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A Utilization of Properties of the Discrete-Time Riccati Equation in Stochastic Realization Theory

Faris A. Badawi*

Abstract. The problem of generating families of (wide sense) stochastic realizations of a discrete-time stationary stochastic process is considered. To do this, it is known that a Riccati Equation has to be solved. In this paper, the non-Riccati Algorithm of Lindquist and Kailath will be used to generate families of realizations, the state covariances of which are totally ordered. Finally, the property of constant directions which the discrete-time Riccati equation enjoys will be utilized to obtain families of realizations, the state covariances of which have the same value in certain "directions".

Key Words. Stochastic realizations, Riccati equation, non-Riccati algorithm, constant directions.

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1. Introduction

Let $\{y(t);\ t \in Z\}$ (Z = {integers}) be an m-dimensional stationary and purely nondeterministic stochastic process defined on an underlying probability space. Let the spectral density $\Phi(z)$ of y be given and have the following properties: rational in z, each element of Φ is analytic on the unit circle, $\Phi(z) = \Phi(z^{-1})'$ (prime denotes transpose), $\Phi(e^{i\omega}) > 0$ for all real ω and $\Phi(\infty) < \infty$. The stochastic realization problem is to find all representations of the type

$$x(t + 1) = Ax(t) + Bw(t) \tag{1a}$$

$$y(t) = Cx(t) + Dw(t) \tag{1b}$$

where A, B, C and D are constant matrices of dimensions $n \times n$, $n \times p$, $m \times n$ and $m \times p$ respectively and n is minimal in the sense that no other representation of type (1) has lower dimension than n such that the output process y of (1) has spectral density Φ. The process x is the state of the model and w is the input. Such a representation is called a (wide sense) stochastic realization of y (Ref. 1).

This problem has applications in and connections to optimal control theory (Ref. 2), spectral factorization (Ref. 3) and estimation theory (Ref.4).

The early contributions to this problem are due to B.D.O. Anderson (Ref. 3) and Faurre (Ref. 2). In these early papers, the problem was studied from a deterministic point of view, the objective being to determine the parameters [A, B, C, D] of the stochastic realizations. These early results have been extended by Clerget (Ref. 5) and Germain (Ref. 6).

More recently, the probabilistic aspects of the realizations have been studied by Akaike (Ref. 7), Lindquist and Picci (Ref. 1), Ruckebusch (Ref. 8), Pavon (Ref. 9) and Gevers and Kailath (Ref. 10).

It is well known (and will be shown) that the above-mentioned problem is equivalent to solving a set of equations (known as the Positive Real Lemma (Ref. 11)). the centerpiece of which is a discrete-time Riccati equation.

In this paper, we shall apply some known properties of this discrete-time Riccati equation to obtain some new results in the context of stochastic realization theory. In Section 2, some preliminaries are presented. The Riccati procedure (which requires solving a Riccati difference equation) to generate families of wide sense realizations is presented. It is remarked here that this procedure is included only for comparison purposes to make the matter easier for the reader. In Section 3, the non-Riccati algorithms of Lindquist (Refs. 12, 13) and Kailath et al (Ref. 14) (for which we have provided a new proof using a Hamiltonian approach in (Ref. 15)) will be used to formulate a new algorithm to generate families of realizations with state covariances that are decreasing (in the sense that $P \geq Q$ if and only if $P - Q \geq 0$) and which converge to the minimum P_* of the set $\mathcal{P}$ of all state covariances of the realizations. Unlike the continuous-time case, however, to generate realizations with covariances that are increasing and which converge to P^* (the maximum element of P), we have to resort to a backward setting since there is no direct procedure to do this. These non-Riccati algorithms have the advantage of avoiding the need to solve a matrix Riccati difference equation for an unwanted quantity. Moreover, these algorithms are the discrete-time analogues of their continuous-time counterparts found in (Ref. 1), by which paper, most of this work has been inspired. Finally, in Section 4, we shall impose some conditions on the given data of the problem by which the resulting Riccati equation will have invariant directions (to be defined later)(Refs. 9, 16, 17, 18, 19). This will allow us to generate families of realizations, not only having the previously mentioned properties, but also their state covariances will have the same value in certain directions. This idea has been suggested in a first version of (Ref. 9). This paper contains as a proper subset our conference paper (Ref. 20).

2. Preliminaries

The wide sense stochastic realization problem is equivalent to the spectral factorization problem (Ref. 3): Given $\Phi(z)$, find all matrices $W(z)$ of proper rational functions of minimal McMillan degree (Ref. 21) with all poles inside the unit circle and satisfying $\Phi(z) = W(z)\,W(z^{-1})'$. Using the method of partial fractions, $\Phi(z)$ can be written $\Phi(z) = S(z) + S(z^{-1})'$, where S is a discrete positive real rational function (Ref. 11). Since S

