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R. J. Gardner

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

CORSON COMPACTS AND RADON SPACES †

R.J. Gardner

Abstract. Assuming the continuum hypothesis, there is a Corson compact space of cardinality ω_1 which is not a Radon space, and a Corson compact Radon space which is not hereditarily weakly θ-refinable.

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1. Introduction. The aim of this note is to provide some examples which shed some light on the structure of Corson compact spaces. We are particularly interested in the behaviour of measures on Corson compacts, but this is inextricably linked with the topological covering properties they have. So, with the help of the continuum hypothesis, we apply constructions from the theory of Radon spaces to exhibit Corson compacts which have unusually complicated topologies. The discussion in the next section suggests that, conversely, the study of Corson compacts may provide answers to some unsolved problems concerning Radon spaces and covering properties.

2. Preliminaries. We write $|A|$ for the cardinality of a set A. An ordinal is identified with the set of all smaller ordinals, and ω and ω_1 denote the first infinite and first uncountable ordinals. A cardinal is identified with its initial ordinal.

All spaces will be Hausdorff. If X is a space, $\mathcal{G}$ and $\mathcal{K}$ are the families of all open and compact subsets of X, respectively. The family of Borel sets, $\mathcal{B}$, is the smallest σ-algebra in X containing $\mathcal{G}$.

In this paper, a Borel measure μ in X is a non-negative countably additive and finite set function defined on $\mathcal{B}$.

A Radon measure μ is a Borel measure which satisfies $\mu(B) = \sup\{\mu(K): K \in \mathcal{K},\ K \subset B\}$ for all $B \in \mathcal{B}$. We say that X is a Radon space if each Borel measure in X is a Radon measure.

If $\mathcal{A}$ is a family of sets in X, and $x \in X$, $st(x, \mathcal{A}) = \{A \in \mathcal{A}: x \in A\}$. Then $\mathcal{A}$ is called point-finite or point-countable if $|st(x,\mathcal{A})| < \omega$ or $|st(x, \mathcal{A})| \leq \omega_1$, respectively, for each $x \in X$. Also, $\mathcal{A}$ is σ-point-finite if $\mathcal{A} = \cup_{n\in\omega}\mathcal{A}_n$ with each $\mathcal{A}_n$ point-finite. A space X is $(\sigma\text{-})$metacompact or metalindelöf if each open cover of X has a $(\sigma\text{-})$point-finite or point-countable open refinement, respectively. A weakly θ-refinable space X is one for which every open cover has an open refinement $\cup_{n\in\omega}\mathcal{U}_n$ (called a weak θ-refinement) such that for each $x \in X$ there is an $n_x \in \omega$ with $1 \leq |st(x, \mathcal{U}_{n_x})| < \omega$. The countable chain condition is denoted by CCC.

The continuum hypothesis states that $2^\omega = \omega_1$; we denote this by CH. Martin's axiom, MA, is the assertion that less than 2^ω dense open sets in a compact space with the CCC have non-empty intersection.

An Eberlein compact is a space which is homeomorphic to a weakly compact (that is, compact in the weak topology) subset of a Banach space. Of several characterizations of Eberlein compacts in the literature, we shall use only one. A family $\mathcal{U}$ of subsets of X is called separating if, whenever $x \neq x'$ are in X, there is a $U \in \mathcal{U}$ such that $x \in U$ and $x' \notin U$, or vice versa. Then (see, for example, [MR], Theorem 1.4) a compact space is Eberlein compact if and only if it has a σ-point-finite, separating collection of open F_σ-subsets.

The less familiar Corson compacts, introduced in [C], are spaces homeomorphic to a compact subset of a Σ-product of closed unit intervals. We avoid the complexities of this definition by using a characterization similar to the one above for Eberlein compacts. A compact space is a Corson compact if and only if it has a point-countable, separating collection of open F_σ-subsets; the proof is given in [MR], Theorem 6.1.

Every Eberlein compact is a Corson compact, but the converse is not generally true (see [Ta], Theorem 4.3, or [AP], Example 7). In terms of the usual covering properties, an Eberlein compact is always hereditarily σ-metacompact (and hence hereditarily weakly θ-refinable), while Corson compacts are hereditarily metalindelöf. These facts are proved in [Y].

Eberlein compacts have been studied extensively, and [W] is a survey of some of their properties. For example, it is not difficult to deduce from their topological characterization above that each Eberlein compact with the CCC is metrizable, and hence separable. Another property interests us more here. Denote by (*) the condition on a space X that it contains no discrete subsets of measurable cardinality (see, e.g., [P], Chapter 18). Then W. Schachermayer proved ([S]) that every Eberlein compact with (*) is a Radon space.

There has been considerable interest in finding topological conditions on a space which imply that it is Radon. For compact spaces, the following theorem (see [G_1], [P] or [GP_3]) is the most general known.

