King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 054

May 1983

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WAVEFORM DISTORTION AND SHOCK DEVELOPMENT IN NONLINEAR RAYLEIGH WAVES

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Abstract

Using an asymptotic theory of nonlinear Rayleigh waves previously developed, numerical solutions are obtained for the variation with distance of the amplitudes of the higher harmonics in an initially sinusoidal wave. The two components of the particle velocity on the free surface are calculated and it is found that after a finite distance a shock develops in the form of a discontinuity in the horizontal velocity component.

1. Introduction

There has in recent years been considerable interest in the effect of nonlinearities on surface Rayleigh waves in elastic solids. One reason for this interest is that, because of the high level of energy density close to the free surface, nonlinear effects are stronger than for bulk elastic waves, and thus there is a greater possibility of using them in the design of acoustic devices.

Based on a method initiated by Kalyanasundaram [1], the present author [2] has given a theory of nonlinear Rayleigh waves on an isotropic solid which allows the distortion of an arbitrary waveform to be calculated. The end result of this theory is a nonlinear coupled system of differential equations for quantities which are related to the Fourier transforms of the displacement components on the free surface. The kernel in this system is algebraically complicated, and it is unlikely that analytic solutions can be found. In [2] approximate solutions were found for the generation of higher harmonics in an initially sinusoidal wave.

A significant question is whether shocks can develop in Rayleigh waves as a result of the nonlinear elasticity. In order to answer this, the approximation technique used in [2] in which only a few harmonics are retained in the solution is clearly insufficient. In the present paper, numerical solutions are obtained with a large number of harmonics retained. The resultant distorting waveform is calculated and it is found that shocks do develop in general.

2. Review of the Equations

We consider an isotropic elastic medium occupying the half-space $y < 0$ with the surface $y = 0$ traction-free. We take units in which the shear-wave velocity is unity, and let c denote the linear Rayleigh wave speed. We consider a surface wave traveling in the positive x-direction with $u(x,y,t)$ and $v(x,y,t)$ denoting the x- and y-components of the displacement. If the wave contains a fundamental with wave number $k = 1$ together with higher harmonics, then the displacement components on the surface are expressible in the following form [2]:

$$u(x,0,t) = q(q-1)\sum_{k=1}^{\infty}\gamma_k(x)e^{ik(x-ct)} + cc \tag{1}$$

$$v(x,0,t) = ip_2(q-1)\sum_{k=1}^{\infty}\gamma_k(x)e^{ik(x-ct)} + cc \tag{2}$$

where

$$q = 1 - \tfrac{1}{2}c^2, \qquad p_1^2 = 1 - c^2, \qquad p_2 = q^2/p_1 \,. \tag{3}$$

The dependence of the amplitudes γ_k on x allows for the nonlinear modulation of the waveform. It turns out that the $\gamma_k(x)$ satisfy a coupled system of equations of the form

$$\frac{d\gamma_k}{dx} + \sum_{k'=1}^{\infty} H_{kk'}k'(k-k')\gamma_{k'}\gamma_{k-k'} = 0 \tag{4}$$

where the kernel $H_{kk'}$ is given in [2]. (Equation (A.20) gives $H_{kk'}$ for $k' < k$ while equation (A.21) gives the quantity $K(k,k') \equiv k'H_{kk'}$ for $k' > k$.)

We shall seek a particular class of solutions in which

$$k\gamma_k(x) = -A_k(x)e^{ik\phi}$$

where $A_k(x)$ is real and ϕ is constant. Equation (4) then becomes

$$\frac{dA_k}{dx} = k\sum_{k'=1}^{\infty} H_{kk'}A_{k'}A_{k-k'} \,. \tag{5}$$

From (1), (2) and (3), the velocity components on the free surface are then given by

$$u_t(x,0,t) = qc^3 \sum_{k=1}^{\infty} A_k(x) \sin k(x-ct+\phi) \tag{6}$$

$$v_t(x,0,t) = p_2 c^3 \sum_{k=1}^{\infty} A_k(x) \cos k(x-ct+\phi) \tag{7}$$

We consider waves generated by a purely harmonic source at $x = 0$. For definiteness we take the boundary condition

$$A_k(0) = 0 \qquad (k \geq 2) \; ; \qquad A_1(0) = \frac{1}{2} \, . \tag{8}$$

Because of the invariance of system (5) under the re-scaling transformation

$$A_k^* = \lambda A_k \; , \qquad x^* = x/\lambda$$

the solution with any other value of $A_1(0)$ can be found directly from the results given below.

