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THE DIRICHLET PROBLEM WITH DENJOY-PERRON INTEGRABLE BOUNDARY CONDITION

M. Benedicks and W.F. Pfeffer*)

Dedicated to the memory of
Professor J.B. Diaz

Abstract. The Poisson integral of a Denjoy-Perron integrable function defined on the boundary of an open disc is harmonic in this disc. Moreover, almost everywhere on the boundary, the nontangential limits of the integral coincide with the boundary condition. This extends the classical result for Lebesgue integrable boundary conditions. By means of conformal maps, a generalization to domains bounded by a sufficiently smooth Jordan curve is also obtained.

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By R and C we shall denote the sets of all real and complex numbers, respectively. We let $D = \{z \in C: |z| < 1\}$ and $T = \{z \in C: |z| = 1\}$.

It is well known that a convolution of the Poisson kernel with a Lebesgue integrable function f on T produces a harmonic function in D whose nontangential limits are equal to f almost everywhere on T (see, e,g. [T, thm.IV.1]). By a limiting procedure this result can be extended to the case when f has only an improper Lebesgue integral. While the idea of such an extension is straightforward, the actual proof, which involves a switch of iterated limits, is unpleasant. How to proceed when the singularities of f cluster so that the improper Lebesgue integral does not exist (see later Example) is unclear.

Since the Denjoy-Perron integral (abbreviated as DP integral) generalizes the improper Lebesgue integral and is, in fact, closed with respect to the formation of improper integrals, it seems appropriate to consider functions which are DP integrable on T. We are obliged to J.B. Diaz for suggesting this to the second author. The second author is also obliged to A.K. Lyzzaik for several stimulating discussions.

To convince the reader that using the DP integral is not extravagant, we briefly recall the Kurzweil and Henstock definition (see [K] and [H]), which shows that the DP integral is only a very natural generalization of the classical Riemann integral.

A partition of an interval $[a,b] \subset R$ is a set

$$\{t_0, \ldots, t_n; \ \tau_1, \ldots, \tau_n\}$$

such that $a = t_0 < \ldots < t_n = b$, and $t_{j-1} \leq \tau_j \leq t_j$ for $j = 1, \ldots, n$. If δ is a strictly positive function on $[a,b]$, we say that the partition $\{t_0, \ldots, t_n; \ \tau_1, \ldots, \tau_n\}$ of $[a,b]$ is δ-fine whenever $t_j - t_{j-1} < \delta(\tau_j)$ for $j = 1, \ldots, n$. A function $f:[a,b] \to R$ is called DP integrable if there is a real number $\int_a^b f(t)dt$ such that given $\varepsilon > 0$, we can find a positive function δ on $[a,b]$ with

$$\left|\sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_a^b f(t)dt\right| < \varepsilon$$

for each δ-fine partition $\{t_0, \ldots, t_n; \tau_1, \ldots, \tau_n\}$ of $[a,b]$.

Basic properties of the DP integral are derived from the above definition in $[P_1]$. They include the relationship between the DP and Lebesgue integrals, and also the equivalence of the Kurzweil-Henstock and Perron definitions of the DP integral.

Since DP integrable functions are in general not absolutely integrable, the usual maximal function approach to the Dirichlet problem (see [G, Chpt.2, Sec.3]) cannot be applied. However, following the proof of [T, Thm.IV.1], which essentially goes back to Fatou, we show that a judicious use of integration by parts for the DP integral (see $[P_2]$ for a simple proof) yields a result completely analogous to the classical one. The critical property we employ is that almost everywhere the derivative of an indefinite DP integral exists and equals the integrand (see [S, Chpt. VI, Thm. (6.1)]).

The usefulness of our result is illustrated by an example of a DP integrable function which appears unmanageable within the limits of Lebesgue integration.

Throughout this paper, integrability always means DP integrability, and all integrals are DP integrals. A function $f:T \to R$ is called integrable if the integral $\int_{-\pi}^{\pi} f(e^{i\theta})d\theta$ exists.

The <u>Poisson kernel</u> is the positive function

$$P_r(t) = \frac{1}{2\pi} \cdot \frac{1 - r^2}{1 - 2r\cos t + r^2}$$

defined for $0 \leq r < 1$ and $t \in R$. Instead of $(\partial/\partial t)P_r(t)$ we shall write $P_r'(t)$. We note that by associating $P(z) = P_r(t)$ to each $z = r\,e^{it}$ in D, we can define a map $P:D \to R$.

<u>Proposition 1</u>. Let $f:T \to R$ be an integrable function. Then the integral

$$u(r\,e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\; f(e^{it})\; dt$$

exists for each $(r,\theta) \in [0,1) \times R$, and the function u is harmonic in D.

