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R.W. Lardner

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

NONLINEAR SURFACE ACOUSTIC WAVES ON AN ELASTIC SOLID OF GENERAL ANISOTROPY

R.W. Lardner
Department of Mathematical Sciences
University of Petroleum and Minerals
Dhahran, Saudi Arabia

Abstract

The effect of elastic nonlinearity on the propagation of Rayleigh waves in an anisotropic elastic solid is considered. A nonlinear integro-differential equation is derived for a quantity which is related to the Fourier transform of the displacement components on the surface. The variation of this quantity along the surface accounts for the slow modulation of the wave through formation and depletion of the different harmonics. Explicit results are given for harmonic generation in an initially sinusoidal wave and for parametric amplification of a weak signal by a pump wave of twice its frequency.

1. Introduction

Because of the high energy density in a layer close to the surface, nonlinear effects occur much more readily for surface waves on an elastic solid (i.e., Rayleigh waves) than they do for bulk waves, in which the energy is dispersed throughout the medium. As a result, there is interest in the possibility of using surface waves in the design of certain nonlinear acoustic devices, such as parametric amplifiers.

In a previous paper [1] we have given the theory of nonlinear Rayleigh waves on an isotropic elastic solid, based on use of the method of multiple scales, originally proposed for this problem by Kalyanasundaram [2]. In subsequent papers [3,4] we have included the effect of thermo-viscous damping and have given numerical results for harmonic generation, parametric amplification and waveform distortion, in the latter case demonstrating the formation of shocks. However, since experimental results as well as acoustical devices often use crystalline materials, it becomes important to extend the analysis to the case of an anisotropic medium. This is the subject of the present paper.

The analysis follows the same pattern as in [1]. While the anisotropy introduces complications of an algebraic nature, it turns out not to require any fundamental changes in the form of solution. Moreover, the final result is of exactly the same type as in the isotropic case: a single integro-differential equation for a quantity γ which is directly related to the Fourier transform of the displacements on the free surface. Because of this, the earlier results on harmonic generation and parametric amplification may be directly transposed to the anisotropic case. These results are summarized in the final section of the paper.

2. Basic Equations

We use (x_1, x_2, x_3) as Cartesian coordinates in the reference configuration of the body and $u_i(x_1, x_2, x_3, t)$ as the components of the particle displacements at time t. The equations of motion are then

$$\rho\ddot{u}_i = \sigma_{ij,j} \tag{1}$$

where ρ is the density in the reference state (assumed to be constant) and σ_{ij} is the Piola-Kirchoff stress tensor. We use dots to denote time derivatives, the comma notation for derivatives with respect to x_j, and the summation convention for repeated suffixes.

The Lagrangian strain tensor is defined by

$$e_{ab} = \frac{1}{2}(u_{a,b} + u_{b,a} + u_{j,a}u_{j,b}).$$

In terms of this, we assume that the strain energy density is given up to cubic terms by

$$U = \frac{1}{2}\, c_{abcd}\, e_{ab}\, e_{cd} + \frac{1}{6}\, c^{(3)}_{abcdef}\, e_{ab}\, e_{cd}\, e_{ef} \tag{2}$$

where c_{abcd} and $c^{(3)}_{abcdef}$ are the second and third order elastic moduli of the medium.

The Piola-Kirchoff stress tensor is given in general by

$$\sigma_{ij} = (\delta_{iq} + u_{i,q}) \frac{\partial U}{\partial e_{(jq)}}$$

where the parentheses round jq indicate symmetrization with respect to those indices and δ_{iq} is the Kronecker delta. From equation (2) we then obtain

$$\sigma_{ij} = c_{ijk\ell} u_{k,\ell} + \frac{1}{2}\, c^{(3)}_{ijk\ell mn}\, u_{k,\ell}\, u_{m,n} + \frac{1}{2}\, c_{ijk\ell}\, u_{m,k}\, u_{m,\ell}$$

$$+ c_{mjk\ell}\, u_{k,\ell}\, u_{i,m}. \tag{3}$$

In order to keep track of the different orders of magnitude, we shall make the replacements $u_i \to \varepsilon u_i$ and $\sigma_{ij} \to \varepsilon \sigma_{ij}$ where ε is a small positive scaling parameter. Equation (1) remains unchanged while (3) can be written in the form

$$\sigma_{ij} = c_{ijk\ell}\, u_{k,\ell} + \varepsilon d_{ijk\ell mn}\, u_{k,\ell}\, u_{m,n} \tag{4}$$

where

$$d_{ijk\ell mn} = \frac{1}{2}[c^{(3)}_{ijk\ell mn} + c_{ijn\ell}\, \delta_{km} + c_{njk\ell}\, \delta_{im} + c_{\ell jmn}\, \delta_{ik}]\,. \tag{5}$$

