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ON THE GEOMETRY OF THE SET OF SOLUTIONS OF A DISCRETE-TIME QUADRATIC MATRIX INEQUALITY[+]

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ABSTRACT. In this paper, we consider the set (P) of all solutions of a Quadratic Matrix Inequality (QMI) that arises in the discrete-time stochastic realization theory and the subset (P_0) of all solutions of the corresponding Algebraic Riccati Equation (ARE). A complete characterization of some subsets of P and P_0 is given. The boundary, interior and extreme points of P are identified. It is shown that elements of P_0 are not only boundary points of the convex compact set P, but are extreme points as well.

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1. Introduction

Quadratic matrix inequalities (QMI) are of central importance in Linear Systems Theory and arise in the study of many specific problems in this theory. Corresponding to every (QMI), is an algebraic Riccati equation (ARE) which is also of considerable importance.

In this paper, we consider a (QMI) and the corresponding (ARE) that arise in the discrete-time stochastic realization problem; the continuous-time counter parts were considered by Badawi (1982). The results in this paper may be considered, in some sense or another, a continuation of those obtained by Willems (1971), Finesso (1982), Faurre (1973) and Germain (1974).

To be specific and to introduce the inequality, we shall briefly review some facts from the discrete-time stochastic realization theory and we shall follow the presentation in Badawi (1980 and 1981) for this purpose.

Let $\{y(t);\ t \in Z\}$ $(Z = \{\text{integers}\})$ be an m-dimensional stationary and purely nondeterministic stochastic process with spectral density $\Phi(z)$ which is rational in z, analytic on the unit disc and para-hermitian. We shall also assume that $\Phi(e^{i\omega}) > 0\ \forall\ \omega \in \mathbb{R}$, and that $0 < \Phi(\infty) < \infty$. These assumptions are to avoid some technicalities which only obscure matters. The stochastic realization problem requires finding all systems of the type

$$x(t + 1) = A\,x(t) + B\,w(t) \tag{1.1a}$$

$$y(t) = C\,x(t) + D\,w(t) \tag{1.1b}$$

where A, B, C and D are constant matrices of dimensions $n \times n$, $n \times p$, $m \times n$ and $m \times m$ respectively, $n := \dim(A)$ is minimal, $|\lambda(A)| < 1$, such that the output process y has spectral density Φ. Each such model is called a stochastic realization of y, x the state and w is the input process of (1.1).

It can be easily seen (Anderson, 1969 and Lindquist, 1979) that this problem is equivalent to finding all spectral factors of Φ i.e., all matrices $W(z)$ of proper rational functions of minimal McMillan degree with all poles inside the unit circle and satisfying

$$\Phi(z) = W(z)\,W(z^{-1})' \,. \tag{1.2}$$

Let $[F, G, H, J]$ be a minimal realization (Brockett, 1970) of W. Then $|\lambda(F)| < 1$, (F, G) is controllable and (H, F) is observable.

Upon using coordinate transformation in the state space, all solutions to the above problem are given by

$$x(t + 1) = F\,x(t) + B_1 u(t) + B_2 v(t) \tag{1.3a}$$

$$y(t) = H\,x(t) + R(p)^{\frac{1}{2}} u(t) \tag{1.3b}$$

where $B = (B_1, B_2)$ and the state covariance matrix $P := E\{x(t)x(t)'\}$, together with R(P) satisfies the equations of the positive real lemma:

$$P = FPF' + B_1B_1' + B_2B_2' \tag{1.4a}$$

$$G = FPH' + B_1R(P)^{1/2} \tag{1.4b}$$

$$R(P) = J + J' - HPH' \tag{1.4c}$$

$$P = P' > 0 \tag{1.4d}$$

Let $\mathcal{P}$ be the set of all symmetric positive definite matrices P which solve (1.4). It can be easily seen that $\mathcal{P}$ consists of all solutions of the Quadratic Matrix Inequality (QMI):

