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A Critical Analysis of Bell's Proof Against Local Hidden Variable Theories

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A CRITICAL ANALYSIS OF BELL'S PROOF AGAINST LOCAL HIDDEN VARIABLE THEORIES

The underlying physical assumptions of Bell's proof are analysed. It is shown that a key assumption of his proof is not valid for realistic hidden variable theories but only for "classical" theories already shown to be invalid by Von Neumann's proof against hidden variable theories.

Quantum theory is unusual among physical theories in that while only statistical predictions can be made about observables of quantum systems, actual measurements can be performed on individual quantum structures. This differs from, say, classical statistical mechanics where both prediction and observation refer to ensembles of systems. This has led to the suggestion that quantum mechanics is not a complete theory. If so, the statistical nature of its predictions could be eliminated, at least theoretically, if certain "hidden variables" could be specified about a quantum system in addition to the information contained in the quantum state vector. A number of such hidden variable theories have been proposed.[1,2,3]

Concomitant with these theories, however, have come alleged proofs of the incompatibility of hidden variables with quantum results. The classical work in this regard is Von Neumann's argument against dispersion free ensembles.[4]

By considering expectation values of non-commuting operators, Von Neumann demonstrated that hidden variables in the classical sense, i.e. variables which are attached to the system and remain unchanged by measurement cannot reproduce quantum predictions. Such a classical hidden variable theory, which uses measurement as a filter to identify non-identical elements of a mixed ensemble without at the same time altering the values of the variables describing those elements ultimately leads to dispersion free ensembles, something which is easily demonstrated to be impossible in quantum terms for observables described by non-commuting operators.

If we remove this restriction on measurement and consider the necessity for measuring apparatus and system to interact in any measurement process, then we must allow the hidden variables to be altered by measurement and hidden variable theory is no longer excluded by Von Neumann's proof. Such a realistic hidden variable theory for electron spin was proposed by Bell.[5]

It is important to note that the Stern-Gerlach measurement of spin alters the value of Bell's hidden parameter. Thus, further Stern-Gerlach measurements after the initial measurement will in general result in different results than if these measurements were performed without the intermediary of the first measurement.

This crucial point seems to be overlooked by Bell when he next examines the possibility of local hidden variable theory explaining correlated composite quantum systems. For his model, Bell takes a spin zero unstable system which decays into two spin ½ particles (labeled U and V) which are then subjected to Stern-Gerlach apparatus (SGA) at different spatial locations. If we let $A(\hat{a},\lambda)$ be the measurement result (in units of $\hbar/2$) of an SGA whose field gradient is oriented in the $\hat{a}$ direction on electron U with hidden variable λ and $B(\hat{b},\lambda)$ be the measurement result of an SGA oriented in the $\hat{b}$ direction on electron V with similar hidden parameter λ, then it is possible to define the mean value product $P(\hat{a},\hat{b})$:

$$P(\hat{a},\hat{b}) = \int d\lambda\ \rho(\lambda)\ A(\hat{a},\lambda)\ B(\hat{b},\lambda) \equiv \langle A(\hat{a})\ B(\hat{b}) \rangle_\lambda$$

where $\rho(\lambda)$ is the density distribution of λ.
From this it follows that

$$P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}') = \langle A(\hat{a})\ B(\hat{b}) - A(\hat{a})\ B(\hat{b}') \rangle_\lambda$$

$$= \langle A(\hat{a})\ B(\hat{b}) - A(\hat{a})\ B(\hat{b}') \rangle_\lambda \pm \langle A(\hat{a})\ B(\hat{b})\ A(\hat{a}')\ B(\hat{b}') \rangle_\lambda \mp \langle A(\hat{a})\ B(\hat{b})\ A(\hat{a}')\ B(\hat{b}') \rangle_\lambda$$

$$= \langle A(\hat{a})\ B(\hat{b})\ [1 \pm A(\hat{a}')\ B(\hat{b}')] \rangle_\lambda$$

$$- \langle A(\hat{a})\ B(\hat{b}')\ [1 \pm A(\hat{a}')\ B(\hat{b})] \rangle_\lambda$$

Since A, B are the results of SGA measurements they will have only the values ±1.[6] Therefore the following inequality may be constructed

$$|P(\hat{a},\hat{b}) - P(\hat{a}',\hat{b}')| \leq \langle 1 \pm A(\hat{a}')\, B(\hat{b}')\rangle_\lambda + \langle 1 \pm A(\hat{a}')\, B(\hat{b})\rangle_\lambda$$

or

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| \leq 2 \pm [P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})]$$