In this last equation, $\underline{d}$ has been symmetrized with respect to the pairs $(k\ell)$ and (mn) of indices, and the result is then automatically symmetrical with respect to all three pairs (ij), $(k\ell)$ and (mn). Using this symmetry and substituting equation (4) into equation (1), we obtain

$$\rho \ddot{u}_i = c_{ijk\ell}\, u_{k,\ell j} + 2\varepsilon d_{ijk\ell mn}\, u_{k,\ell}\, u_{m,nj}. \tag{6}$$

We consider a body which in its reference state occupies the region $x_2 < 0$ and in which the deformation consists of a plane surface wave, independent of x_3, travelling in the positive x_1 direction. We seek the solution in the form

$$u_j(x,y,t) = \int_{-\infty}^{\infty} \tilde{u}_j(k,y,\xi,\eta,\tau) e^{ik(x-ct)} dk \tag{7}$$

where we use (x,y) synonymously with (x_1,x_2), c denotes the Rayleigh wave speed and the integration over k allows an arbitrary spectrum of wave numbers to be included. The introduction of the slow scales $\xi = \varepsilon x$, $\eta = \varepsilon y$, $\tau = \varepsilon t$ accounts for slow modulation of the wave in space and time.

Substituting (7) into equations (6) we then obtain

$$c_{i2j2}\, \tilde{u}_{j,yy} + ik(c_{i1j2} + c_{i2j1})\tilde{u}_{j,y} + k^2(\rho c^2 \delta_{ij} - c_{i1j1})\tilde{u}_j +$$

$$+ \varepsilon\{2c_{i2j2}\,\tilde{u}_{j,y\eta} + (c_{i1j2} + c_{i2j1})(\tilde{u}_{j,y\xi} + ik\tilde{u}_{j,\eta})$$

$$+ 2c_{i1j1}\,ik\tilde{u}_{j,\xi} + 2i\rho ck\tilde{u}_{i,\tau} + 2N_i\} = 0 \tag{8}$$

where N_i contains the nonlinear terms and will be given below. It is assumed that the boundary $y = 0$ is fraction-free, and the resulting conditions $\sigma_{i2} = 0$ lead to the equation

$$c_{i2j2}\,\tilde{u}_{j,y} + ikc_{i2j1}\,\tilde{u}_j + \varepsilon\{c_{i2j2}\,\tilde{u}_{j,\eta} +$$

$$c_{i2j1}\,\tilde{u}_{j,\xi} + N_i'\} = 0 \tag{9}$$

where again the nonlinear term N_i' will be given below.

We now seek a solution in the form of a perturbation expansion,

$$\tilde{u}_j = A_j(k,y,\xi,\eta,\tau) + \varepsilon B_j(k,y,\xi,\eta,\tau) + \ldots .$$

Substituting into equation (8) and comparing coefficients of the different powers of ε, we obtain

$$c_{i2j2}\,A_{j,yy} + ik(c_{i1j2} + c_{i2j1})A_{j,y} +$$

$$k^2(\rho c^2\delta_{ij} - c_{i1j1})A_j = 0 \tag{10}$$

$$c_{i2j2}\, B_{j,yy} + ik(c_{i1j2} + c_{i2j1})B_{j,y} + k^2(\rho c^2\delta_{ij} - c_{i1j1})B_j$$

$$+ 2c_{i2j2}\, A_{j,y\eta} + (c_{i1j2} + c_{i2j1})(A_{j,y\xi} + ikA_{j,\eta})$$

$$+ 2c_{i1j1}\, ikA_{j,\xi} + 2i\rho ck\, A_{i,\tau} + 2N_i = 0\ . \tag{11}$$

Similarly from equation (9) we obtain the following boundary conditions on $y = \eta = 0$,

$$c_{i2j2}\, A_{j,y} + ikc_{i2j1}\, A_j = 0 \tag{12}$$

$$c_{i2j2}\, B_{j,y} + ikc_{i2j1}\, B_j + c_{i2j2} A_{j,\eta} + c_{i2j1} A_{j,\xi} + N_i' = 0\ . \tag{13}$$