<u>THEOREM A</u>. A compact space with (*), which is also hereditarily weakly θ-refinable, is a Radon space.

Since Eberlein compacts with (*) have the required properties, we immediately deduce that they are Radon spaces, and sidestep Schachermayer's direct proof in [S].

Attempts to generalize Theorem A lead to the consideration of hereditarily metalindelöf spaces. In [GP_1] it was noted that a theorem of Tall shows that under MA + $\neg$CH, each compact space which is also hereditarily metalindelöf and locally CCC is hereditarily paracompact. Therefore under MA + $\neg$CH Theorem A implies that each such space with (*) is a Radon space. On the other hand, ([GP_1], Section 2), if CH is assumed, there are compact spaces with (*), which are in addition hereditarily metalindelöf and CCC, but which are not Radon.

Since Corson compacts are generalizations of Eberlein compacts, it is natural to ask whether they have similar nice properties. It is known that if MA + $\neg$CH is assumed, a Corson compact with the CCC must be separable. To see this, we invoke another theorem of Tall ([Tl], Theorem 6.2) that under MA + $\neg$CH each compact, hereditarily metalindelöf space with the CCC must be hereditarily Lindelöf; then separability follows (using MA + $\neg$CH again) by a result of Juhász (see [Ro], 6.2). From the topological characterization given above, it is not hard to see that a separable Corson compact is metrizable. So, under MA + $\neg$CH, Corson compacts share some of the behaviour of Eberlein compacts. However, there are known examples ([CN], p.206; see also Theorem 4.1 below), constructed using CH, of non-separable Corson compacts with the CCC.

The question arises: are Corson compacts with (*) Radon spaces? Contrary to the hasty assertion in [G_2], 19.4, it is not obvious that assuming MA + $\neg$CH they are. The stumbling block is the CCC. From what we have said, no 'real' example of a non-metrizable Corson compact has the CCC. Furthermore, there are also Corson compacts which do not even satisfy the CCC locally. Example 4 of [Y], an Eberlein compact which is not hereditarily metacompact, shows this. For, if this space satisfied the CCC locally, it would (under MA + $\neg$CH) have to be hereditarily paracompact. It is not known if, assuming MA + $\neg$CH, compact spaces with (*) which are hereditarily metalindelöf are Radon spaces.

The main purpose here is to show that CH implies that there are Corson compacts with (*) which are not Radon, and Corson compact Radon spaces which are not hereditarily weakly θ-refinable. Therefore, they differ markedly in this case from Eberlein compacts. Along the way we also use CH to build a relatively simple compact Radon space with (*) which is not hereditarily weakly θ-refinable, which shows that under CH the converse of Theorem A is not true. The constructions are based on those in [GG] and [GP_1]. There is still some hope that under MA + $\neg$CH Theorem A can be reversed; this is an interesting open problem. We do not know if examples such as ours exist without extra set-theoretic axioms being assumed.

Since the results of this paper were obtained, there have been further developments. Strictly between the classes of Eberlein and Corson compacts lie two others, called Talagrand compacts and Gul'ko compacts, the latter class being larger. In [Ta], Theorem 6.11, it is proved that each Talagrand compact with (*) is a Radon space. Gary Gruenhage ([Gr_1]) has improved this by showing that Gul'ko compacts are hereditarily weakly θ-refinable. It follows from Theorem A that Gul'ko compacts with (*) are Radon spaces. In another paper ([Gr_2]) he shows that a Corson compact space first constructed by Todorčević is not hereditarily weakly θ-refinable, but is Radon if it has (*). No extra axioms are needed for this, but the space contains discrete subsets of cardinality c, so one needs to assume that c is not measurable to deduce that the space is Radon. This assumption is much weaker than CH, so the example is a considerable improvement on Theorem 4.6 in this respect. However, the space constructed in Theorem 4.6 has the additional advantage of satisfying the CCC.

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3. Background Material. We shall need two constructions, both of which need CH. The first is Kunen's space.

<u>PROPOSITION 3.1</u> ([Ku])(CH). There is a first countable, hereditarily Lindelöf, non-separable compact space Z with $|Z| = \omega_1$. There is a Radon measure ν in Z such that $\nu(Z) = 1$. In this space, the classes of Borel sets which are separable, second countable, nowhere dense, or which have ν-measure zero, all coincide.

We shall call the second construction the JKR machine.

<u>PROPOSITION 3.2</u> ([JKR])(CH). Given a first countable space Y' with $|Y'| = \omega_1$, there is a first countable, regular, locally compact and locally countable space Y whose topology is finer than that of Y'. Since $|Y| = \omega_1$, Y cannot be Lindelöf. Furthermore, if Y' is hereditarily separable, then Y is also, and in this case Y has the same Borel sets as Y'.

4. <u>The examples</u>.

<u>THEOREM 4.1</u>. Kunen's space Z is a Corson compact.