For the spin zero composite system U+V, quantum mechanics predicts that $P(\hat{a},\hat{b})=-(\hat{a}\cdot\hat{b})$. Therefore the inequality may finally be written as

$$|(\hat{a}\cdot\hat{b}) - (\hat{a}\cdot\hat{b}')| \leq 2 \pm [(\hat{a}'\cdot\hat{b}') + (\hat{a}'\cdot\hat{b})]$$

It is possible by properly selecting the values of $\hat{a},\hat{a}', \hat{b},\hat{b}'$ to construct a situation where this inequality is violated. From this Bell concludes that any local hidden variable theory could not describe quantum mechanics.

The implications of this result are extremely significant. H.P. Stapp[8] labels it "the most profound discovery of science". Fox[9] suggests it might imply the existence of tachyons. Actually Bell's theorem contains explicit assumptions which allow it to reject not realistic local hidden variable theories but only classical models already shown to be false by Von Neumann's proof. If we look at Bell's (generalized) proof we see it depends on the addition of the term

$$\langle Q\rangle_\lambda = \int d\lambda\ \rho(\lambda)\ Q(\lambda) = \int d\lambda\ \rho(\lambda)[A(\hat{a},\lambda)\ B(\hat{b},\lambda)\ A(\hat{a}',\lambda)\ B(\hat{b}',\lambda)]$$

but this term is meaningless for any realistic hidden variable model since a B measurement of the V electron with a SGA oriented in the $\hat{b}$ direction will change the value of λ that will determine the value of B when measured by the SGA oriented in the b' direction. There will be no single λ value for the four measurements as implied in this term unless we return to the classical model which implies no change in λ through measurement, i.e. measurement as a filtration process. This, of course, has been shown to be invalid. A more proper definition of $\langle Q\rangle_\lambda$ might be

$$\langle Q\rangle_\lambda = \int d\lambda\ \rho(\lambda)\ A_1(\hat{a},\lambda)\ B_1(\hat{b},\lambda)\ A_2(\hat{a},\lambda,\hat{a}',\lambda')\ B_2(\hat{b},\lambda,\hat{b}',\lambda'')$$

$$= \langle\ A_1(\hat{a})\ B_1(\hat{b})\ A_2(\hat{a},\hat{a}')\ B_2(\hat{b},\hat{b}')\ \rangle_\lambda$$

where 1(2) refers to the first (second) measurement on the U or V electron and λ' and λ'' refer to the new values of the hidden variable. Since A_2 measurement is performed after a SGA measurement in the $\hat{a}$ direction, its value will in general depend not only on $\hat{a}'$ but also on $\hat{a}$. A similar condition applies to B_2.

If $\langle Q \rangle_\lambda$ is defined in this manner, as it must be for any realistic hidden variable theory, Bell's inequality cannot be derived since

$$P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}') = \langle A_1(\hat{a})\ B_1(\hat{b})\rangle_\lambda - \langle A_1(\hat{a})\ B_2(\hat{b},\hat{b}')\rangle_\lambda$$

$$= \langle A_1(\hat{a})\ B_1(\hat{b})[1 \pm A_2(\hat{a},\hat{a}')\ B_2(\hat{b},\hat{b}')]\rangle_\lambda$$

$$- \langle A_1(\hat{a})\ B_2(\hat{b},\hat{b}')[1 \pm A_2(\hat{a},\hat{a}')\ B_1(\hat{b})]\rangle_\lambda$$

Therefore

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| \leq \langle 1 \pm A_2(\hat{a},\hat{a}')\ B_2(\hat{b},\hat{b}')\rangle_\lambda + \langle 1 \pm A_2(\hat{a},\hat{a}')\ B_1(\hat{b})\rangle_\lambda$$

$$\leq 2 \pm [\langle A_2(\hat{a},\hat{a}')\ B_2(\hat{b},\hat{b}')\rangle_\lambda + A_2(\hat{a},\hat{a}')\ B_1(\hat{b})\rangle_\lambda]$$

But

$$\langle A_2(\hat{a},\hat{a}')\ B_2(\hat{b},\hat{b}')\rangle_\lambda \neq \langle A_1(\hat{a}')\ B_1(\hat{b}')\rangle_\lambda = -(\hat{a}'\cdot\hat{b}')$$

since the measurements at $\hat{a},\hat{b}$ destroyed the correlation between the spins of the two electrons U,V. Similarly

$$\langle A_2(\hat{a},\hat{a}')\ B_1(\hat{b})\rangle_\lambda \neq \langle A_1(\hat{a}')\ B_1(\hat{b})\rangle_\lambda = -(\hat{a}'\cdot\hat{b})$$

Hence the inequality cannot be constructed.