To obtain the nonlinear terms in equations (11) and (13) we must replace $\tilde{u}_i$ by A_i in the corresponding terms in (8) and (9). The result is as follows:

$$N_i = \int_{-\infty}^{\infty} \{ik'A_j(k')[-d_{i1j1m1}(k - k')^2 A_m(k - k') +$$

$$+ (d_{i1j1m2} + d_{i2j1m1})i(k - k')A_{m,2}(k - k')$$

$$+ d_{i2j1m2}\, A_{m,22}(k - k')]$$

$$+ A_{j,2}(k')[-d_{i1j2m1}(k - k')^2 A_m(k - k') +$$

$$+ (d_{i1j2m2} + d_{i2j2m1}) i(k - k')A_{m,2}(k - k')$$

$$+ d_{i2j2m2} A_{m,22}(k - k')]\}dk' \tag{14}$$

$$N_i' = \int_{-\infty}^{\infty} \{ik'A_j(k')[d_{i2j1m1}\, i(k - k')A_m(k - k')$$

$$+ d_{i2j1m2} A_{m,2}(k - k')]$$

$$+ A_{j,2}(k')[d_{i2j2m1}\, i(k - k')A_m(k - k')$$

$$+ d_{i2j2m2} A_{m,2}(k - k')]\}dk'. \tag{15}$$

3. First Order Solution

Equations (10) and (12) for the first order amplitudes A_j are the standard equations of linear Rayleigh wave theory except for the additional parameters ξ, η and τ. We seek solutions of equation (10) of the form $A_j = a_j \exp(isky)$ and obtain the conditions

$$L_{ij}(s)a_j = 0$$

where

$$L_{ij}(s) = c_{i2j2}s^2 + (c_{i2j1} + c_{i1j2})s + c_{i1j1} - \rho c^2 \delta_{ij} \tag{16}$$

The condition $\det[L_{ij}(s)] = 0$ is an equation for s of sixth degree with roots occurring in complex conjugate pairs. We denote by $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$ the three roots with negative imaginary parts; since $A_j \to 0$ as $y \to -\infty$, these are the relevant roots when $k > 0$, while the conjugate roots are relevant for $k < 0$. For $\ell = 1,2,3$, let $a_j^{(\ell)}$ denote any particular solution of the system

$$L_{ij}(s^{(\ell)})a_j^{(\ell)} = 0 \ . \tag{17}$$

Then the general solution of equation (10) has the form

$$\begin{aligned} A_j(k,y,\xi,\eta,\tau) &= \sum_{\ell=1}^{3} \alpha_\ell(k,\xi,\eta,\tau) a_j^{(\ell)} e^{is^{(\ell)}ky} \qquad (k > 0) \\ &= \sum_{\ell=1}^{3} \alpha_\ell(k,\xi,\eta,\tau) \overline{a_j^{(\ell)}} \, e^{i\overline{s^{(\ell)}}ky} \qquad (k < 0) \end{aligned} \tag{18}$$

Furthermore we note that $\alpha_\ell(-k) = \overline{\alpha_\ell(k)}$ in order that $u_j(x,y,t)$ given by (7) should be real.

The boundary conditions (12) then lead to the equations

$$M_{i\ell} \, \alpha_\ell(k,\xi,0,\tau) = 0 \tag{19}$$

where

$$M_{i\ell} = (c_{i2j1} + s^{(\ell)} c_{i2j2}) a_j^{(\ell)} . \tag{20}$$

The condition $\det(M_{i\ell}) = 0$ then determines the Rayleigh wave speed c, as usual. If we denote by β_ℓ any particular (constant) solution of the system $M_{i\ell}\,\beta_\ell = 0$, then we can write

$$\alpha_\ell(k,\xi,0,\tau) = \gamma(k,\xi,\tau)\beta_\ell \tag{21}$$

for a certain function γ.

The function γ is directly related to the Fourier transform of the displacements on the boundary. For, setting $y = \eta = 0$ in (7) and including only the lowest order terms (18), we obtain with use of (21) that

$$u_j(x,0,t) = (\sum_\ell \beta_\ell a_j^{(\ell)}) \int_0^\infty \gamma(k,\xi,\tau) e^{ik(x-ct)} dk + CC \tag{22}$$

Thus γ is directly related to observable quantities, and our main aim will be to obtain the equation which determines the ξ and τ dependence of this function.