Before proceeding, we review some of the material constants occuring in the equations. The Rayleigh speed c is the solution of the equation

$$(2 - c^2)^2 = 4(1 - c^2)^{\frac{1}{2}}(1 - r^{-1}c^2)^{\frac{1}{2}}$$

where

$$r = (\lambda + 2\mu)/\mu = 2(1 - \nu)/(1 - 2\nu) \; ,$$

λ and μ being the Lamé constants and ν Poisson's ratio. The kernel $H_{kk'}$ involves four constants c_1^2, c_2^2, c_3^2 and c_4^2 which are expressed in terms of the third-order elastic constants α, β and γ by equations (14) of [1]. (To conform to the present units we must set $v_0 = 1$ and $\rho_0 = \mu$ in these equations.) However we note that if we define the two combinations of third-order constants,

$$\chi_1 = (\alpha + \beta + \gamma)/\mu \ , \qquad \chi_2 = (\beta + \tfrac{3}{2}\gamma)/\mu \ ,$$

then

$$c_1^2 = \frac{3r}{2} + 3\chi_1 \qquad c_2^2 = \tfrac{1}{2}r - 1 + 3\chi_1 - 2\chi_2$$

$$c_3^2 = 1 + \chi_2 \qquad c_4^2 = \tfrac{1}{2}(r + \chi_2) \ .$$

By this means the number of independent material parameters is reduced simply to three: ν, χ_1 and χ_2.

The kernel $H_{kk'}$ depends only on the ratio k'/k. The graphs of this function when $\nu = 1/3$ are shown in Figure 1 for the two cases $\chi_1 = 1$, $\chi_2 = 0$ and $\chi_1 = 0$, $\chi_2 = 1$. For different values of ν, the form of these graphs is qualitatively similar to those shown in the Figure over the range $0.25 \leq \nu \leq 0.4$, with $H_{kk'}$ decreasing monotonically as ν increases. Thus the solutions for other values of ν can be expected to be similar to those given in the next section for $\nu = 1/3$ but with the scale of x-values over which distortion of the waveform occurs increasing as ν increases.

3. Numerical Solutions

The basic numerical procedure has been to truncate the system (5) at some value $k' = k_m$ then solve using a fourth-order Runge-Kutta routine. A number of checks have been made on the solution: the truncation point has been varied from $k_m = 30$ to $k_m = 60$; the integration stepsize has been varied between $\Delta x = 0.01$ to $\Delta x = 0.001$; and the integration routine has been replaced by a modified Euler method. None of these changes has produced an appreciable change in the solution up to the value of x at which shocks form.

As a second type of check, the numerical procedure has been applied to the system (5) with $H_{kk'}$ replaced by 0.5 if $k' < k$ and -1.0 if $k' > k$. This system can be solved exactly*, and with the initial conditions (8), the solution is given by

$$A_k(x) = J_k(kx)/kx$$

where $J_k(z)$ is the Bessel function of the first kind of order k. The shock formation point is $x = 1$ in this case. The exact solution is reproduced numerically with errors smaller than 2×10^{-5} in every $A_k(x)$ up to $x = 0.9$ and up to $x = 1.0$ the errors are still only of order 10^{-4}.

Figures 2(a) and (b) show typical results for the two surface velocity components u_t and v_t computed from equations (6) and (7). These figures correspond to the material parameters $\nu = 1/3$, $\chi_1 = 1$ and $\chi_2 = 0$ and are computed with $k_m = 40$, $\Delta x = 0.01$. The velocities are given as functions of $\theta = x - ct + \phi$ for several values of x. With the initial condition (8), there is a purely harmonic wave at $x = 0$,

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* It corresponds to the partial differential equation $\partial h/\partial x = h\, \partial h/\partial\theta$ where $h(x,\theta) = \sum_{k=1}^{\infty} 2A_k(x) \sin k(\theta + \pi/2)$. The initial condition (8) implies $h(0,\theta) = \cos\theta$, and the solution is $h(x,\theta) = \cos\xi$ where $\xi = \theta + x\cos\xi$.

and the figures clearly show the distortion of the waveform as x increases and the development of a shock at approximately $x = 4$.

