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A RIEMANN INTEGRAL AND THE DIVERGENCE THEOREM*)

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By integration, we want to obtain the flux of a vector field from its divergence. The Denjoy-Perron integral does this in dimension one (see [10, ch.4, §6 and ch.8, §3]), but its higher dimensional generalizations, including the most recent ones (see [4] and [5]), do not. Elaborating on the ideas of Henstock and Kurzweil (see [2] and [3]), we define an m-dimensional Riemann-type integral for which a satisfactory divergence theorem holds. This is accomplished by employing partitions with a weak Vitali property. For $m = 1$ our definition coincides with that of [2].

The sets of all reals and positive reals are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Throughout, we fix a Euclidean space $\mathbb{R}^m$, $m \geq 1$ an integer, and for $A \subset \mathbb{R}^m$, we denote by A^o, ∂A, $d(A)$, and $|A|$ the interior, boundary, diameter, and outer measure of A, respectively. If $A^o \neq \emptyset$, then the number $r(A) = m^{m/2}|A|/[d(A)]^m$ is called the <u>regularity</u> of A (cf. [10, ch. 4, sec.2, p.106]). An <u>interval</u> is a set $A = \Pi_{i=1}^m [a_i, b_i]$ where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, $i = 1, \ldots, m$. Clearly, $r(A) = 1$ if and only if $b_1 - a_1 = \ldots = b_m - a_m$.

A <u>division</u> of an interval A is a finite collection of intervals whose interiors are disjoint, and whose union is A. For $\varepsilon > 0$, an <u>ε-partition</u> of a division $\mathcal{D}$ of an interval A is a set $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ satisfying the following conditions:

(i) $\{A_1, \ldots, A_p\}$ is a division of A;

(ii) $x_i \in A_i$, $i = 1, \ldots, p$;

(iii) for each $D \in \mathcal{D}$ there are intervals $B_1, \ldots, B_p$ such that $r(B_i) \geq \varepsilon$ and $B_i \cap D = A_i \cap D$, $i = 1, \ldots, p$.

If $\delta : A \to \mathbb{R}_+$, we say that P is <u>δ-fine</u> whenever $d(A_i) < \delta(x_i)$, $i = 1, \ldots, p$. A simple compactness argument reveals that each division of an interval A has a δ-fine 1-partition for any $\delta : A \to \mathbb{R}_+$.

<u>Definition</u>. Let A be an interval. We say that $f : A \to \mathbb{R}$ is <u>integrable</u> on A if there is a real number $\int_A f$ with the following property: given a division $\mathcal{D}$ of A and an $\varepsilon > 0$, we can find a $\delta : A \to \mathbb{R}_+$ such that

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - \int_A f \right| < \varepsilon$$

for each δ-fine ε-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ of $\mathcal{D}$. The number $\int_A f$, which is uniquely defined, is called the <u>integral</u> of f over A.

Mimicking the one-dimensional proofs of [7, §§ 1, 2], it is easy to see that the set $R(A)$ of all integrable functions on an interval A is a vector space which is closed with respect to limits of dominated sequences, and that the integral is a non-negative continuous linear functional on $R(A)$ which is <u>finitely</u> additive as a function of subintervals of A. Interpreting the integral as a Perron-type integral (see [7, app. B] for the one-dimensional analogy) and using [8] and [10, ch.4, §4], we can show that integrable functions are measurable. Moreover, the Lebesgue integral $(L)\int_A f$ of a finite function f on an interval A exists if and only if both f and $|f|$ are integrable on A, in which case $(L)\int_A f = \int_A f$. This is established by techniques of [6].

The Euclidean norm of $x \in \mathbb{R}^m$ is denoted by $|x|$. By a <u>k-plane</u>, $k = 0, \ldots, m-1$, we mean a k-dimensional linear submanifold of $\mathbb{R}^m$ which is parallel to k distinct coordinate axes. The word <u>differentiable</u> is used in the sense of [9, def.9.11, p.212]; in particular, differentiable does not imply continuously differentiable. If A is an interval and v is a vector field on A, we denote by $(L)\int_{\partial A} v\cdot n$ the Lebesgue surface integral (if it exists) of v in the direction of the exterior normal n.

<u>Theorem</u>. Let A be an interval, $S_{-1} = \emptyset$, and for $k = 0,\ldots,m-1$, let S_k be a countable union of k-planes. Further, let $v = (f_1,\ldots,f_m)$ be a vector field in A which satisfies the following conditions:

(i) v is differentiable in $A^o - S_{m-1}$;

(ii) v is continuous in $A - S_{m-2}$;

(iii) for each $z \in A - S_{m-k}$, $k = 3, \ldots, m+1$, there is a decreasing $\alpha_z : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^1 \alpha_z < +\infty$, and

$$|v(x) - v(z)| = O[\alpha_z(|x - z|)|x - z|^{3-k}]$$

as $x \to z$.

Then $(L)\int_{\partial A} v\cdot n$ exists, and if $\nabla\cdot v: A \to \mathbb{R}$ is such that $\nabla\cdot v(x) = \Sigma_{i=1}^{m} \partial_i f_i(x)$ for each $x \in A^o - S_{m-1}$, then $\nabla\cdot v$ is integrable on A and $\int_A \nabla\cdot v = (L)\int_{\partial A} v\cdot n$.

The theorem is obtained by means of the variational integral (see [1]). Its proof will appear elsewhere.

It seems worthwhile to formulate separately the important special case of $S_0 = \ldots = S_{m-1} = \emptyset$.

Corollary. Let A be an interval, and let v be a continuous vector field in A which is differentiable in A^o. Then $\nabla\cdot v$ is integrable on A and $\int_A \nabla\cdot v = (L)\int_{\partial A} v\cdot n$.

Remark. The unavoidable price we must pay for a general divergence theorem is the loss of Fubini's theorem (cf. [11]). Indeed, a vector field may be everywhere differentiable and the iterated integrals of its divergence may have different values.

It is easy to see that the integral we defined is affine-invariant but not PL-invariant. However, our definition will produce a PL-invariant integral if convex linear cells are used instead of intervals. Then we can show that the theorem remains valid for a vector field v which is differentiable everywhere in A. Even this weaker version of the Theorem is false for the integral defined in [4] or [5] (cf. the Remark). Whether a general divergence theorem can be proved for an integral invariant with respect to the diffeomorphism group is an open problem.

Note. After this writing was finished, the author saw reference [12] which approaches the divergence theorem from the point of view similar to his. However, the integral defined in [12] is not an additive function of intervals in the usual sense (see [12, prop. 2]). Also, the divergence theorem proved in [12] (thm. 2) is considerably weaker than the Theorem stated above; in fact, it is even weaker than the Corollary.

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