$$\mathcal{P} = \{P = P' > 0 \,|\, \Lambda(P) \leq 0\}, \tag{1.5}$$

where

$$\Lambda(P) = -P + FPF' + (G - FPH')R(P)^{-1}(G - FPH')' \, . \tag{1.6}$$

Moreover, the set $\mathcal{P}$ contains as a subset a set $\mathcal{P}_0$, the elements of which are the solutions of the corresponding Algebraic Riccati Equation (ARE):

$$\mathcal{P}_0 = \{P \in \mathcal{P} \,|\, \Lambda(P) \equiv 0\}. \tag{1.7}$$

It was shown by Faurre (1973) and Germain (1974) that $\mathcal{P}$ is convex and compact and that there are two elements P_* and P^* in $\mathcal{P}_0$ such that $P_* \leq P \leq P^*$ for all $P \in \mathcal{P}$ and that $\mathcal{P}_0$ is the set of all solutions of (1.4) for which $B_2 = 0$.

Finally, to compute the minimum element P_* and the maximum element P^* of $\mathcal{P}$, the following algorithm, due to Faurre (1973) may be used.

Proposition 1.1. Let $\{\Pi(t);\ t \in Z^+\}$ $(Z^+ = \{0,1,2,\dots\})$ and $\{\overline{\Pi}(t),\ t \in Z^+\}$ be the solutions of the $n \times n$-matrix difference equations

$$\Pi(t+1) - \Pi(t) = \Lambda(\Pi(t))\ ; \qquad \Pi(0) = 0 \tag{1.8a}$$

$$\overline{\Pi}(t+1) - \overline{\Pi}(t) = \overline{\Lambda}(\overline{\Pi}(t))\ ; \qquad \overline{\Pi}(0) = 0\ , \tag{1.8b}$$

respectively, where Λ is given by (1.6) and $\overline{\Lambda}$ by

$$\overline{\Lambda}(P) = -P + F'PF + (H'-F'PG)(J+J'-G'PG)^{-1}(H'-F'PG)'\ . \tag{1.8c}$$

Then, $\Pi(t) \to P_*$ and $\overline{\Pi}(t)^{-1} \to P^*$ as $t \to \infty$.

In this paper, we shall study the structures of the sets $\mathcal{P}$ and $\mathcal{P}_0$ and characterise the elements of some subsets $\mathcal{P}^+$ and $\mathcal{P}^-$ of $\mathcal{P}$ and $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ of $\mathcal{P}_0$. In particular, we shall study the boundary, interior and extreme points of $\mathcal{P}$ and find connections between these and $\mathcal{P}_0$.

2. Characterization of the Sets $\mathcal{P}^+, \mathcal{P}^-, \mathcal{P}_0^+$ and $\mathcal{P}_0^-$.

Let $\mathcal{P}^+ = \{P \in \mathcal{P} | P > P_*\}$ and $\mathcal{P}^- = \{P \in \mathcal{P} | P < P^*\}$, where P_* and P^* are the minimum and maximum elements of $\mathcal{P}$. Since $\Phi(e^{i\omega}) > 0$, $P^* - P_* > 0$ (Germain, 1974) and consequently $\mathcal{P}^+$ and $\mathcal{P}^-$ are both nonempty. For each $P \in \mathcal{P}$, define the feedback matrix

$$\Gamma = F - (G - FPH')R(P)^{-1}H . \tag{2.1}$$

Let Γ_* and Γ^* be the feedback matrices corresponding to P_* and P^* respectively. Then $|\lambda(\Gamma_*)| < 1$ and $|\lambda(\Gamma^*)| > 1$ (Faurre, 1973) and Γ_* is nonsingular (Pavon, 1980). Moreover, let $\mathcal{P}_0^+ :=$ $\{P \in \mathcal{P}_0 | P > P_*\}$ and $\mathcal{P}_0^- := \{P \in \mathcal{P}_0 | P < P^*\}$.