is proper (i.e., $S(\infty) < \infty$), it has a minimal realization $[F, G, H, J]$, i.e., $S(z) = H(zI - F)^{-1}G + J$ for some constant matrices F, G, H and J of dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively. Hence $|\lambda(F)| < 1$, (F, G) is controllable and (H, F) is observable. Several procedures are available for determining $[F, G, H, J]$ and hence we shall assume them to be part of the data. Using the fact that S is discrete positive real and the Positive Real Lemma, Anderson (Ref. 22) has shown that, modulo a coordinate transformation in the state space, all solutions to this problem are given by

$$[A, B, C, D] = [F, (B_1, B_2), H, (R(P)^{\frac{1}{2}}, 0)]$$

where B, the state covariance matrix $P := E\{x(t)\, x(t)'\}$ and $R(P)$ satisfy

$$P = FPF' + B_1B_1' + B_2B_2' \tag{2a}$$

$$G = FPH' + B_1R(P)^{\frac{1}{2}} \tag{2b}$$

$$R(P) = J + J' - HPH' \tag{2c}$$

$$P = P' > 0 \quad \text{(an } n\times n\text{-matrix)} \tag{2d}$$

Since $\Phi(\infty) < \infty$, and assuming $\dim F = n \geq 1$, it is easy to see that F is nonsingular (Ref. 9), $R(P) > 0$ (Ref. 8), and

$$R(P) = G'F'^{-1}H' + \Phi(\infty) - HPH'. \tag{2e}$$

Hence all stochastic realizations of y are of the form

$$x(t+1) = Fx(t) + B_1u(t) + B_2v(t) \tag{3a}$$

$$y(t) = Hx(t) + R(P)^{1/2}\, u(t) \tag{3b}$$

where $w = \begin{bmatrix} u \\ v \end{bmatrix}$.

Let $\mathcal{P}$ be the set of all symmetric n×n-matrices which solve (2). For each $P \in \mathcal{P}$, define the map

$$\Lambda(P) = -P + FPF' + (G - FPH')\, R(P)^{-1}\, (G - FPH')'. \tag{4}$$

It is well known (Refs. 2, 6, 9) that $\mathcal{P}$ is convex and compact and that $\mathcal{P} = \{P = P' \mid \Lambda(P) \leq 0\}$. Furthermore, let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ defined by $\mathcal{P}_0 = \{P \in \mathcal{P} \mid \Lambda(P) = 0\}$. Then, there are two elements P_* and P^* in $\mathcal{P}_0$ such that $P_* \leq P \leq P^*$ for all $P \in \mathcal{P}$ and finally, $\mathcal{P}_0$ is the set of all solutions of (2) for which $B_2 = 0$.

To compute the minimum P_* and the maximum P^*, the following algorithm, due to Faurre (Ref. 2) may be used.

<u>Proposition 2.1</u>. <u>Let</u> $\{\pi(t);\ t \in Z^+\}$ $(Z^+ = \{0, 1, 2, \ldots\})$ <u>and</u> $\{\bar{\pi}(t);\ t \in Z^+\}$ <u>be the solutions of the</u> n×n-<u>matrix difference equations</u>

$$\pi(t + 1) - \pi(t) = \Lambda(\pi(t)); \quad \pi(0) = 0 \tag{5a}$$

$$\bar{\pi}(t + 1) - \bar{\pi}(t) = \bar{\Lambda}(\bar{\pi}(t)); \quad \bar{\pi}(0) = 0, \tag{5b}$$

<u>respectively, where</u> Λ <u>is given by</u> (4) <u>and</u> $\bar{\Lambda}$ <u>by</u>

$$\bar{\Lambda}(P) = -P + F'PF + (H' - F'PG)\,(J + J' - G'PG)^{-1}\,(H' - F'PG)' \tag{5c}$$

<u>Then</u>, $\pi(t) \to P_*$ <u>and</u> $\bar{\pi}(t)^{-1} \to P^*$ <u>as</u> $t \to \infty$.

In fact, P_* can be regarded as the state covariance of the steady-state Kalman-Bucy filter. We refer the reader to (Ref. 2) for a proof of this proposition and for more discussion of the above convergence. In the rest of this section, we shall see how one obtains all wide sense stochastic realizations using what we call a Riccati procedure.

Let $\mathcal{P}_+ = \{P \in \mathcal{P} | P > P_*\}$ and $\mathcal{P}_- = \{P \in \mathcal{P} | P < P^*\}$. Since $\Phi(e^{i\omega}) > 0$, $P^* - P_* > 0$ (Ref. 6) and consequently, $\mathcal{P}_+$ and $\mathcal{P}_-$ are both nonempty. For each $P \in \mathcal{P}$, define the <u>feedback matrix</u>

$$\Gamma = F - (G - FPH')R(P)^{-1}H. \tag{6}$$

Then the feedback matrices Γ_* and Γ^* corresponding to P_* and P^* respectively satisfy $|\lambda(\Gamma_*)| < 1$, $|\lambda(\Gamma^*)| > 1$ (Ref. 2), and Γ_* is nonsingular (Ref. 9).

The following result, the proof of which can be found in (Ref. 4) gives a complete characterization of all elements of the subsets $\mathcal{P}_+$ and $\mathcal{P}_-$. It is a slight modification of a similar result in (Ref. 6). Unlike (Ref. 6), however, we did not rely on continuous-time theory to prove it.