At the moment of shock formation, one or both of the expansions (6) and (7) ceases to be uniformly convergent in the neighbourhood of $\theta = 0$. Figure 2(a) shows that at $x = 3.5$, ripples are already forming in the computed u_t near to $\theta = 0$, indicating that, while the shock has not quite formed at this value of x, the series (6) is so slowly convergent that the truncation at $k_m = 40$ is insufficiently accurate. As x increases beyond 3.5, the ripples grow and the solution loses accuracy.

Figures 3(a) and (b) show similar results for the physical parameters $\nu = 1/3$, $\chi_1 = 0$, $\chi_2 = 1$. In this case the shock develops somewhat faster and the solution for $x = 3.5$, one side of which is shown in Figure 3(a), already shows a clear loss of accuracy close to $\theta = 0$.

References

[1] N. Kalyanasundaram, Int. J. Engg. Sci. 19, 279-286 (1981).

[2] R.W. Lardner, Int. J. Engg. Sci., Vol.20, to be published.

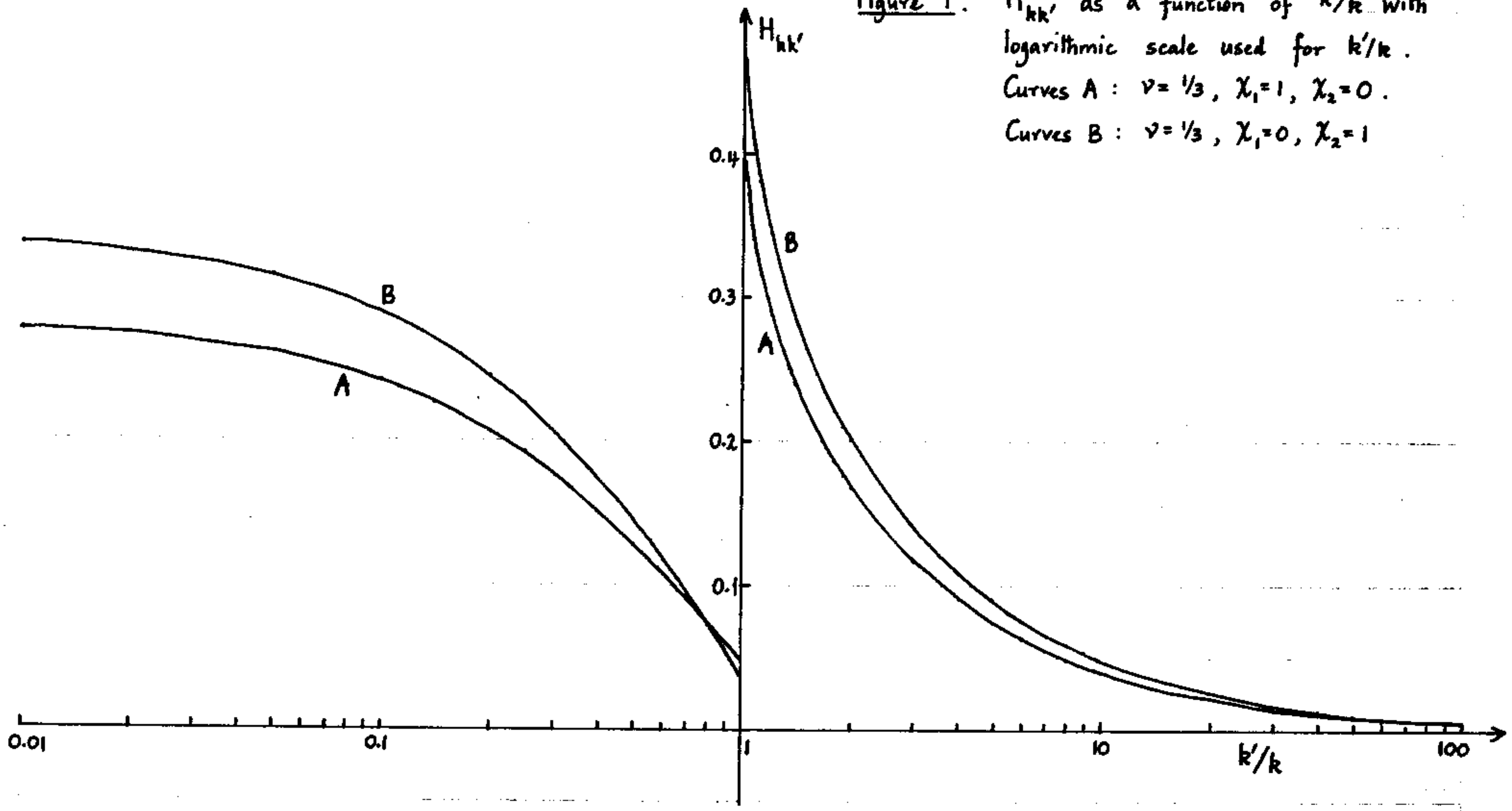


Figure 1. $H_{kk'}$ as a function of k'/k with logarithmic scale used for k'/k.
Curves A : $\nu = 1/3$, $\chi_1 = 1$, $\chi_2 = 0$.
Curves B : $\nu = 1/3$, $\chi_1 = 0$, $\chi_2 = 1$