The following lemma, the idea of the proof of which is due to (Morf, 1974), is of central importance to this paper.

Lemma 2.1. *Let P_1 and P_2 be arbitrary elements of $\mathcal{P}_0$ and let Γ_1 and Γ_2 be the corresponding feedback matrices* (2.1). *Then*

$$\Delta P = \Gamma_2 \Delta P \Gamma_1' = \Gamma_1 \Delta P \Gamma_2' \tag{2.2}$$

and

$$-\Delta P + \Gamma_1 \Delta P \Gamma_1' = -\Gamma_1 \Delta P H' R_2^{-1} H \Delta P \Gamma_1' \tag{2.3a}$$

$$-\Delta P + \Gamma_2 \Delta P \Gamma_2' = \Gamma_2 \Delta P H' R_1^{-1} H \Delta P \Gamma_2' \tag{2.3b}$$

where $\Delta P = P_2 - P_1$ *and* $R_i = R(P_i)$, $i = 1,2,$. *Moreover, if* $\Delta P > 0$, *then*

$$-(\Delta P)^{-1} + \Gamma_1'(\Delta P)^{-1}\Gamma_1 = -H'R_1^{-1}H \ , \tag{2.4a}$$

$$(\Delta P)^{-1} - \Gamma_2'(\Delta P)^{-1}\Gamma_2 = -H'R_2^{-1}H \ . \tag{2.4b}$$

In particular,

$$-(P^* - P_*)^{-1} + \Gamma_*'(P^* - P_*)^{-1}\Gamma_* = -H'R_*^{-1}H \ , \tag{2.5a}$$

$$(P^* - P_*)^{-1} - \Gamma^{*'}(P^* - P_*)^{-1}\Gamma^* = -H'R^{*-1}H, \tag{2.5b}$$

where $R_* = J + J' - HP_*H'$ *and* $R^* = J + J' - HP^*H'$.

Proof. We shall prove (2.3a) first. Since $\Lambda(P_1) = \Lambda(P_2) = 0$,

$$P_1 = FP_1F' + K_1R_1K_1' \quad \text{and} \quad P_2 = FP_2F' + K_2R_2K_2'$$

where $K_iR_i = G - FP_iH'$ and $R_i = J + J' - HP_iH'$; $i = 1,2$. Substracting the first from the second, we obtain $\Delta P = F\Delta PF' + K_2R_2K' - K_1R_1K_1'$. Using the differencing technique in (Morf, 1974) and observing that $R_2 = R_1 - H\Delta PH'$, and that $K_2 = [K_1R_1 - F\Delta PH']R^{-1}$, we obtain $K_2 - K_1 = (K_1H - F)\Delta PH'R^{-1}$. Then noting that $K_2 - K_1 + \Delta K$, where $\Delta K = K_2 - K_1$, we have

$$\Delta P = F\Delta PF' + (K_1 + \Delta K)R_2(K_1 + \Delta K)' - K_1R_1K_1' \ .$$

After long algebraic calculation the above reduces to (2.3a). To prove (2.2), we show that

$$\Gamma_1 = \Gamma_2(I - \Delta PH'R_1^{-1}H)$$

$$\Gamma_2 = \Gamma_1(I - \Delta PH'R_2^{-1}H) \ .$$

To show the first of these, observe that, $\Gamma_1 = F - K_1H' = F - (K_2R_2 + F\Delta PH')R_1^{-1}H$. Using the fact that $R_2 = R_1 - H\Delta PH'$ and some algebra, we obtain the above. Substituting the second equation of the above in (2.3a), (2.2) follows. Relations (2.4) may be obtained from the matrix inversion lemma and (2.5) are an immediate consequence of (2.4). □

The following corollary is an immediate consequence of the above lemma.