<u>Theorem 2.1.</u> (a) <u>Set</u> $R_* = J + J' - HP_*P'$. <u>Let</u> $M_*(N)$ <u>be the positive definite solution of the Liapunov equation</u>

$$-M_* + \Gamma_*' M_* \Gamma_* + H' R_*^{-1} H + N = 0. \tag{7a}$$

<u>Then, the matrix</u> $P = P_* + [M_*(N)]^{-1}$ <u>belongs to</u> $\mathcal{P}_+$ <u>if and only if</u> $N \geq 0$.

(b) <u>Set</u> $R^* = J + J' - HP^*H'$. <u>Let</u> $M^*(N)$ <u>be the positive definite solution of the Liapunov equation</u>

$$M^* - \Gamma^{*'} M^* \Gamma^* + H' R^{*-1} H + N = 0. \tag{7b}$$

<u>Then, the matrix</u> $P = P^* - [M^*(N)]^{-1}$ <u>belongs to</u> $\mathcal{P}_-$ <u>if and only if</u> $N \geq 0$.

(c) $P^* - P_* = [M_*(0)]^{-1} = [M^*(0)]^{-1}$. (7c)

In addition to its significance in parameterizing the set $\mathcal{P}$, Theorem 2.1, together with Proposition 2.1, provides us with a procedure to generate stochastic realizations of y corresponding to an arbitrary element $P \in \mathcal{P}_+ \cup \mathcal{P}_-$: First, use (5a) to compute P_*; P^* will be obtained from Theorem 2.1(c), and varying N over the nonnegative cone will generate the other elements of $\mathcal{P}_+ \cup \mathcal{P}_-$. The realization $[F, B, H, (R(P)^{1/2}, 0)]$ corresponding to $P \in \mathcal{P}_+ \cup \mathcal{P}_-$ can be computed via

$$B_1 = (G - FPH')R(P)^{-1/2} \tag{8a}$$

$$B_2 B_2' = -\Lambda(P). \tag{8b}$$

This procedure of generating stochastic realizations requires solving a matrix Riccati equation (5a) in order to determine P_*, in addition to the burden of determing $P \in \mathcal{P}_+ \cup \mathcal{P}_-$. In the following section, another procedure that eliminates the intermediate step of computing P will be given.

3. Non-Riccati Algorithm inside $\mathcal{B}$.

Each $P \in \mathcal{P}$ can be interpreted as the state covariance matrix of the corresponding realization (3) (Ref. 22). Consequently, there is a minimum-variance (P_*) and a maximum-variance (P^*) realization for each of which $B_2 = 0$.

Faurre's algorithm (5) shows that the solutions $\pi(t)$ and $\bar{\pi}(t)^{-1}$ converge to P_* and P^* respectively as $t \to \infty$. However, these solutions start outside the set $\mathcal{P}$ (for $0 \notin \mathcal{P}$). In this section, P_* and P^* will be approached from inside $\mathcal{P}$. In particular, for a given $P_0 \in \mathcal{P}_-$ ($\mathcal{P}_+$), we shall construct a trajectory extending from P_0 to $P_*(P^*)$ so this trajectory is a totally ordered set of matrices satisfying (2). Such a result will enable us to construct a countable family of realizations of y, the state covariance matrices of which are totally ordered, yielding a procedure to obtain realizations without resort to the intermediate step of determining the auxiliary quantity P.

This procedure is based on the following (non-Riccati) factorization of the matrix Riccati equation, a new proof of which can be found in (Ref. 15), where we have used a Hamiltonian approach. Let P_0 be any $n \times n$-symmetric matrix, and let $r := \text{rank}\ \Lambda(P_0)$; then $\Lambda(P_0)$ can be written NSN', where N is $n \times r$ and S is $r \times r$.

Theorem 3.1. Let $\{P(t);\ t \in Z^+\}$ be the solution of

$$P(t+1) - P(t) = \Lambda(P(t));\ P(0) = P_0 \tag{9}$$

where Λ is given by (4) and let N and S be as above. Then P can be determined from the system

$$P(t+1) = P(t) - Q(t)\, Z(t)\, Q(t)';\quad P(0) = P_0 \tag{10a}$$

where the matrix sequences $\{Q(t);\ t \in Z^+\}$ and $\{Z(t);\ t \in Z^+\}$ are generated by

$$Q(t+1)=[F-U(t+1)R(t+1)^{-1}H]Q(t) \quad ; \quad Q(0)=N \tag{10b}$$

$$U(t+1)=U(t)+FQ(t)Z(t)Q(t)'H' \quad ; \quad U(0)=G-FP_0H' \tag{10c}$$

$$R(t+1)=R(t)+HQ(t)Z(t)Q(t)'H' \quad ; \quad R(0)=J+J'-HP_0H' \tag{10d}$$

$$Z(t+1)=Z(t)+Z(t)Q(t)'H'R(t)^{-1}HQ(t)Z(t); \quad Z(0)=-S \tag{10e}$$

The version of the above factorization corresponding to the Kalman-Bucy filter equations $(P_0 \equiv 0)$ was originally presented by Lindquist (Refs. 12, 13) and Kailath et al (Ref. 14).