4. Second Order Equations

The next step is to substitute the solution (18) into the equations (11) and (13) satisfied by the second-order amplitudes B_j. For simplicity we consider $k > 0$ only. First consider the nonlinear terms given in equations (14) and (15). Taking for example the first term in (14), we obtain

$$-id_{i1j1m1}\sum_{\ell,n}\{\int_0^k \Omega_{\ell n}a_j^{(\ell)}a_m^{(n)}e^{i[s^{(\ell)}k'+s^{(n)}(k-k')]y}dk'$$

$$+\int_k^\infty \Omega_{\ell n}a_j^{(\ell)}\overline{a_m^{(n)}}\ e^{i[s^{(\ell)}k'+\overline{s^{(n)}}(k-k')]y}\ dk'$$

$$+\int_{-\infty}^0 \Omega_{\ell n}\ \overline{a_j^{(\ell)}}\ a_m^{(n)}\ e^{i[\overline{s^{(\ell)}}k'+s^{(n)}(k-k')]y}\ dk'\}$$

where we use the abbreviation $\Omega_{\ell n} = k'(k - k')^2\alpha_\ell(k')\alpha_n(k - k')$ for the quantity common to each integrand. We have similar expressions for the remaining terms in (14). Let us define the following constants:

$$D_{ijm\ell} = d_{i1j1m1} + s^{(\ell)}d_{i1j2m1}$$

$$D'_{ijm\ell} = (d_{i1j1m2} + d_{i2j1m1}) + s^{(\ell)}(d_{i1j2m2}+d_{i2j2m1}) \tag{23}$$

$$D''_{ijm\ell} = d_{i2j1m2} + s^{(\ell)}d_{i2j2m2}$$

$$N^{(1)}_{i\ell n} = \sum_{j,m} a_j^{(\ell)}a_m^{(n)}[D_{ijm\ell}+s^{(n)}D'_{ijm\ell}+s^{(n)2}D''_{ijm\ell}] \tag{24}$$

$$N^{(2)}_{i\ell n} = \sum_{j,m} \overline{a_j^{(\ell)}}\ a_m^{(n)}[\overline{D}_{ijm\ell}+s^{(n)}\overline{D'_{ijm\ell}}+s^{(n)2}\ \overline{D''_{ijm\ell}}\ ] \tag{25}$$

$$N^{(3)}_{i\ell n} = \sum_{j,m} a_j^{(\ell)}\ \overline{a_m^{(n)}}[D_{ijm\ell} + \overline{s^{(n)}}D'_{ijm\ell} + \overline{s^{(n)2}}D''_{ijm\ell}] \tag{26}$$

Then N_i takes the following form:

$$N_i = -i\sum_{\ell,n}\{N^{(1)}_{i\ell n}\int_0^k e^{i[s^{(\ell)}k'+s^{(n)}(k-k')]y}\,k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')dk'$$

$$+ N^{(2)}_{i\ell n}\int_{-\infty}^0 e^{i[\overline{s^{(\ell)}}k'+s^{(n)}(k-k')]y}\,k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')dk'$$

$$+ N^{(3)}_{i\ell n}\int_k^\infty e^{i[s^{(\ell)}k'+\overline{s^{(n)}}(k-k')]y}\,k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')dk'\} \quad (27)$$

In the first integral, the terms with $n = \ell$ lead to an exponential factor $e^{is^{(\ell)}ky}$ and we pick this out explicitly. The remaining terms we denote symbolically by

$$-i\sum_L\int R_{iL}(k,k')e^{iT_L(k,k')ky}\,dk'.$$

A list of the different $R_{iL}(k,k')$ and $T_L(k,k')$ is contained in Table 1. With this notation we then have, after some manipulation,

$$N_i = \sum_\ell -\frac{1}{2}\,ikN^{(1)}_{i\ell\ell}\,e^{is^{(\ell)}ky}I_{\alpha_\ell}(k) - i\sum_L\int R_{iL}(k,k')e^{iT_L(k,k')ky}\,dk' \quad (28)$$

where for any function $\phi(k)$ we denote

$$I_\phi(k) = \int_0^k k'(k-k')\phi(k')\phi(k-k')dk'. \quad (29)$$

Equation (11) now takes the following form:

$$c_{i2j2}B_{j,yy} + ik(c_{i1j2} + c_{i2j1})B_{j,y} + k^2(\rho c^2\delta_{ij} - c_{i1j1})B_j$$

$$+ ik\sum_{\ell}\Big\{[2c_{i2j2}s^{(\ell)}\alpha_{\ell,\eta} + (c_{i1j2} + c_{i2j1})(s^{(\ell)}\alpha_{\ell,\xi} + \alpha_{\ell,\eta})$$