Corollary 2.2. The feedback matrices Γ_* and $\Gamma^{*'-1}$ are similar. (Consequently, if λ_i are the eigenvalues of Γ_*, then $\frac{1}{\lambda_i}$ are the eigenvalues of Γ^*; $i = 1,2,\ldots,n$.)

Proof. Since $P^* - P_* > 0$, (2.2) implies $(P^* - P_*)\Gamma^{*'-1}(P^* - P_*)^{-1} = \Gamma_*$. □

Now, we give a necessary and sufficient condition for a matrix P to belong to $\mathcal{P}^+$ or to $\mathcal{P}^-$.

Proposition 2.3. (a) Let $M_*(N)$ be the solution of the Liapunov equation

$$-M_* + \Gamma_*' M_* \Gamma_* + H'R_*^{-1}H + N = 0 \tag{2.6}$$

Then $M_*(N)$ is positive definite and the matrix $P:=P_* + [M_*(N)]^{-1}$ belong to $\mathcal{P}^+$ if and only if N is nonnegative definite. Moreover,

$$\Lambda(P) = -[M_*(N)]^{-1} N [M_*(N)]^{-1} . \tag{2.7}$$

(b) Let $M^*(N)$ be the solution of the Liapunov equation

$$M^* - \Gamma^{*'} M^* \Gamma^* + H'R^{*-1}H + N = 0 \tag{2.8}$$

Then $M^*(N)$ is positive definite and the matrix $P:=P^* - [M^*(N)]^{-1}$ belongs to $\mathcal{P}^-$ if and only if N is nonnegative definite. Moreover,

$$\Lambda(P) = -[M^*(N)]^{-1} N [M^*(N)]^{-1} \tag{2.9}$$

(c) Finally, $P^* - P_* = [M_*(0)]^{-1} = [M^*(0)]^{-1}$.

<u>Proof</u>. (a) Let $P \varepsilon P^+$. In view of (1.4a) and (1.8a), we have

$$P = FPF' + (G - FPH')R(P)^{-1}(G - FPH')' + B_2B_2'$$

$$P_* = FP_*F' + (G - FP_*H')R_*^{-1}(G - FP_*H')' \ .$$

Upon subtracting the second of these relations from the first and setting $M_* = (P - P_*)^{-1}$ which is positive definite (since $P \varepsilon P^+$), we have

$$M_*^{-1} = FM_*^{-1}F' + KR(P)K' - K_*R_*K_*' + B_2B_2'$$

where K and K_* are defined by $KR(P) = G - FPH'$ and $K_*R_* = G - FP_*H'$ respectively. After long, but simple calculation similar to that used in proving Lemma 2.1, we obtain

$$M_*^{-1} = \Gamma_*M_*^{-1}\Gamma_*' + \Gamma_*M_*^{-1}H'R(P)^{-1}HM_*^{-1}\Gamma_*' + B_2B_2'$$

or

$$M_*^{-1} \geq \Gamma_*[M_*^{-1} + M_*^{-1}H'R(P)^{-1} + HM_*^{-1}]\Gamma_*' \ .$$

which yields (2.6) for some $N \geq 0$. Conversely, let $M_*(N)$ be the solution of (2.6) with $N \geq 0$. The pair (Γ_*, H) is observable for (F,H) is (Wonham, 1968). Recalling that Γ_* is a stability matrix, a standard result in stability theory (Brockett, 1970) implies $M_*(N) > 0$.

Consequently, the matrix $P_* + [M_*(N)]^{-1} \varepsilon \mathcal{P}^+$. To prove (2.7), observe that $\Lambda(P)$ can be written as

$$\Lambda(P) = -P + \Gamma P \Gamma' - FPH'R(P)^{-1}HPF' + GR(P)^{-1}G' .$$

Adding $\Lambda(P_*)$ (which is $\equiv 0$) to this, using the fact that $\Gamma = \Gamma_*(I - [M_*(N)]^{-1}H'R(P)^{-1}H)$ (see the proof of Lemma 2.1) and using (2.6), relation (2.7) follows. This completes the proof of (a). The proof of (b) is analogous and that of (c) is immediate. □

In particular, we are able to see that the sets $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ are singletons.

