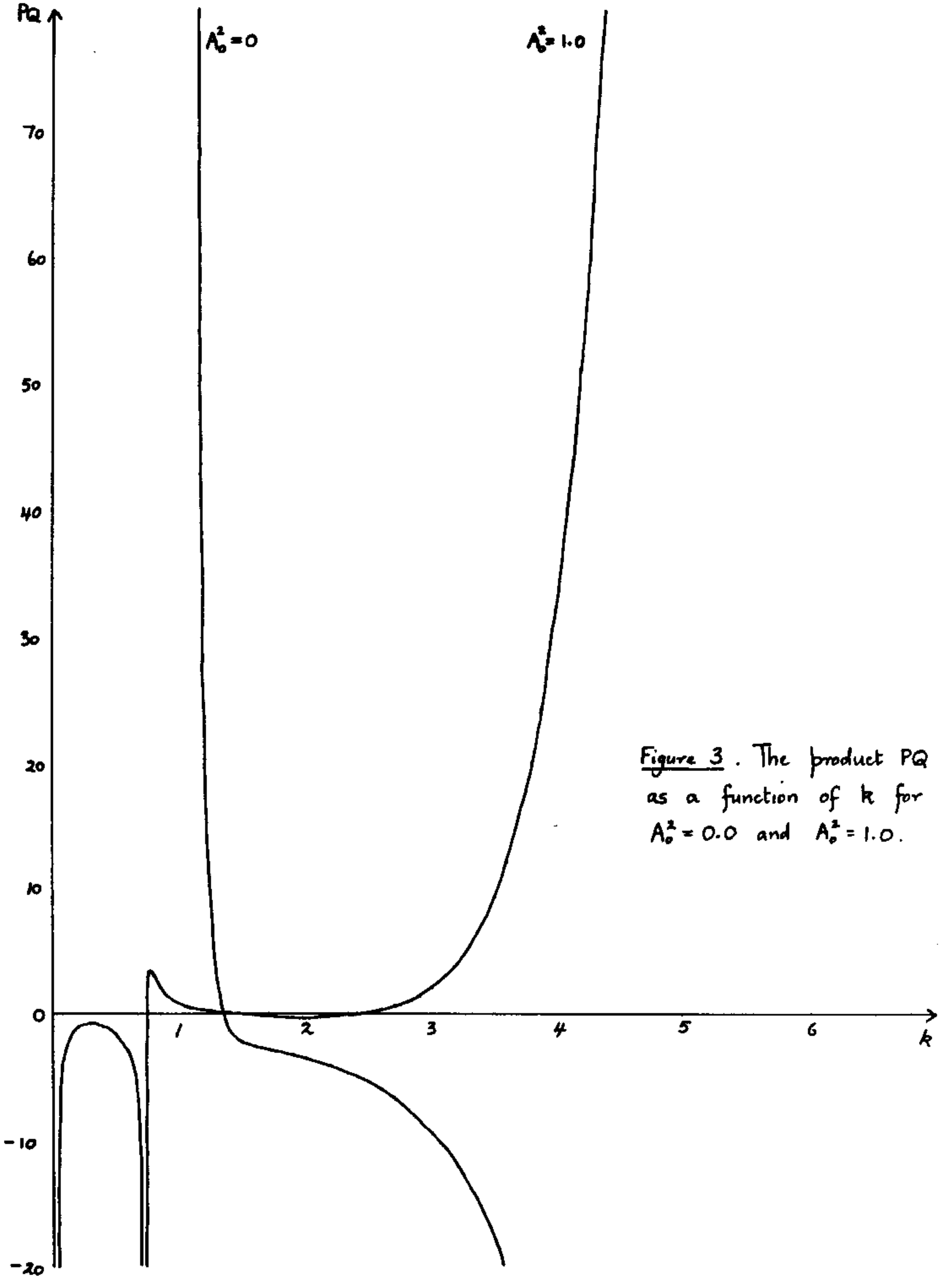


Figure 2. The coefficient Q as a function of k.



<u>Figure 3</u>. The product PQ as a function of k for $A_0^2 = 0.0$ and $A_0^2 = 1.0$.