$$+ 2c_{i1j1}\alpha_{\ell,\xi} + 2\rho c\delta_{ij}\alpha_{\ell,\tau}]a_j^{(\ell)} - N_{i\ell\ell}^{(1)}I_{\alpha_\ell}(k)\Big\}e^{is^{(\ell)}ky}$$

$$- 2i\sum_{L}\int R_{iL}(k,k')e^{iT_L(k,k')ky}\,dk' = 0. \qquad (30)$$

Next, substituting solution (18) into equation (15) and using the expression (21), we find the following expression for the nonlinear term in the boundary conditions

$$N_i' = - C_i^{(1)}\int_0^k k'(k - k')\gamma(k')\gamma(k - k')dk'$$

$$- C_i^{(2)}\{\int_{-\infty}^0 + \int_k^\infty k'(k - k')\gamma(k')\gamma(k - k')dk'\}$$

where the following notation is used:

$$E_j^{(1)} = \sum_\ell \beta_\ell a_j^{(\ell)} \quad , \qquad E_j^{(2)} = \sum_\ell s^{(\ell)}\beta_\ell a_j^{(\ell)} \qquad (31)$$

$$C_i^{(1)} = \sum_{p,q=1}^{2} d_{i2jpmq}\, E_j^{(p)}E_m^{(q)} \qquad (32)$$

$$C_i^{(2)} = \sum_{p,q=1}^{2} d_{i2jpmq}\, \overline{E_j^{(p)}}E_m^{(q)} = \sum_{p,q=1}^{2} d_{i2jpmq}E_j^{(p)}\overline{E_m^{(q)}}. \qquad (33)$$

Thus using the notation (29) and writing also

$$J_\phi(k) = \int_k^\infty k'(k-k')\phi(k')\phi(k-k')dk' = \int_{-\infty}^0 k'(k-k')\phi(k')\phi(k-k')dk', \qquad (34)$$

we have

$$N_i' = -C_i^{(1)} I_\gamma(k) - 2C_i^{(2)} J_\gamma(k) . \qquad (35)$$

The boundary condition (13) then takes the form

$$c_{i2j2}\, B_{j,y} + ikc_{i2j1}\, B_j + \sum_\ell [c_{i2j2}\, \alpha_{\ell,\eta} + c_{i2j1}\, \alpha_{\ell,\xi}\}_{\eta=0}\, a_j^{(\ell)}$$
$$- C_i^{(1)} I_\gamma(k) - 2C_i^{(2)} J_\gamma(k) = 0 \qquad (36)$$

5. <u>Second Order Solution</u>

We seek the solution of equation (30) in the form

$$B_j = \sum_\ell b_j^{(\ell)}(\xi,\eta,\tau)e^{iks^{(\ell)}y} + \sum_L \int W_{jL}(k,k')e^{ikT_L(k,k')y}\, dk'. \qquad (37)$$

This form of solution avoids secular terms of type $y\exp[is^{(\ell)}ky]$, and the assumed absence of such terms will lead to conditions on the function α_ℓ. Substituting (37) into (30) and comparing first the coefficient of $\exp[ik\,T_L\,y]$, we get

$$k^2 L_{ij}(T_L)W_{jL} = -2i\, R_{iL}$$

where $L_{ij}(s)$ is given by (16). Introducing the inverse matrix $L^{-1}_{ij}(s)$, we can therefore write

$$W_{iL} = \frac{2}{ik^2} L^{-1}_{ij}(T_L)R_{jL}. \tag{38}$$

Next, comparing the coefficients of $\exp[iks^{(\ell)}y]$, we get

$$\begin{aligned} &ikL_{ij}(s^{(\ell)})b_j^{(\ell)} + \{[2c_{i2j2}s^{(\ell)} + c_{i1j2} + c_{i2j1}]\alpha_{\ell,\eta} \\ &\quad + [(c_{i1j2} + c_{i2j1})s^{(\ell)} + 2c_{i1j1}]\alpha_{\ell,\xi}\}a_j^{(\ell)} \\ &\quad + 2\rho c\alpha_{\ell,\tau}\, a_i^{(\ell)} - N^{(1)}_{i\ell\ell}\, I_{\alpha_\ell}(k) = 0. \end{aligned} \tag{39}$$