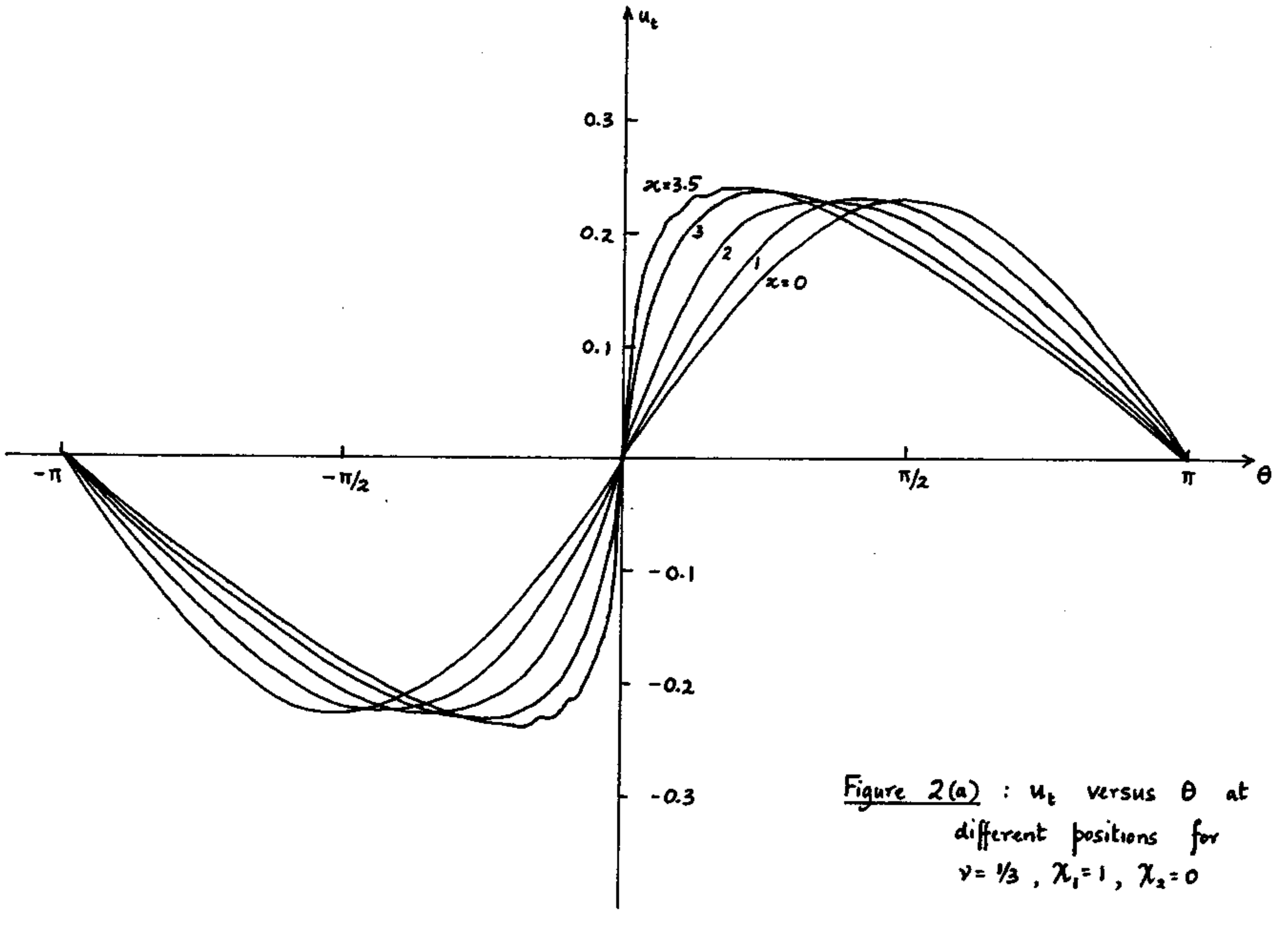


Figure 2(a) : u_t versus θ at different positions for $\nu = 1/3$, $\chi_1 = 1$, $\chi_2 = 0$