Proof. For $t \in R$, let $F(t) = \int_{-\pi}^{t} f(e^{i\theta})d\theta$. Integrating by parts (see [P_2]), we see that the Poisson integral defining the function u exists, and that

$$u(r\,e^{i\theta}) = P_r(\theta - \pi)\,F(\pi) + \int_{-\pi}^{\pi} P_r'(\theta - t)\,F(t)\,dt.$$

Now F is a continuous function, and P_r and P_r' are harmonic in D. Hence the proposition follows by differentiation under the integral sign (note that this is the Lebesgue integral).

For the remainder of this note, we shall assume that $f: T \to R$ is an integrable function. For $(r,t) \in [0,1) \times R$ we let

$$F(t) = \int_0^t f(e^{i\theta})\,d\theta$$

$$P{*}f(r\,e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\,f(e^{it})\,dt.$$

Proposition 2. Let $\theta_0 \in [-\pi,\pi]$ be such that $F'(\theta_0) = f(e^{i\theta_0})$. If $u = P{*}f$, then

$$u(r\,e^{i\theta}) \to f(e^{i\theta_0})$$

whenever $r \to 1-$, and there is a $c > 0$ such that $|\theta - \theta_0| \leq c(1 - r)$.

Proof. It suffices to prove the special case of $f(e^{i\theta_0}) = \theta_0 = 0$; for this yields the general result when applied to the function

$$g(z) = f(z\,e^{i\theta_0}) - f(e^{i\theta_0}).$$

By the continuity of F,

$$M = \sup\{|F(t)| : |t| \leq \pi\} < +\infty .$$

Choose $c > 0$, $\varepsilon > 0$, and let

$$\eta = \varepsilon(5M + c + 1)^{-1} .$$

As $F'(0) = 0$, there is a $\delta \in (0, \pi/2)$ such that $|F(t)| \leq \eta|t|$ for each $t \in [-\delta,\delta]$. Find a $\rho \in (0,1)$ so that $c(1 - \rho) \leq \delta/2$ and $P_r(t) \leq \eta$ whenever $\rho \leq r < 1$ and $\delta/2 \leq |t| \leq \pi$. We show that if $\rho \leq r < 1$ and $|\theta| \leq c(1 - r)$, then $|u(r\, e^{i\theta})| \leq \varepsilon$.

Let $\rho \leq r < 1$ and $|\theta| \leq c(1 - r) \leq \delta/2$. An integration by parts yields

$$\begin{aligned} |u(r\, e^{i\theta})| = |P_r(\theta - \pi)\ F(\pi) - P_r(\theta + \pi)\ F(-\pi) \\ + \textstyle\int_{-\pi}^{\pi} P_r'(\theta - t)\ F(t)\ dt| \leq 2M\eta \\ + M\textstyle\int_{\delta\leq|t|\leq\pi} |P_r'(\theta - t)|dt + \eta\int_{-\delta}^{\delta} |P_r'(\theta - t)t|dt . \end{aligned}$$

Observing that $P_r'(t)\sin t \leq 0$, we shall estimate each integral separately. Since $P_r'(-t) = -P_r'(t)$, both integrals are even functions of θ, and hence it suffices to estimate them for $\theta \geq 0$. Given $\theta \geq 0$, we have

$$\begin{aligned} \textstyle\int_{\delta\leq|t|\leq\pi} |P_r'(\theta - t)|dt = \int_{\delta}^{\pi} P_r'(\theta - t)dt + \int_{-\pi}^{-\pi+\theta} P_r'(\theta - t)dt \\ - \textstyle\int_{-\pi+\theta}^{-\delta} P_r'(\theta - t)dt = P_r(\theta - \pi) + 2P_r(\pi) \\ - P_r(\theta - \delta) - P_r(\theta + \pi) - P_r(\theta + \delta) \leq 3\eta, \end{aligned}$$

and similarly

$$\int_{-\delta}^{\delta} |P_r'(\theta - t)t\, dt| = \int_{-\delta}^{0} P_r'(\theta - t)t\, dt - \int_0^{\theta} P_r'(\theta - t)t\, dt$$

$$+ \int_{\theta}^{\delta} P_r'(\theta - t)t\, dt = 2\theta P_r(0) - \delta P_r(\theta + \delta) - \delta P_r(\theta - \delta)$$

$$+ \int_{-\delta}^{0} P_r(\theta - t)dt - \int_0^{\theta} P_r(\theta - t)dt + \int_{\theta}^{\delta} P_r(\theta - t)dt$$

$$\leq 2\theta P_r(0) + \int_{-\pi}^{\pi} P_r(\theta - t)dt$$

$$= \frac{\theta}{\pi}\frac{1 + r}{1 - r} + 1 \leq c + 1 \;;$$

for $1 + r < 2$, and $\theta < c(1 - r)$. Combining the previous estimates, we see that

$$|u(r\, e^{i\theta})| \leq \eta(5M + c + 1) = \varepsilon,$$

and the proposition is proved.