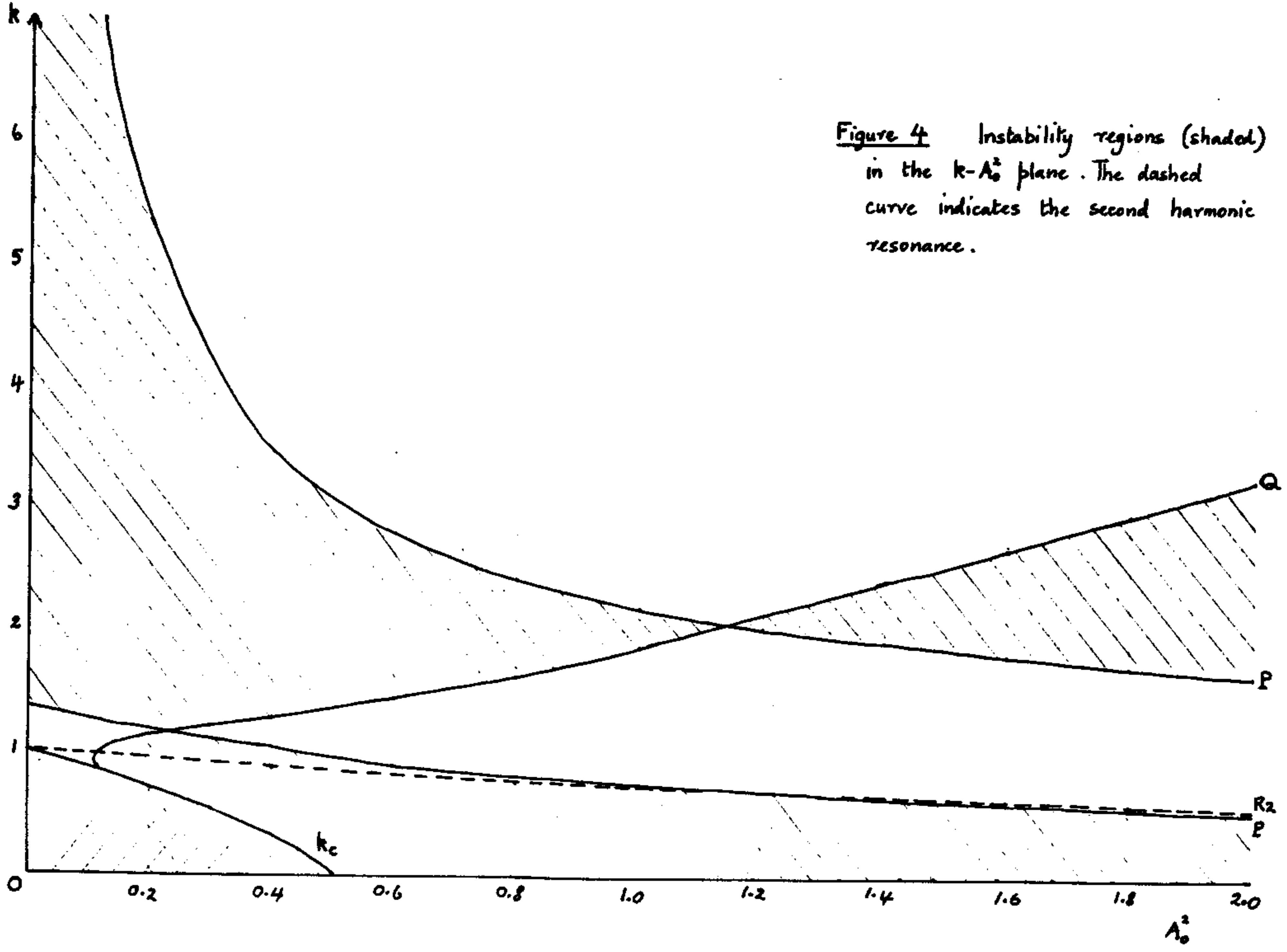


Figure 4 Instability regions (shaded) in the k-A_0^2 plane. The dashed curve indicates the second harmonic resonance.