Proposition 2.4. Let $P \varepsilon \mathcal{P}_0$. Then $P \varepsilon \mathcal{P}_0^+(\mathcal{P}_0^-)$ if and only if N in (2.6) ((2.8)) is zero.

Proof. In either case, in view of (2.7) and (2.9), $\Lambda(P) = 0$, in which case $M_*(0) = P^* - P_*$ and this in turn implies $P = P^*$ $(P = P_*)$ $\varepsilon \mathcal{P}_0^+$ $(\mathcal{P}_0^-)$. □

In fact, we have another characterizatøn of $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$, namely

Propositon 2.5. Let $P \varepsilon \mathcal{P}_0$. Then $P \varepsilon \mathcal{P}_0^+$ $(\mathcal{P}_0^-)$ if and only if the feedback matrix Γ corresponding to P is similar to Γ_*^{-1} (Γ_*).

Proof. Let $P \in \mathcal{P}_0^+$. Then, in view of (2.2), $\Gamma = \Delta P \Gamma_*'^{-1} \Delta P$, where $\Delta P = P - P_*$. Conversely, if Γ is similar to Γ_*^{-1}, then $|\lambda(\Gamma)| > 1$. Setting $P^* = P_2$ and $P = P_1$ in (2.2) we obtain $-(P^* - P) + \Gamma^*(P^* - P)\Gamma' = 0$, which has the solution $P^* - P = 0$ since Γ and Γ^{*-1} have no eigenvalues in common (Gantmacher, 1960). Hence $P = P^*$ which belongs to $\mathcal{P}_0^+$. □

Indeed, the above two propositions may be reformulated to read

Corollary 2.6. $\mathcal{P}_0^+ = \{P^*\}$ and $\mathcal{P}_0^- = \{P_*\}$.

Proof. Let $P \in \mathcal{P}_0^+$ be such that $P \neq P^*$. Then, by (2.4), $-(\Delta P)^{-1} + \Gamma_*'(\Delta P)^{-1}\Gamma_* = -H'R_*^{-1}H$, where we set $\Delta P = P - P_*$. But this Liapunov equation is of the type (2.5), which has the unique solution $P^* - P_*$ since $|\lambda(\Gamma_*)| < 1$. Therefore $P = P^*$, which is a contradiction to the assumption. □

Below is a more clear proof for the above corollary using the eigenvector method to construct solutions of the (ARE). The author would like to thank the referee who pointed this out.

Another Proof of Corollary 2.6. Let $P_1 = Y_1X_1^{-1}$ and $P_2 = Y_2X_2^{-1}$, where $[X_1Y_1]'$ and $[X_2Y_2]'$ are $2n \times n$ matrices consisting of eigenvectors of the associated Hamiltonian matrix. Then $P_1 - P_2$ is nonsingular if and only if the column spaces of $[X_1Y_1]'$ and $[X_2Y_2]'$ are disjoint. If $P_1 = P^*$, then $[X_1Y_1]'$ corresponds to the eigenvalues inside the unit circle i.e., if and only if $P_2 = P_*$. □

3. RELATIONSHIPS BETWEEN (ARE) AND (QMI)

In this section, a closer look at the connections between the solutions of the (ARE) and the (QMI), will be taken. We shall study the boundary, interior and extreme points of $\mathcal{P}$ and see the connections of these with $\mathcal{P}_0$.