The convergence $u(r\, e^{i\theta}) \to f(e^{i\theta_0})$ described in Proposition 2 is usually referred to as <u>nontangential convergence</u>. Using [S, Chpt. VI, Thm. (6.1)], we can summarize our results.

<u>Theorem 1</u>. If f is a DP integrable function on T, then $P*f$ is harmonic in D, and it converges nontangentially to f almost everywhere in T.

<u>Example</u>. Let $I = (a,b)$ with $a,b \in R$ and $a < b$. Set

$$F_I(t) = (t-a)^2 \cdot (t-b)^2 \cdot \sin \frac{\pi}{2} \left(\frac{b-a}{t-a}\right)^2 \cdot \sin \frac{\pi}{2} \left(\frac{b-a}{t-b}\right)^2$$

if $t \in I$, and $F_I(t) = 0$ if $t \in R - I$. Then the finite derivative $F_I'(t)$ exists for each $t \in R$, and it is easy to see that F_I' is not Lebesgue integrable over I (though, the improper Lebesgue integral exists).

Let K be a Cantor discontinuum in the interval $[-\pi,\pi]$, and let I_n, $n = 1,2,\ldots,$ be the connected components of $(-\pi,\pi) - K$. As

$$|F_I(t)| \leq (t-a)^2(t-b)^2 ,$$

it is not difficult to check that the function

$$F = \sum_{n=1}^{\infty} F_{I_n}$$

has a finite derivative F'. Now define $f:T \to R$ by setting $f(z) = F'(\arg z)$ for each $z \in T$. Then f is DP integrable, and

$$F(t) = \int_{-\pi}^{t} f(e^{i\theta})d\theta$$

for each $t \in R$ (see [P_1, Thm. A_3]). Consequently, $P{*}f$ is a harmonic function in D whose nontangential limits are equal to f everywhere in T.

However, $f(e^{it})$ is not Lebesgue integrable in any open interval containing points of K. In particular, $f(e^{it})$ does not have an improper Lebesgue integral over $[-\pi,\pi]$.

Remark 1. Following our proofs, it is easy to verify that Theorem 1 holds for each function $f:T \to R$ for which we can find a bounded measurable function F with $F'(t) = f(e^{it})$ for almost all $t \in [-\pi,\pi]$. Indeed, we only need to define $P*f$ in the distribution sense, i.e., by setting

$$P*f(r\,e^{i\theta}) = P_r(\theta - \pi)\ F(\pi) + \int_{-\pi}^{\pi} P_r'(\theta - t)\ F(t)dt$$

for each $r\,e^{i\theta} \in D$. This, of course, links with the standard fact of harmonic analysis that the convolution of P and a tempered distribution is harmonic in D. However, it is the important feature of DP integrable functions that the convolution $P*f$ can be defined directly.

Using conformal maps, Theorem 1 can be extended to more general domains. We shall describe this next.

As usual, we say that a map $\gamma:T \to C$ is of class $C^{1,\varepsilon}$, $0 < \varepsilon \leq 1$, if it has a derivative which is Lipschitz of order ε (see [Du, Sec. 5.1]).

Let Ω be a Jordan domain of class $C^{1,1}$, i.e., a bounded open subset of C whose boundary $\partial\Omega$ is a Jordan curve of class $C^{1,1}$, and let $\gamma:[0,a] \to \partial\Omega$ be a counter-clockwise parametrization of $\partial\Omega$ by arclength. A function $f:\partial\Omega \to R$ is called integrable if the integral $\int_0^a f[\gamma(s)]ds$ exists. By [P_1, Coroll. A6], the integrability of f does not depend on the parametrization γ of $\partial\Omega$.

By the Riemann mapping theorem and the Caratheodory extension theorem (see [T, Thm. IX.2]), there is a homeomorphism $\phi: D \cup T \to \Omega \cup \partial\Omega$ such that $\phi(T) = \partial\Omega$ and ϕ is conformal in D. Each such homeomorphism is called a <u>Riemann map</u>.

<u>Theorem 2</u>. Let Ω be a Jordan domain of class $C^{1,1}$, let f be a DP integrable function on $\partial\Omega$, and let $\phi: D \cup T \to \Omega \cup \partial\Omega$ be a Riemann map. Then $f \circ \phi$ is a DP integrable function on T. Moreover, $[P * (f \circ \phi)] \circ \phi^{-1}$ is harmonic in Ω, and it converges nontangentially to f almost everywhere in $\partial\Omega$.