The work presented in the rest of this paper is the discrete-time version of Section 6 in (Ref. 1). The procedure is more complicated than its continuous-time counterpart. This is natural, however, and is largely due to the fact that $R(P)$ depends on P; its continuous-time counterpart does not.

Lemma 3.1. Let $\Lambda(P)$ be defined by (4). Then, for each $P_0 \in P$, the solution $\{P(i);\ i \in Z^+\}$ of the matrix difference equation

$$P(i + 1) - P(i) = \Lambda(P(i)); \quad P(0) = P_0 \tag{11}$$

satisfies (a) $P(i) \in P$ for all $i \in Z^+$, (b) $P(i_2) \leq P(i\ )$ for $i_1 \leq i_2$ and (c) if $P_0 \in P_-$, $P(i) \to P_*$ as $i \to \infty$.

Proof. Since $P_0 \in P$, $\Lambda(P_0) = -B_2 B_2'$ (see (8)). Then $P(i)$ satisfies (10) with $N = B_2$ and $S = -I$. Then $Z(0) = I$ and consequently $Z(i) \geq 0$ for all $i \in Z^+$. Therefore, in view of (10a), $\Lambda(P(i)) \leq 0$. Hence $P(i) \in P$ for all $i \in Z^+$. This proves (a) and (b). To prove (c), let $M_i = P^* - P(i)$. An argument similar to that used in proving Theorem 2.1 (see Refs. 4, 14) yields

$$M_{i+1} = \Gamma^* M_i \Gamma^{*\prime} - \Gamma^* M_i H' R_i^{-1} H M_i \Gamma^{*\prime}; \quad M_0 = P^* - P_0 .$$

Since $M_0 > 0$ (for $P_0 \in P\_$) and $M_{i+1} - M_i \geq 0$ by (10a), $M_i > 0$ for all $i \in Z^+$. Consequently M_i^{-1} exists. Define $V_i = M^*(0) - M_i^{-1}$, where $M^*(N)$ is given by Theorem 2.1. Then $V_{i+1} - V_i = -(M_{i+1}^{-1} - M_i^{-1})$. By the matrix inversion lemma, we have

$$V_{i+1} - V_i = M_i^{-1} - \Gamma^{*\prime -1} M_i^{-1} \Gamma^{*-1} - \Gamma^{*\prime -1} H' R^{*-1} H \Gamma^{*-1}.$$

Adding (7b) for $N = 0$ to the above, we obtain

$$V_{i+1} - V_i = - V_i + \Gamma^{*'-1} V_i \Gamma^{*-1} .$$

Since $|\lambda\{\Gamma^{*-1}\}| < 1$, $V_i \to 0$ as $i \to \infty$ and consequently $M_i \to [M^*(0)]^{-1} = P^* - P_*$. Hence $P(i) \to P_*$ as $i \to \infty$. □

Now, we are ready to state the first main result of this section: the non-Riccati algorithm. Since the realizations can be determined by the matrix B, the algorithm will be given in terms of this parameter.

Let $\mathcal{B} = \{B = (B_1, B_2) | B_1, B_2$ are given by (8) with $P \in \mathcal{P}\}$. Let $\mathcal{B}_0$, $\mathcal{B}_-$ and $\mathcal{B}_+$ be defined analogously in terms of $\mathcal{P}_0$, $\mathcal{P}_-$ and $\mathcal{P}_+$. It is clear that $\mathcal{B}_0 = \{B \in \mathcal{B} | B_2 = 0\}$. In particular, let B_* and B^* denote those elements of $\mathcal{B}_0$ corresponding to P_* and P^*.

Theorem 3.2. Let $[F, B_0, H, (R_0, 0)]$ be an arbitrary realization of y, and, for each $i \in Z^+$, let $B(i) = [B_1(i), B_2(i)]$ be given by

$$B_1(i) = U(i) R(i)^{-\frac{1}{2}} ; \quad B_2(i) = Q(i) Z(i)^{\frac{1}{2}}, \tag{12}$$

where the matrix sequences $U(i)$, $Q(i)$, $Z(i)$ and $R(i)$ are given by (10) with initial conditions $U(0) = (B_0)_1 R_0^{\frac{1}{2}}$, $Q(0) = (B_0)_2$ and $Z(0) = I$. For each $i \in Z^+$, let $P(i)$ be the solution

$$- P + FPF' + B(i) B(i)' = 0. \tag{13}$$

Then, for all $i \in Z^+$, $[F, B(i), H, (R(i)^{\frac{1}{2}}, 0)]$ is a realization of y with state covariance $P(i)$. Moreover, if $B_0 \in \mathcal{B}_-$, $B(i) \to (B_*, 0)$ as $i \to \infty$.