Let us start with the boundary points of $\mathcal{P}$ first. For any $\varepsilon > 0$ and any matrix M, let $U(M,\varepsilon) = \{L | L = M + N, \|N\| < \varepsilon\}$, where $\|N\|$ is the usual matrix norm. Any $n \times n$ symmetric matrix P belongs to the <u>boundary</u> of $\mathcal{P}$ (denoted $\partial\mathcal{P}$) if, for any $\varepsilon > 0$, there exist two matrices P_1 and P_2 belonging to $U(P,\varepsilon)$ such that $P_1 \in \mathcal{P}$ and $P_2 \notin \mathcal{P}$. Since $\mathcal{P}$ is closed, $\partial\mathcal{P} \subset \mathcal{P}$.

<u>Theorem 3.1</u>. Germain (1974). <u>Let</u> $P \in \mathcal{P}$ and set $Q = P - FPF'$ <u>and</u> $S = G - FPH'$. <u>Then</u> $P \in \partial\mathcal{P}$ <u>if and only if the matrix</u> $M = \begin{bmatrix} Q & S \\ S' & R(P) \end{bmatrix}$ <u>is singular</u>.

<u>Corollary 3.2</u>. $\mathcal{P}_0 \subset \partial\mathcal{P}$.

<u>Proof</u>. Let $P \varepsilon \mathcal{P}_0$. Then $\Lambda(P) = 0$. Consequently, M in the previous theorem is singular. □

Recall that the output process y is m-dimensional and the dimension of the state process n. If $m < n$, which is usually the case in application, we have the following stronger result.

<u>Corollary 3.3</u>. <u>Let</u> $m < n$ <u>and let</u> P_1 <u>and</u> P_2 <u>be arbitrary elements of</u> $\mathcal{P}_0$. <u>Then the segment</u> $[P_1,P_2] \subset \partial\mathcal{P}$. (<u>In particular</u> $[P_*, P^*] \subset \partial\mathcal{P}$.)

<u>Proof</u>. We shall prove that $P(\alpha) := \alpha P_1 + (1 - \alpha)P_2$ for $\alpha \varepsilon [0,1]$ belongs to $\partial\mathcal{P}$. For this purpose, it can be shown, after long algebraic manipulations that

$$\Lambda(P(\alpha)) = \alpha\Lambda(P_1) + (1-\alpha)\Lambda(P_2) - \alpha(1-\alpha)\Gamma_2\Delta PH'R^{-1}H\Delta P\Gamma_{\frac{1}{2}} \tag{3.1}$$

Since $P_1, P_2 \varepsilon \mathcal{P}_0$, $\Lambda(P_1) = \Lambda(P_2) = 0$. Let $Q(\alpha)$, $S(\alpha)$, $R(\alpha)$ and $M(\alpha)$ be as defined in Theorem 3.1 corresponding to $P(\alpha)$. Then $-\Lambda(P(\alpha)) = Q(\alpha) - S(\alpha)\, R(\alpha)^{-1}S(\alpha)' = \alpha(1-\alpha)\Gamma_2\Delta PH'R_1^{-1}H\Delta P\Gamma_{\frac{1}{2}}$. If $m < n$, then $H'R_1^{-1}H$ is not full rank and hence the matrix $M(\alpha)$ is singular. Hence, in view of Theorem 3.1, $P(\alpha) \varepsilon \partial\mathcal{P}$ for any $\alpha \varepsilon [0,1]$. □

The second geometrical aspect of the set $\mathcal{P}$, that we shall study is the <u>interior</u> points (denoted by Int $\mathcal{P}$) and defined by

$$\text{Int } \mathcal{P} := \{P \in \mathcal{P} | \Lambda(P) < 0\}. \tag{3.2}$$

First, we shall show that the interior of $\mathcal{P}$ is nonempty.

<u>Proposition 3.4.</u> Int $\mathcal{P}$ <u>is nonempty</u>.

<u>Proof</u>. Set $N = I$ in (2.6). Then the matrix $P := P_* + M_*(I)^{-1}$ belongs to $\mathcal{P}^+$ and $\Lambda(P)$ in (2.7) is equal to $-M_*(I)^{-2}$ which is negative definite. Therefore $P \in \text{Int } \mathcal{P}$. □

The next theorem the proof of which, in view of (2.7) and (2.9), is immediate, gives a necessary and sufficient condition for a matrix P to belong to Int $\mathcal{P}$.