There is a circumstance where $Q(\lambda)$ can be defined in the sense used by Bell. This would occur if we let λ, instead of being a hidden variable, refer to the trial number of a series of SGA measurements on an ensemble of 2-electron systems. Then if we have four SGA oriented in the $\hat{a},\hat{a}',\hat{b},\hat{b}'$ directions, we could define $Q(\lambda)$ as simply the product of the four measurements for the coupled system of electrons on the λth trial. Then we can rewrite Bell's proof as follows

$$N[P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')] = \sum_{\lambda=1}^{N} A(\hat{a},\lambda)\, B(\hat{b},\lambda) - \sum_{\lambda=1}^{N} A(\hat{a},\lambda)\, B(\hat{b}',\lambda)$$

where $A(\hat{a},\lambda)$ refers to the measurement result of the λth trial on electron 1 through a SGA oriented in the $\hat{a}$ direction. If we add and subtract $\pm \sum_{\lambda=1}^{N} Q(\lambda)$ to the right side and factor we get

$$\sum_{\lambda} A(\hat{a},\lambda)\, B(\hat{b},\lambda) - \sum_{\lambda} A(\hat{a},\lambda)\, B(\hat{b}'\lambda) = \sum_{\lambda} A(\hat{a},\lambda)\, B(\hat{b},\lambda)[1 \pm A(\hat{a}',\lambda)\, B(\hat{b}',\lambda)]$$

$$- \sum_{\lambda} A(\hat{a},\lambda)\, B(\hat{b}',\lambda)[1 \pm A(\hat{a}',\lambda)\, B(\hat{b},\lambda)]$$

Now the results of any measurement at any of the SGA can be only ± 1 (in units of $\hbar/2$). Therefore we can write the following inequality

$$\left|\sum_{\lambda} A(\hat{a},\lambda)\, B(\hat{b},\lambda) - \sum A(\hat{a},\lambda)\, B(\hat{b}',\lambda)\right| \leq 2N \pm [\sum_{\lambda} A(\hat{a}',\lambda)\, B(\hat{b}',\lambda) + \sum_{\lambda} A(\hat{a}',\lambda)\, B(\hat{b},\lambda)]$$

Consider the ith trial (λ=i). Then take the more strict inequality

$$|A(\hat{a},i)\, B(\hat{b},i) - A(\hat{a},i)\, B(\hat{b}',i)| \leq 2 \pm [A(\hat{a}',i)\, B(\hat{b}',i) + A(\hat{a}',i)\, B(\hat{b},i)]$$

The reader is free to choose any combination of values ± 1 for A,B and he will find that the inequality is always satisfied. Therefore the more general preceding inequality is also always satisfied. But (as we let $N \to \infty$ and divide by N), this is the same as Bell's inequality. Hence, it would seem that not only local hidden variable theories but also experimental results could not reproduce the predictions of quantum mechanics! The solution of this dilemma, of course, is to realize that Bell's inquality combines several different ensembles of measurement results; $P(\hat{a}',\hat{b})$ refers to measurements performed on an ensemble of systems (U,V) with two SGA, one in the $\hat{a}'$ direction and one in the $\hat{b}$ direction. $P(\hat{a}',\hat{b}')$ refers to a different ensemble of measurement results with the 2 SGA in different directions. This same interpretation must be applied for any realistic hidden variable theory and then the derivation of the inequality, involving as it does terms such as $\langle Q\rangle_\lambda$, becomes invalid.

1. D. Bohm, Phys. Rev. 85; 166 (1952).

2. D. Bohm and J. Bubb, Rev. Mod. Phys. 38; 453 (1966).

3. L. de Broglie, Non-Linear Wave Mechanics, A Casual Interpretation (Elsevier, Amsterdam, 1960).

4. J. Von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1955); chapter IV, secs. 1 and 2.

5. J.S. Bell, Rev. Mod. Phys. 38; 447 (1966).

6. Bell considers the more general case where only (A) $\leq$ 1, (B) $\leq$ 1 but this generalization is not needed in what follows.

7. This is a more general form of Bell's original inequality and is due to J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. 26; 880 (1969).

8. H.P. Stapp, Il Nuouo Cimento 29; 270 (1975).

9. Fox, Phys. Rev. D 5; 329 (1972).