<u>Finally, the sequence</u> $\{P(i);\ i \in Z^{+}\}$ <u>satisfies conditions</u> (a) - (c) <u>of</u> <u>Lemma</u> 3.1 <u>and the difference equation</u>

$$P(i+1) - P(i) = -B_2(i)\ B_2(i)'. \tag{14}$$

<u>Proof.</u> Let P_0 be the state covariance of the initial realization $[F,\ B_0,\ H,\ (R_0^{\frac{1}{2}},\ 0)]$, and let $\{P(i);\ i \in Z^{+}\}$ be the trajectory through P_0 defined by Lemma 3.1. Then $P(i) \in \mathcal{P}$ for all $i \in Z^{+}$. Define $B_1(i)$ and $B_2(i)$ by

$$B_1(i) = [G - FP(i)H']R(i)^{-\frac{1}{2}};\ B_2(i) = Q(i)Z(i)^{\frac{1}{2}}.$$

Then, in view of (10a), (14) follows. From Theorem 3.1, we see that $U(i) = G - FP(i)H'$. Hence, the first of relations (12) follows. Equations (11) and (14) imply $B_2(i)\ B_2(i)' = -\Lambda(P(i))$, which together with the above definition of $B_1(i)$ yields (13). Since $|\lambda(F)| < 1$ and $(F,\ B(i))$ is controllable (since (F, B_0) is), the solution of (13) is symmetric and positive definite. This fact together with (13) and the definition of $B_1(i)$ insure that $(P(i),\ B(i))$ satisfies (2). Hence $[F,\ B(i),\ H, (R(i)^{\frac{1}{2}}, 0)]$ is a realization of y with state covariance $P(i)$. By Lemma 3.1, $P(i)$ satisfies conditions (a) - (c). Finally, by the same lemma, $P(i) \to P_*$ as $i \to \infty$ if $P_0 \in \mathcal{P}_-$. Hence, if $B_0 \in \mathcal{B}_-$, $B_1(i) \to B_*$ as $i \to \infty$ and, in view of (14), $B_2(i)\ B_2(i) \to 0$ i.e. $B_2(i) \to 0$. □

<u>Remark 3.1.</u> Throughout this section, we have used the parameter i rather than t to stress the fact this quantity has nothing to do with time.

The next task is to construct a sequence belonging to set $\mathcal{P}$ which is increasing (rather than decreasing) in i and which converges to P^*. In the continuous-time case, this can be done using the same Riccati equation; the analogue of (11). Here, unfortunately, to achieve this, we shall have to follow an indirect procedure through a "backward" approach. To this end, let us review certain facts about backward realizations.

We would like to consider realizations of y that evolve backward in time of the form

$$\bar{x}(t - 1) = \bar{A}\bar{x}(t) + \bar{B}\bar{w}(t) \tag{15a}$$

$$y(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{w}(t), \tag{15b}$$

where $\bar{w}$ is a normalized white noise sequence such that, for each t, $\bar{w}(t)$ is uncorrelated to <u>future</u> (rather than past) values of x.

This problem has been studied by Pavon (Ref. 9) and earlier in (Refs. 2, 6, 8). We shall outline some of the results of (Refs. 9, 8) here, since we shall need them below.

Analogous to the forward setting, it can be shown that all <u>backward realizations</u> of y are of the form

$$\bar{x}(t - 1) = F'\bar{x}(t) + \bar{B}_1\bar{u}(t) + \bar{B}_2\bar{v}(t) \tag{16a}$$

$$y(t) = G'\bar{x}(t) + \bar{R}(\bar{P})^{\frac{1}{2}}\bar{u}(t) \tag{16b}$$

where $w = [\frac{\bar{u}}{v}]$, $\bar{B}$, the state covariance matrix $\bar{P} := E\{\bar{x}(t)\,\bar{x}(t)'\}$ and $\bar{R}(\bar{P})$ satisfy

$$\bar{P} = F'\bar{P}F + \bar{B}_1\bar{B}_1' + \bar{B}_2\bar{B}_2 \tag{17a}$$

$$H' = F'\bar{P}G + \bar{B}_1 R(\bar{P})^{\frac{1}{2}} \tag{17b}$$

$$\bar{R}(\bar{P}) = J + J' - G'\bar{P}G \tag{17c}$$

$$\bar{P} = \bar{P}' > 0 \quad \text{(an n×n symmetric matrix.)} \tag{17d}$$

Let $\bar{\mathcal{P}}$ be the set of all solutions of (17), define the map

$$\bar{\Lambda}(\bar{P}) = -\bar{P} + F'\bar{P}F + (H' - F'\bar{P}G)\bar{R}(\bar{P})^{-1}(H' - F'\bar{P}G)', \tag{18}$$

and let $\bar{\mathcal{P}}_0 = \{\bar{P} \mid \bar{\Lambda}(\bar{P}) = 0\}$. Then $\bar{\mathcal{P}} = \{\bar{P} \mid \bar{\Lambda}(\bar{P}) \leq 0\}$ and it has the same properties as $\mathcal{P}$. Hence, there exist two elements $\bar{P}_*$ and $\bar{P}^*$ in $\bar{\mathcal{P}}_0$ such that $\bar{P}_* \leq \bar{P} \leq \bar{P}^*$ for every $\bar{P} \in \bar{\mathcal{P}}$. It is well-known (Refs. 2, 6) that $\bar{\mathcal{P}}$ is related to $\mathcal{P}$ by $\bar{\mathcal{P}} = \{P^{-1} \mid P \in \mathcal{P}\}$. Thus $\bar{P}_* = (P^*)^{-1}$ and $\bar{P}^* = (\bar{P}_*)^{-1}$. This explains the choice of (5b) by Faurre.