<u>Theorem 3.5</u>. <u>Let</u> $M_*(N)$ $(M^*(N))$ <u>be as in Proposition</u> 2.3. <u>Then, the matrix</u> $P := P_* + [M_*(N)]^{-1}$ $(P := P^* - [M^*(N)]^{-1})$ <u>belongs to</u> Int $\mathcal{P}$ <u>if and only if</u> $N > 0$.

The final geometrical aspect of the set $\mathcal{P}$ to be looked at in this section is the extreme points of $\mathcal{P}$. We shall prove that solutions of the (ARE) are among the extreme points of $\mathcal{P}$, which is a much stronger result than Corollary 3.2. (The extreme points of a set are contained in its boundary.) Initially, the aim was to show that

$\mathcal{P}_0$ = {Extreme points of $\mathcal{P}_0$} in order to use Krein-Milman theorem and conclude that $\mathcal{P}$ = convex hull of $\mathcal{P}_0$ (since $\mathcal{P}$ is convex and compact). However, it turned out that one inclusion is true and the other is not, i.e., $\mathcal{P}_0 \subset$ {Extreme points of $\mathcal{P}$} and {Extreme points of $\mathcal{P}$} $\not\subset \mathcal{P}_0$. To show this one inclusion, the following lemma is needed.

Lemma 3.6. Let $P \in \mathcal{P}_0$. Suppose there exist two elements P_1 and P_2 in $\mathcal{P}$ such that $P = \alpha P_1 + (1 - \alpha)P_2$ for some $\alpha \in (0,1)$. Then, $P_1, P_2 \in \mathcal{P}_0$ and $\Gamma_2 \Delta P H' R_1^{-1} H \Delta P \Gamma_2 = 0$.

Proof. Let P_1, P_2 and P be as in the lemma. Then, in (3.1), the last term $-\alpha(1 - \alpha)\Gamma_2 \Delta P H' R_1^{-1} H \Delta P \Gamma_2 \leq 0$. On the other hand, since $\Lambda(P) = 0$ (for $P \in \mathcal{P}_0$), $\Lambda(P_1) \leq 0$, $\Lambda(P_2) \leq 0$ (for both belong to $\mathcal{P}$), $-\alpha(1 - \alpha)\Gamma_2 \Delta P H' R_1^{-1} H \Delta P \Gamma_2 \geq 0$. This implies that this quantity is identically zero. Consequently, $\Lambda(P_1) = \Lambda(P_2) = 0$, which implies that P_1 and $P_2 \in \mathcal{P}_0$. □

The next theorem is the main result in this paper, and to our knowledge, no proof of it is available anywhere. However, its continuous-time counterpart can be found in Badawi (1982) and Faurre (1979).

Theorem 3.7. Let $P \in \mathcal{P}_0$. Then P is an extreme point of $\mathcal{P}$.

Proof. Let $P \in \mathcal{P}_0$ and assume that there are P_1 and $P_2 \in \mathcal{P}$ such that $P = \alpha P_1 + (1 - \alpha)P_2$ for some $\alpha \varepsilon (0,1)$. We shall show that $P=P_1=P_2$. By Lemma 3.6, $\Gamma_2 \Delta PH'R_1^{-1}H\Delta P\Gamma_2' = 0$. This implies, after long algebraic manipulations that $(G - FP_1H')R_1^{-1} = (G - FP_2H')R_2^{-1}$. This in turn, implies $-\Delta P + F\Delta PF' = 0$. But $|\lambda(F)| < 1$, then F and F'^{-1} do not have any eigenvalue in common, which implies that the only solution to this equation (Gantmacher, 1960) is $\Delta P = 0$. □

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