The matrix $L_{ij}(s^{(\ell)})$ is singular. Since it is symmetric, the consistency condition can be obtained by pre-multiplying by $a_i^{(\ell)}$ and contracting over i (cf. (17)). We obtain therefore that

$$\Gamma^{(\ell)}\alpha_{\ell,\xi} + \Delta^{(\ell)}\alpha_{\ell,\eta} + \Theta^{(\ell)}\alpha_{\ell,\tau} - \Lambda^{(\ell)}I_{\alpha_\ell}(k) = 0 \tag{40}$$

where

$$\begin{aligned} &\Gamma^{(\ell)} = [2c_{i1j1} + (c_{i1j2} + c_{i2j1})s^{(\ell)}]a_i^{(\ell)}a_j^{(\ell)} \\ &\Delta^{(\ell)} = [2c_{i2j2}s^{(\ell)} + c_{i1j2} + c_{i2j1}]a_i^{(\ell)}a_j^{(\ell)} \\ &\Theta^{(\ell)} = 2\rho c a_i^{(\ell)}a_i^{(\ell)}, \qquad \Lambda^{(\ell)} = N^{(1)}_{i\ell\ell}\, a_i^{(\ell)} \end{aligned} \tag{41}$$

By using (16) and (17) we can obtain the following relation between these constants:

$$\Gamma^{(\ell)} + s^{(\ell)}\Delta^{(\ell)} = c\ \Theta^{(\ell)} \tag{42}$$

We note that (40) is a first order integro-differential equation which determines α_ℓ in the region $\eta < 0$ once the boundary value on $\eta = 0$ is known - that is, once $\gamma(k,\xi,\tau)$ is known. The solution of equations of the type (40) is discussed in the final section of reference [1]. We shall not pursue the solution here since our aim is to determine simply the boundary function γ.

We suppose that the general solution of the system

$$L_{ij}(s^{(\ell)})b_j^{(\ell)} = f_i^{(\ell)} \tag{43}$$

can be written as

$$b_j^{(\ell)} = \delta_\ell a_j^{(\ell)} + U_{ji}^{(\ell)} f_i^{(\ell)} \tag{44}$$

where δ_ℓ is an arbitrary constant and $U_{ji}^{(\ell)}$ is the inverse of $L_{ij}(s^{(\ell)})$ restricted to the subspace orthogonal to $a_j^{(\ell)}$. For example, $\underline{U}^{(\ell)}$ may be expressed in terms of the two non-zero eigenvalues $\lambda^{(2)}$, $\lambda^{(3)}$ of $L_{ij}(s^{(\ell)})$ and their corresponding eigenvectors $a_{1j}^{(\ell)}$, $a_{2j}^{(\ell)}$ $[L_{ij}(s^{(\ell)})a_{pj}^{(\ell)} = \lambda^{(p)}\ a_{pj}^{(\ell)},\ p = 1,2]$ as follows:

$$U_{ij}^{(\ell)} = \sum_{p=1,2} \frac{1}{\lambda^{(p)}} \frac{a_{pi}^{(\ell)} a_{pj}^{(\ell)}}{a_{pk}^{(\ell)} a_{pk}^{(\ell)}} . \tag{45}$$

Then the solution of equations (39) is given by

$$b_j^{(\ell)} = \delta_\ell a_j^{(\ell)} - \frac{1}{ik} \{F_j^{(\ell)} \alpha_{\ell,\eta} + G_j^{(\ell)} \alpha_{\ell,\xi} - H_j^{(\ell)} I_{\alpha_\ell}(k)\} \tag{46}$$

where

$$F_j^{(\ell)} = U_{ji}^{(\ell)} a_m^{(\ell)} [2c_{i2m2} s^{(\ell)} + c_{i1m2} + c_{i2m1}]$$

$$G_j^{(\ell)} = U_{ji}^{(\ell)} a_m^{(\ell)} [(c_{i1m2} + c_{i2m1}) s^{(\ell)} + 2c_{i1m1}] \tag{47}$$

$$H_j^{(\ell)} = U_{ji}^{(\ell)} N_{i\ell\ell}^{(1)} .$$

The first two of these constants are related, for using (17) and the fact that $U_{ji}^{(\ell)} a_i^{(\ell)} = 0$, as follows for example from (45), we get that

$$G_j^{(\ell)} = - s^{(\ell)} F_j^{(\ell)} . \tag{48}$$

The final step is to substitute (37) into the second order boundary conditions (36). We obtain

$$\sum_\ell ik[c_{i2j2} s^{(\ell)} + c_{i2j1}] b_j^{(\ell)} + ik \sum_L \int [c_{i2j2} T_L + c_{i2j1}] W_{jL} dk'$$