In fact there is a one-one correspondence between forward and backward realizations. It was shown in (Ref. 9) how to compute the backward elements $\bar{B}$ and $\bar{R}(\bar{P})$ from the knowledge of the forward ones. In the following proposition, we also include the converse.

Proposition 3.1. (a) (Ref. 9) Assume the quadruplet $[F, B, H, (R(P)^{\frac{1}{2}}, 0)]$ solves the forward problem. Set

$$\bar{B} = -P^{-1} F^{-1} B(I - B' P^{-1} B)^{\frac{1}{2}}, \text{ and} \tag{19a}$$

$$(\bar{R}(\bar{P})^{\frac{1}{2}}, 0) = [(R(P)^{\frac{1}{2}}, 0) - HF^{-1}B] [I - B'P^{-1}B]^{\frac{1}{2}}. \tag{19b}$$

<u>Then</u> $[F', \bar{B}, G', (\bar{R}(\bar{P})^{\frac{1}{2}}, 0)]$ <u>solves the backward one</u>.

(b) <u>Assume the quadruplet</u> $[F', \bar{B}, G', (\bar{R}(\bar{P})^{\frac{1}{2}}, 0)]$ <u>solves the backward problem. Set</u>

$$B = -\bar{P}^{-1}F'^{-1}(I - \bar{B}'\bar{P}^{-1}\bar{B})^{\frac{1}{2}}, \text{ and} \tag{19c}$$

$$(R(P)^{\frac{1}{2}}, 0) = [(\bar{R}(\bar{P})^{\frac{1}{2}}, 0) - G'F'^{-1}\bar{B}][I - \bar{B}'\bar{P}^{-1}\bar{B}]^{\frac{1}{2}}. \tag{19d}$$

<u>Then</u> $[F, B, H, (R(P)^{\frac{1}{2}}, 0)]$ <u>solves the forward one</u>.

In light of this duality between the forward and the backward settings, we present now a summary for the procedure that may be used to generate a family of realizations $[F, B(i), H, (R(i)^{\frac{1}{2}}, 0)]$ from an initial realization $[F, B_0, H, (R_0^{\frac{1}{2}}, 0)]$ such that the state covariances $\{P(i)\}$ of this family are increasing and (if $P_0 \in \mathcal{P}^+$) converge to P^* as $i \to \infty$.

Let $[F, B_0, H, (R_0^{\frac{1}{2}}, 0)]$ be a given realization with state covariance P_0.

(1) Define $\bar{P}_0 := P_0^{-1}$.

(2) Using $\bar{P}_0$ as an initial condition, obtain the sequence $\{\bar{P}(i)\}$ as the solution of the matrix difference equation

$$\bar{P}(i + 1) - \bar{P}(i) = \bar{\Lambda}(\bar{P}(i)); \quad \bar{P}(0) = \bar{P}_0, \tag{20}$$

which is the dual of (11). Then, $\bar{P}(i) \in \bar{\mathcal{P}}$, $\bar{P}(i_2) \leq \bar{P}(i_1)$ for $i_1 \leq i_2$ and if $P_0 \in \mathcal{P}^+$, $\bar{P}(i) \to P^{*-1}$.

(3) Obtain the family of realizations $[F', \bar{B}(i), G', (\bar{R}(i)^{\frac{1}{2}}, 0)]$ corresponding to $\bar{P}(i)$ in a manner analogous to that of the forward setting presented in Theorem 3.2.

(4) Finally, use (19c) and (19d) to obtain the family $[F, B(i), H, (R(i)^{\frac{1}{2}}, 0)]$ from the family $[F', \bar{B}(i), G', (\bar{R}(i)^{\frac{1}{2}}, 0)]$. Since $P(i) = [\bar{P}(i)]^{-1}$, therefore $P(i) \to P^*$ as $i \to \infty$ if $P_0 \in P^+$.

Remark 3.2. Theorem 3.2 and the above summary have the following interpretation.

Let

$$x_0(t+1) = Fx_0(t) + (B_0)_1\, u(t) + (B_0)_2\, v(t) \tag{21a}$$

$$y(t) = Hx_0(t) + R_0^{\frac{1}{2}}\, u(t) \tag{21b}$$

be an arbitrary realization of y with state covariance P_0.