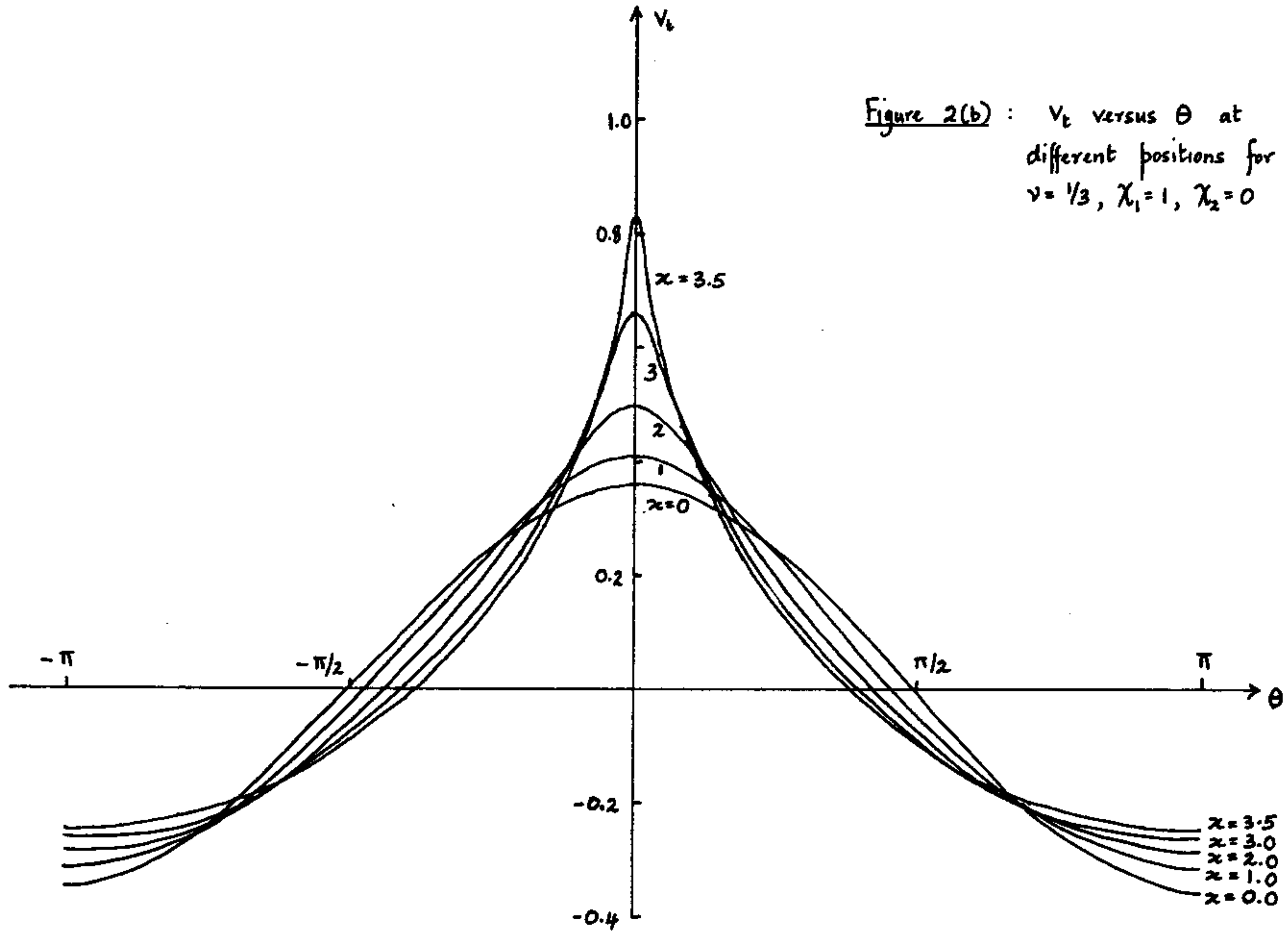
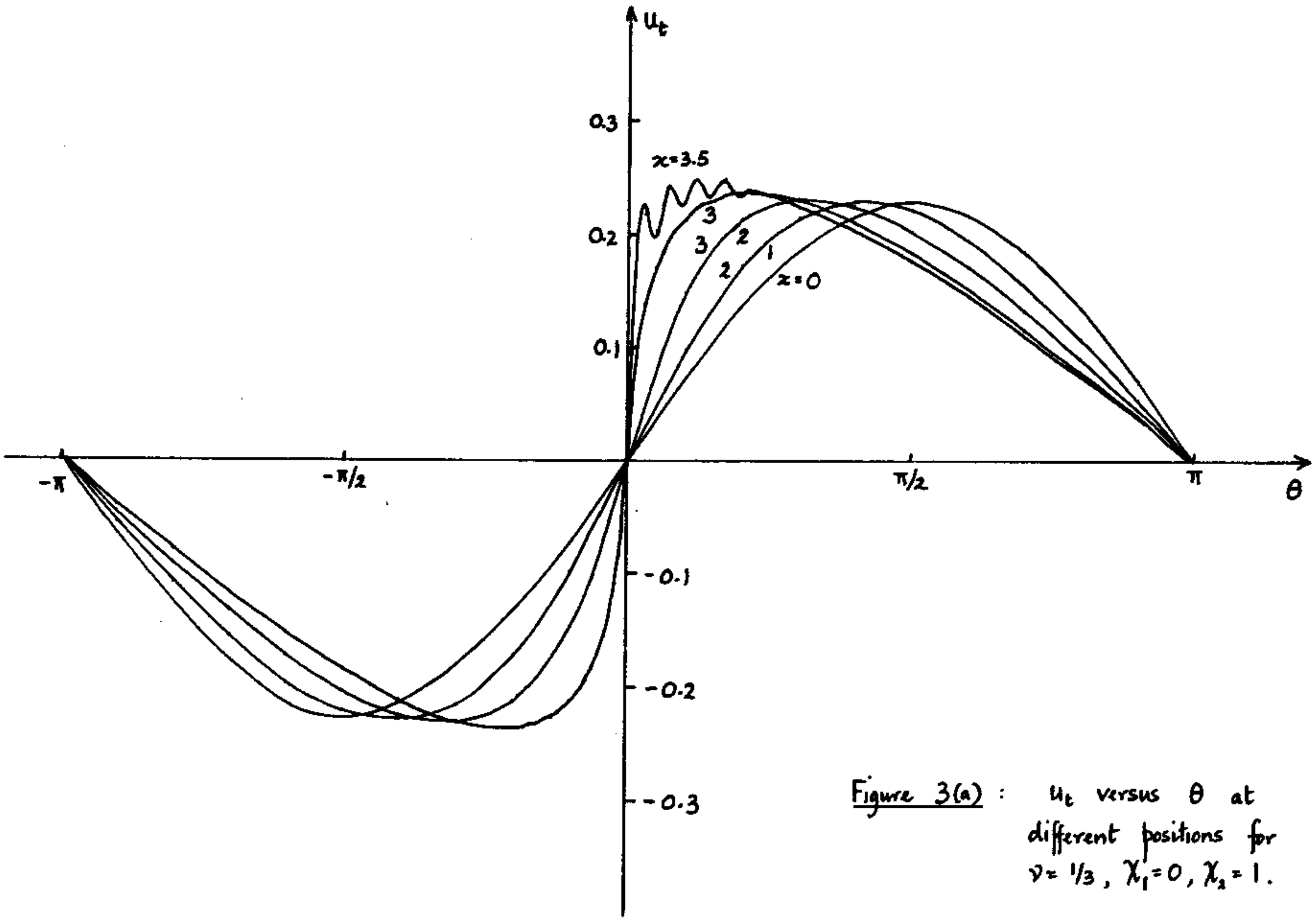


Figure 2(b) : V_t versus θ at different positions for $\nu = 1/3$, $\chi_1 = 1$, $\chi_2 = 0$



<u>Figure 3(a)</u> : u_t versus θ at different positions for $\nu = 1/3$, $\chi_1 = 0$, $\chi_2 = 1$.