$$+ \sum_\ell [c_{i2j2} \alpha_{\ell,\eta} + c_{i2j1} \alpha_{\ell,\xi}] a_j^{(\ell)}$$

$$- C_i^{(1)} I_\gamma(k) - 2C_i^{(2)} J_\gamma(k) = 0$$

where all quantities are evaluated on $\eta = 0$. After substituting from (38) and (46) for W_{jL} and $b_j^{(\ell)}$ and using (20) and (48) we obtain

$$ik\, M_{i\ell}\delta_\ell + \sum_\ell [c_{i2j2}\, a_j^{(\ell)} - (c_{i2j2}\, s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]\alpha_{\ell,\eta}$$

$$+ \sum_\ell [c_{i2j1}\, a_j^{(\ell)} + s^{(\ell)}(c_{i2j2}\, s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]\alpha_{\ell,\xi}$$

$$+ \sum_\ell (c_{i2j2}\, s^{(\ell)} + c_{i2j1})H_j^{(\ell)} I_{\alpha_\ell}(k) - C_i^{(1)}\, I_\gamma(k) - 2C_i^{(2)}\, J_\gamma(k)$$

$$+ \frac{2}{k}\sum_L \int [c_{i2j2}\, T_L + c_{i2j1}]L_{jn}^{-1}(T_L)R_{nL}dk' = 0$$

These equations for δ_ℓ are singular by virtue of the Rayleigh wave speed condition. Let β_i' denote any particular solution of the adjoint homogeneous system $M_{i\ell}\, \beta_i' = 0$. Then multiplying throughout by β_i' and contracting over i we obtain the consistency condition for the existence of a solution. Substituting from (40) for $\alpha_{\ell,\eta}$ and then using (21) to express the α_ℓ in terms of γ, the result is obtained in the form

$$P\gamma_\tau + Q\gamma_\xi + P'I_\gamma(k) + Q'J_\gamma(k)$$

$$+ \frac{2}{k}\sum_L \int \beta_i'[c_{i2j2}\, T_L + c_{i2j1}]L_{jn}^{-1}(T_L)R_{nL}dk' = 0 \tag{49}$$

where

$$P = \sum_\ell \beta_i'[-c_{i2j2}\, a_j^{(\ell)} + (c_{i2j2}\, s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]\beta_\ell\, \theta^{(\ell)}/\Delta^{(\ell)} \tag{50}$$

$$Q = \sum_{\ell}\beta_i'\beta_\ell\{[c_{i2j1}a_j^{(\ell)} + s^{(\ell)}(c_{i2j2}s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]$$

$$- [c_{i2j2}a_j^{(\ell)} - (c_{i2j2}s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]\Gamma^{(\ell)}/\Delta^{(\ell)}\} = cP \quad (51)$$

$$P' = - \beta_i'C_i^{(1)} + \sum_{\ell}\beta_i'\beta_\ell^2\{(c_{i2j2}s^{(\ell)} + c_{i2j1})\theta_j^{(\ell)}$$

$$+ [c_{i2j2}\, a_j^{(\ell)} - (c_{i2j2}s^{(\ell)} + c_{i2j1})F_j^{(\ell)}]\Lambda^{(\ell)}/\Delta^{(\ell)}\} \quad (52)$$

$$Q' = - 2\beta_i'C_i^{(2)} \quad (53)$$

(The result $Q = cP$ is obtained using (42) and (20) and that $M_{i\ell}\beta_\ell = 0$.) After substituting the values of R_{nL} and T_L in the various regions of integration from Table 1, the integral terms can be combined and the resulting equation written in the form

$$\gamma_\xi + c^{-1}\gamma_\tau + \int_0^\infty H(k,k')k'(k-k')\gamma(k')\gamma(k-k')dk' = 0 \quad (54)$$

where $H(k,k')$ is given as follows.

For $0 < k' < k$,

$$H(k,k') = \frac{P'}{Q} + \frac{2}{Qk}\sum_{\ell\neq n}\beta_i'\beta_\ell\beta_n(k-k')\{c_{i2j2}\frac{s^{(\ell)}k' + s^{(n)}(k-k')}{k} + c_{i2j1}\}\cdot$$

$$\cdot N_{i\ell n}^{(1)} L_{jn}^{-1}(\frac{s^{(\ell)}k' + s^{(n)}(k-k')}{k}). \quad (55)$$