(a) Let $B(i) = [B_1(i), B_2(i)]$ and $R(i)$ be given by (12) and (10). Then, for each $i \in Z^+$,

$$x_i(t+1) = Fx_i(t) + B_1(i)\, u(t) + B_2(i)\, v(t) \tag{22a}$$

$$y(t) = Hx_i(t) + R(i)^{\frac{1}{2}}\, u(t) \tag{22b}$$

is a realization of y with state covariance $P(i) = E\{x_i(t)\, x_i(t)'\}$ given by

$$P(i+1) - P(i) = -B_2(i)\, B_2(i)'; \quad P(0) = P_0. \tag{22c}$$

Furthermore, $\{P(i);\, i \in Z^+\}$ is a decreasing sequence in i such that, if $B_0 \in \mathcal{B}_-$, $P(i) \to P_*$ and $B(i) \to (B_*, 0)$ as $i \to \infty$, where B_* is, in fact, the Kalman gain of the steady-state Kalman-Bucy filter.

(b) Similarly, let $B(i)$ and $R(i)$ be as in the above summary. Then (22a, b) is a realization of y with state covariance $P(i)$ given by (22c). But now, the sequence $\{P(i);\ i \in Z^+\}$ is <u>increasing</u> in i such that, if $B_0 \in \mathcal{B}_+$, $P(i) \to P^*$ and $B(i) \to (B^*, 0)$ as $i \to \infty$. Indeed, B^* is the forward counterpart (in the sense of Proposition (3.1)) of the gain of the steady-state <u>backward</u> Kalman-Bucy filter.

4. Invariant Directions of the Riccati Equation and Stochastic Realizations

Given any $P_0 \in \mathcal{P}$, in the previous section, we constructed a family of stochastic realizations of the process y with state covariances $P(i)$ starting at P_0 and converging to P_* or P^*. In this section we shall employ some known properties of the Riccati equation to generate a family of realizations corresponding to each $P_0 \in \mathcal{P}$ such that the $P(i)$'s have the same value in certain directions.

Thus, let (3) be an arbitrary realization of y. Recall that the Riccati equation (11) is the difference equation for the trajectory $P(i)$ corresponding to $P_0 \in \mathcal{P}$. For easy reference, we reproduce that equation here:

$$P(i+1)=FP(i)F'+(G-FP(i)H')R(P(i))^{-1}(G-FP(i)H')'. \qquad (23)$$

We shall denote the solution of (23) at stage i with initial condition P_0 by $P(i, P_0)$.

Definition 4.1. (Refs. 16, 17, 18, 19). The n-vector e is called a k-invariant direction for (23) if and only if $P(i, P_0)e = P(k, 0)e$ for all $i \geq k$ and all symmetric nonnegative definite matrices P_0.

Remark 4.1. By Lemma 3.1, the solution $P(i, P_0)$ of (23) (for the initial condition P_0 which belongs to $\mathcal{P}$) belongs to $\mathcal{P}$ for all $i \in Z^+$ and $P(i, P_0) \to P_*$ as $i \to \infty$, if $P_0 \in \mathcal{P}_-$. However, it is important to remark that $P(k, 0)$ does not belong to $\mathcal{P}$ (for $0 \notin \mathcal{P}$).

Before we exploit this definition in the context of stochastic realization theory, we shall answer the following two questions:

(1) What conditions (if any) should be imposed on the system parameters in order to guarantee the existence of nontrivial invariant directions?

(2) If there are nontrivial directions, how many and what are these directions?

These questions will be answered for a special case $(\Phi(\infty) = 0)$ that is most suitable for our purposes. The general case is rather involved and will not be pursued here. Moreover, since $\Phi(z)$ (or $[F,G,H,J]$) is the apriori available data, the answers will be given in terms of it.

To answer the first question, recall that $R(P) = G'F'^{-1}H' + \Phi(\infty) - HPH'$ (see 2e)). Let $\Phi(\infty) = 0$ and consider the n-vector

$$e = F'^{-1} H' \lambda \tag{24}$$

for an arbitrary m-vector λ. If $\Phi(\infty) = 0$, then $P(1, 0)e_1 = G(G'F'^{-1}H')^{-1}G'e_1 = G\lambda$, and for any $P_0 \geq 0$. It is easy to check that $P(1,P_0)e_1 = P(2, P_0)\, e_1 = \ldots = P(1,0)e_1 = G\lambda$. Hence e_1 is 1-invariant for (23). Thus the condition $\Phi(\infty) = 0$ implies the existence of at least one nontrivial invariant direction. From now on, assume this condition holds.

To answer the second question, we need the following.

Lemma 4.1. _Let_ $\Phi(z)$ _be the spectral density of_ y. _Then_ $\Phi(z)$ _can be written_

$$\Phi(z) = \sum_{i=0}^{\infty} A_i z^{-1}, \tag{25}$$

where

$$A_0 = J + J' - G'F'^{-1} H' \equiv \Phi(\infty) \tag{26a}$$

and

$$A_i = HF^{i-1} G - G' F'^{-i-1} H' \quad \text{for } i \geq 1. \tag{26b}$$

Proof. Recall that (see Section 2) $\Phi(z) = S(z) + S(z^{-1})'$, where $S(z) = H(zI - F)^{-1} G + J$. It is not hard to see that $S(z)$ and $S(z^{-1})'$ can be expanded in the following manner

$$S(z)=J+HGz^{-1}+HFGz^{-2}+HF^2Gz^{-3}+\ldots+HF^{n-1}Gz^{-n}+\ldots \tag{27}$$

and

$$S(z^{-1})'=J'-G'F'^{-1}H'-G'F'^{-2}H'Z^{-1}-\ldots-G'F'^{n-1}H'Z^{-n}-\ldots \quad (28)$$

Adding (27) and (28), we obtain (25) and (26). □

The following proposition gives the answer to question (2).