<u>Proof</u>. Let $\gamma:[0,a] \to \partial\Omega$ be a counter-clockwise parametrization of $\partial\Omega$ by arclength such that $\gamma(0) = \gamma(a) = \phi(-1)$. By Kellog's theorem (see [T, Thm. IX.7.]), the derivative ϕ' can be extended continuously from D to $D \cup T$, and this extension (also denoted by ϕ') is nowhere equal to zero. Letting

$$s(\theta) = \int_{-\pi}^{\theta} |\phi'(e^{it})| dt ,$$

we have $\gamma[s(t)] = \phi(e^{it})$ for each $t \in [-\pi,\pi]$. The change of variable theorem (see [P_1, Coroll. A6]) yields

$$\begin{aligned}\int_0^a f[\gamma(s)]ds &= \int_{-\pi}^{\pi} f \circ \gamma[s(t)]\, s'(t)dt \\ &= \int_{-\pi}^{\pi} f \circ \phi(e^{it}) |\phi'(e^{it})| dt ,\end{aligned}$$

and so $f \circ \phi \cdot |\phi'|$ is integrable. The integrability of $f \circ \phi$ is now a consequence of the following claim.

<u>Claim</u>. The function $\phi'(e^{it})$ is absolutely continuous in $[-\pi,\pi]$.

Indeed, as $|\phi'| > 0$ is continuous in $D \cup T$, the claim implies that $|\phi'(e^{it})|^{-1}$ has a finite variation in $[-\pi,\pi]$. By $[P_2$, Propos.], this guarantees the integrability of

$$f \circ \phi = [f \circ \phi \cdot |\phi'|] \cdot |\phi'|^{-1} .$$

Now giving the obvious meaning to "nontangential convergence" and "almost everywhere" in $\partial\Omega$, and using the conformity of ϕ, Theorem 2 follows from Theorem 1.

It remains to establish the claim.

<u>Proof of the Claim</u>. For $|z| \leq 1$, let

$$G(z) = \log \phi'(z) = u(z) + iv(z)$$

where $u(z) = \log|\phi'(z)|$, and $v(z) = \arg \phi'(z) = -i \log \frac{\phi'(z)}{|\phi'(z)|}$.
If $g(t) = v(e^{it})$, $t \in [-\pi,\pi]$, then by [Du, Sec.1.1],

$$v(r\, e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\, g(t)\, dt$$

for each $z = r\, e^{i\theta}$ in D. By differentiating the equation

$$\gamma[s(t)] = \phi(e^{it}) ,$$

$t \in [-\pi,\pi]$, we obtain

$$\gamma'[s(t)] \cdot |\phi'(e^{it})| = i\, e^{it}\, \phi'(e^{it}) ,$$

and hence

$$g(t) = -i \log\{-i\, e^{-it}\, \gamma'[s(t)]\} .$$

Since γ is of class $C^{1,1}$ and $s'(t) = |\phi'(e^{it})|$ is bounded, we see that g is Lipschitz of order 1. In particular, g has a bounded derivative g' almost everywhere in $[-\pi,\pi]$. For $z = r\, e^{i\theta}$ in D, we have

$$\begin{aligned} iz\frac{\phi''(z)}{\phi'(z)} = iz\, G'(z) &= \frac{\partial}{\partial\theta} G(z) \\ &= \frac{\partial}{\partial\theta} u(z) + i \int_{-\pi}^{\pi} P_r'(\theta - t)\, g(t)\, dt \\ &= \frac{\partial}{\partial\theta} u(z) + i \int_{-\pi}^{\pi} P_r(\theta - t)\, g'(t)\, dt. \end{aligned}$$

As g' is essentially bounded,

$$w(r\, e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\, g'(t)\, dt$$

is bounded in D, and hence w is of Hardy class $h^\infty \subset h^2$ (for the definition of Hardy's classes, we refer to [Du, Sec.1.1]). An application of M. Riesz' theorem (see [Du, Thm. 4.1]) shows that G', and hence $\phi'' = \phi' G'$ is in Hardy class $H^2 \subset H^1$. Now the claim follows from [Du, Thm. 3.11].

<u>Remark 2</u>. If the function f is Lebesgue integrable on $\partial\Omega$, then it is easy to see that Theorem 2 holds whenever Ω is a Jordan domain of class $C^{1,\varepsilon}$. Recently, it was shown in [D_1], [D_2], and by

a different method in [FJR], that $C^{1,\epsilon}$ can be relaxed to C^1 (i.e., continuously differentiable boundary with no Lipschitz conditions imposed) if $|f|^p$ is Lebesgue integrable on $\partial\Omega$ for some $p > 1$. If f is merely DP integrable, this generality is not possible for the following reason: a function which is Lipschitz of order $\epsilon < 1$ may be of infinite variation, and hence its product with a DP integrable function need not be DP integrable.

Remark 3. If $\partial\Omega$ is only piecewise $C^{1,1}$ it is not difficult to show that Theorem 2 still holds provided Ω is locally convex at each corner of $\partial\Omega$. The same condition must be imposed even if the function f is Lebesgue integrable on $\partial\Omega$.

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