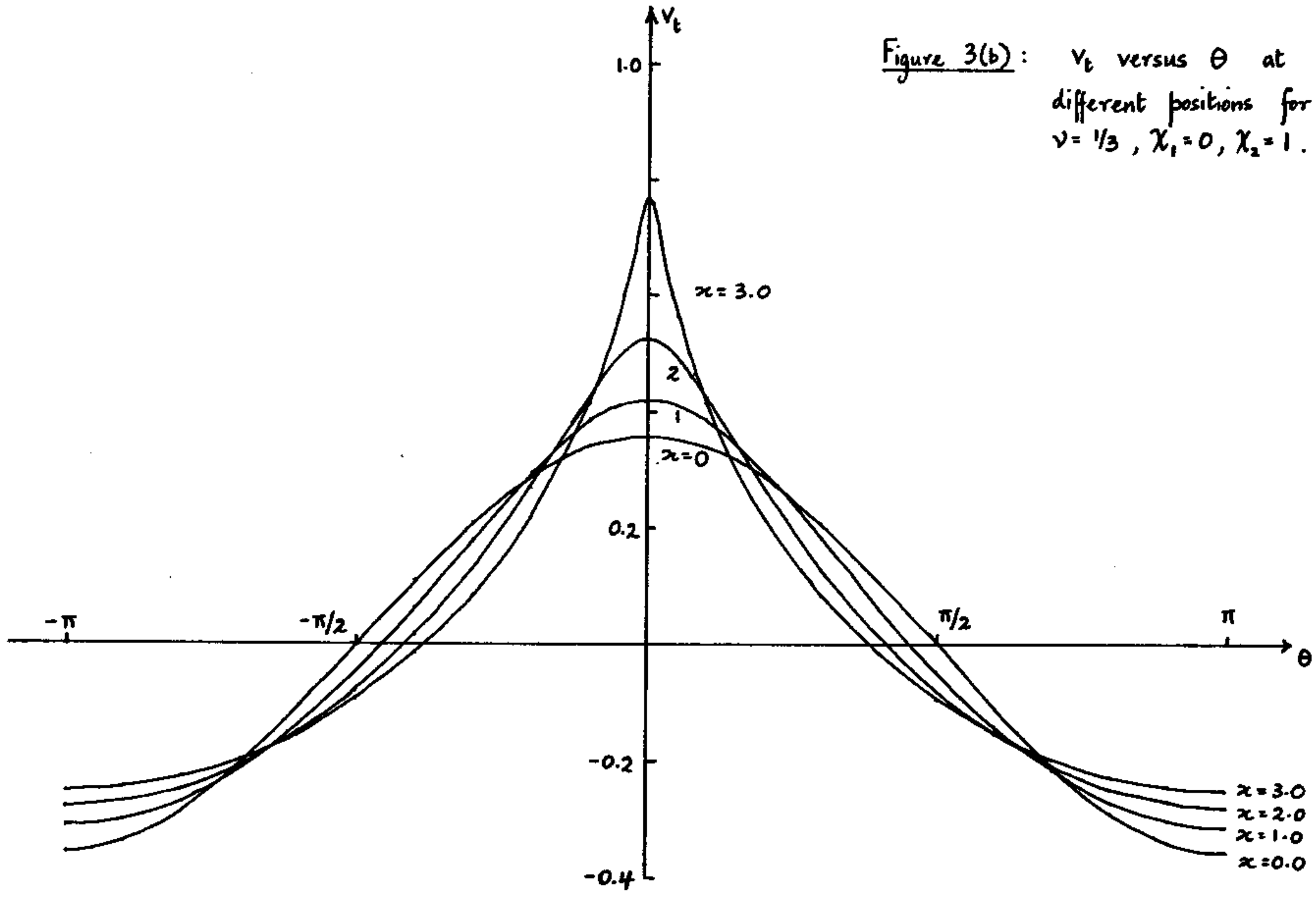


Figure 3(b): v_t versus θ at different positions for $\nu = 1/3$, $\chi_1 = 0$, $\chi_2 = 1$.