Proposition 4.1. *Let* A_i *be given by* (26). *If* $A_i = 0$ *for* $i = 0, 1, \ldots, r-1$, *then*

$$e_j = F'^{-j}\, H'\lambda\ ; \quad j = 1, 2, \ldots, r \quad (29)$$

where λ *is an arbitrary m-vector, is a j-invariant direction for* (23) *and*

$$P(i,\ P_0)e_j = P(i,\ 0)e_j = F^{j-1}G\lambda \quad \text{for all} \quad i \geq j. \quad (30)$$

Proof. The proof is by induction on j.

First, we show that $e_1 = F'^{-1}H'\lambda$ is 1-invariant and that $P(i,\ P_0)e_1 = G\lambda$ for all $i \geq 1$ and any initial condition $P_0 \geq 0$. From (23), we have

$$P(i,P_0)e_1 = FP(i-1,P_0)F'F'^{-1}H'\lambda +$$
$$(G-FP(i-1,P_0)H')(G'F'^{-1}H'-HP(i-1,P_0)H')^{-1}(G'e_1-HP(i-1,P_0)F'e_1) \quad (31)$$

But

$$G'e_1-HP(i-1,P_0)F'e_1=G'F'^{-1}H'\lambda-HP(i-1,P_0)F'F'^{-1}H'\lambda$$
$$=(G'F'^{-1}H'-HP(i-1,P_0)H')\lambda$$

Hence, (31) becomes

$$P(i,P_0)e_1=FP(i-1,P_0)H'\lambda+G\lambda - FP(i-1,P_0)H'\lambda = G\lambda.$$

Now, assume that $e_{j-1} = F'^{-j+1}H'\lambda$ is j-1-invariant and $P(i,P_0)e_{j-1} = F^{j-2}G\lambda$ for all $i \geq j-1$ and any $P_0 \geq 0$. Then, by (23),

$$P(i,P_0)e_j = FP(i-1,P_0)F'e_j +$$
$$(G-FP(i-1,P_0)H')R(P(i-1,P_0))^{-1}(G'e_j-HP(i-1,P_0)F'e_j). \quad (32)$$

But

$$G'e_j - HP(i-1,P_0)F'e_j = G'F'^{-j}H'\lambda - HP(i-1,P_0)F'F'^{-j}H'\lambda$$

$$= G'F'^{-j}H'\lambda - HP(i-1,P_0)e_{j-1} = (G'F'^{-j}H' - HF^{j-2}G)\lambda = A_{j-1}\lambda = 0.$$

Also,

$$\begin{aligned} FP(i-1,\ P_0)e_j &= FP(i-1,\ P_0)F'F'^{-j}H'\lambda \\ &= FP(i-1,\ P_0)F'^{-j+1}H'\lambda \\ &= FP(i-1,\ P_0)e_{j-1} \\ &= FF^{j-2}G\lambda = F^{j-1}G\lambda. \end{aligned}$$

This completes the proof. □

Remark 4.2. The number r is equal to the relative order of the process (Ref. 18), which is the number of the differencing operations on the output equation required to produce a Kronecker delta term in the covariance of y when it is nonstationary process and is equal to the difference between the degrees of the denominator and numerator polynomials of $\Phi(z)$ when y is stationary.

The main result of this section is the next theorem which describes how we can exploit Definition 4.1 in order to generate families of realizations with state covariances that have the same value in r directions.

Theorem 4.1. Let $A_i = 0$, $i = 0,1,\ldots, r-1$, *where* A_i *is given by* (26). *Let* (3) *be an arbitrary realization of* y *with state covariance* P_0. *Let* $P(i, P_0)$ *be the solution of* (23) *at stage* i *with initial condition* P_0 *and define*

$$B_1(i) = (G - FP(i)\ H')\ R(P(i))^{-\frac{1}{2}}$$

and

$$B_2(i)\ B_2(i)' = -\Lambda(P(i)).$$

Then

$$x_i(t+1) = Fx_i(t) + B_1(i)\ u_i(t) + B_2(i)\ v_i(t)$$

$$y(t) = Hxi(t) + R(P(i))^{\frac{1}{2}}\ u_i(t)$$

is a stochastic realization of y *with state covariance* $P(i, P_0)$ *satisfying* (a) - (c) *of Lemma* 3.1 *and the property that for* $j = 1, 2, \ldots, r$, $P(i, P_0)e_j = P(j, 0)e_j$ *for all* $i \geq j$ *and* e_j *is as in* (29).

Of course, we can equally well obtain r independent directions for the dual Riccati equation (20).

Remark 4.3. If P_0 is chosen from $\mathcal{P}_0$ then the above-obtained family $P(i, P_0)$ shrinks down to one element.

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