For $k' > k$,

$$H(k,k') = \frac{Q'}{Q} + \frac{2}{Qk}\sum_{\ell,n} \beta_i'\beta_\ell\bar{\beta}_n\{c_{i2j2}\frac{s^{(\ell)}k' + \overline{s^{(n)}}(k-k')}{k} + c_{i2j1}\}\cdot$$

$$\cdot\{(k - k')N^{(3)}_{i\ell n}\, L^{-1}_{jn}(\frac{s^{(\ell)}k' + \overline{s^{(n)}}(k - k')}{k})$$

$$+ k'N^{(2)}_{in\ell}\, L^{-1}_{j\ell}(\frac{s^{(\ell)}k' + \overline{s^{(n)}}(k - k')}{k})\} \quad (56)$$

6. Discussion and Summary

The displacement components throughout the medium are given by (7) where in the lowest order, $\tilde{u}_j$ may be replaced by A_j which is given by (18). In particular the displacement components on the free surface are given in terms of the function γ by equation (22). The ξ and τ dependence of this function is determined by the final integro-differential equation (54).

Consider the case of a signalling problem in which the displacements are given on a boundary $x = \xi = 0$ and the wave propagates into the region $x > 0$. For such problems the τ dependence of γ is not needed. Furthermore, let us restrict to waveforms which contain only a fundamental and its harmonics with the wave numbers $k = 1,2,\ldots$. Then the integrals over k must be replaced by sums, and we have

$$u_j(x,0,t) = \sum_\ell \beta_\ell a_j^{(\ell)} \sum_{k=1}^{\infty} \gamma_k(\xi)e^{ik(x-ct)} + CC \quad (57)$$

where

$$\frac{d\gamma_k(\xi)}{d\xi} + \sum_{k'=1}^{\infty} H_{kk'}\, k'(k - k')\gamma_{k'}(\xi)\gamma_{k-k'}(\xi) = 0 \,, \qquad (58)$$

$\gamma_{-k}(\xi) = \overline{\gamma_k(\xi)}$ and $H_{kk'}$ is given in (55) and (56).

This system of equations has precisely the same form as equations (50) in reference [3] and the particular solutions obtained there can be taken over directly to the anisotropic case. For the generation of harmonics by an initially sinusoidal wave, we have the boundary conditions $\gamma_1 = a$, $\gamma_k = 0$ $(k > 1)$ at $\xi = 0$ and the first few terms in the power series solution of (58) then have the form

$$\gamma_1 = a\{1 - H_{12}H_{21}a\bar{a}\xi^2 + \ldots\}$$

$$\gamma_2 = - H_{21}a^2\xi\{1 - [\tfrac{2}{3} H_{12}H_{21} + H_{23}(H_{31} + H_{32})]a\bar{a}\xi^2 + \ldots\}$$

$$\gamma_3 = H_{21}(H_{31} + H_{32})a^3\xi^2 + \ldots \;.$$

From these expressions, the linear growth of the amplitude of the second harmonic as well as the cubic derivation from linearity may be calculated. The quadratic depletion of the fundamental and the quadratic growth of the third harmonic can also be found.

A second problem analyzed in [3] is the parametric amplification of a weak signal by a strong pump signal of twice the frequency. Here the boundary conditions associated with the system (58) are $\gamma_1 = \kappa a$, $\gamma_2 = a$, $\gamma_k = 0$ $(k > 2)$ on $\xi = 0$, where $\kappa \ll 1$. Expanding the solution of (58) in powers of both ξ and κ, we find the following solution for the γ_1:

$$\gamma_1 = \kappa a + 2H_{12}\bar{\kappa} a \bar{a}\xi + \{2H_{12}\bar{H}_{12} - 6H_{13}(H_{31} + H_{32})\}\kappa a^2\bar{a}\xi^2 + \ldots$$

The linear and quadratic growth in the signal amplitude with distance can be calculated for any given material from this expression.

TABLE 1

Range of k'	Range of ℓ,n	$R_{iL}(k,k')$	$T_L(k,k')$
$0<k'<k$	$\ell \neq n$	$N^{(1)}_{i\ell n}k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')$	$k^{-1}[s^{(\ell)}k' + s^{(n)}(k-k')]$
$k'<0$	All ℓ,n	$N^{(2)}_{i\ell n}k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')$	$k^{-1}[\overline{s^{(\ell)}}k' + s^{(n)}(k-k')]$
$k'>k$	All ℓ,n	$N^{(3)}_{i\ell n}k'(k-k')^2\alpha_\ell(k')\alpha_n(k-k')$	$k^{-1}[s^{(\ell)}k' + \overline{s^{(n)}}(k-k')]$

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