<u>Proof</u>. Let $Z = \{z_\alpha\}_{\alpha<\omega_1}$, and for each $\alpha < \omega_1$, let $U_\alpha = Z - c\ell(\cup_{\beta<\alpha}\{z_\beta\})$. Since $c\ell(\cup_{\beta<\alpha}\{z_\beta\})$ is separable, by Proposition 3.1 $\{U_\alpha\}_{\alpha<\omega_1}$ is a decreasing family of open sets with $\nu(U_\alpha) = 1$ for each α, which is clearly point-countable. For each $\alpha < \omega_1$, let $H_\alpha = U_\alpha - U_{\alpha+1}$.

Then $\nu(H_\alpha) = 0$, so H_α is second countable. Therefore, there are open in Z sets $\{G_n^\alpha\}_{n\in\omega}$ with $G_n^\alpha \subset U_\alpha$ such that $\{G_n^\alpha \cap H_\alpha\}_{n\in\omega}$ is a countable open base in H_α. Consider the family $\{G_n^\alpha : \alpha < \omega_1,\ n < \omega\}$. This is a point-countable family of open sets, and each is F_σ because Z is hereditarily Lindelöf. Suppose $z \neq z'$ are in Z. If $z \in H_\alpha$ and $z' \in H_\gamma$, with $\gamma > \alpha$, then there is an n such that $z' \in G_n^\gamma$ and $z \notin G_n^\gamma$. If z and z' are in H_α, there is an n with $z \in G_n^\alpha$ and $z' \notin G_n^\alpha$. Therefore the family is also point-separating, and the proposition is proved. |||

The next example is essentially that of [GP_1], Section 2.

<u>THEOREM</u> 4.2 (CH) There is a Corson-compact with (*) which is not a Radon space.

<u>Proof</u>. Let ω_1 be the space of countable ordinals, and let $\{U_\alpha\}_{\alpha<\omega_1}$ be a base for ω_1 consisting of countable open sets. Let Z be Kunen's space and suppose we relabel the point-countable, separating family of open F_σ-sets of Theorem 4.1 as $\{V_\alpha\}_{\alpha<\omega_1}$. Let $X_0 = \cup_{\alpha<\omega_1}(U_\alpha \times V_\alpha)$ with the product topology inherited from $\omega_1 \times Z$. Then X_0 is a locally compact space; let X be the one-point compactification of X_0.

Since $\{V_\alpha\}_{\alpha<\omega_1}$ is point-countable and separating in Z, and $\{U_\alpha\}_{\alpha<\omega_1}$ is a base in ω_1, $\{U_\alpha \times V_\alpha\}$ is point-countable and separating in X_0, and therefore also in X. Each U_α is countable, and each V_α is σ-compact, so $(U_\alpha \times V_\alpha)$ is F_σ in X for $\alpha < \omega_1$. It follows that X is a Corson compact. Also, X has (*) since ω_1 is not a measurable cardinal.

The proof that X is not Radon is exactly that of [GP_1], Section 2, and we omit it. A rather simpler proof of this can be found in [GP_2], Example 11.20. |||

Before proving the next theorem, we need a definition. A space in which there is no nontrivial Borel measure which vanishes on singletons is called <u>universally measure zero</u>. Under CH, there are uncountable Lusin subsets of [0,1], and these are universally measure zero ([L], p.151). There are also 'real' examples of uncountable universally measure zero sets (see [L], Theorem 1.2). Obviously universally measure zero spaces are Radon.

<u>THEOREM 4.3</u> (CH) There is a compact, hereditarily separable, Radon (even universally measure zero) space which is not hereditarily weakly θ-refinable.

Proof. Let Y' be an uncountable Lusin subset of $[0,1]$, and let Y be the space obtained from Y' by the JKR machine (Proposition 3.2). Then Y is locally compact and hereditarily separable, so the one-point compactification X of Y is also hereditarily separable. Since Y' is universally measure zero, and Y has the same Borel sets as Y', it follows that Y, and so X, is also universally measure zero.

Suppose the subset Y of X were weakly θ-refinable. Then any open cover of Y has a weak θ-refinement $\cup_{n\in\omega}\mathcal{U}_n$. For each $n \in \omega$, let $Y_n = \{y \in Y : 1 \leq |st(y,\mathcal{U}_n)| < \omega\}$, so that $Y = \cup_{n\in\omega}Y_n$. Since Y_n is separable, and $\mathcal{U}_n$ is point-finite in Y_n, a countable subfamily of $\mathcal{U}_n$ covers Y_n. Thus $\cup_{n\in\omega}\mathcal{U}_n$ also has a countable open subcover, which means that Y is Lindelöf. This is impossible, by Proposition 3.2. |||

We turn now to the problem of finding a Corson compact Radon space without nice covering properties. Of course, a hereditarily separable Corson compact is hereditarily Lindelöf, but the next theorem shows that the example we seek must also differ in another way from the previous one.

THEOREM 4.4. A universally measure zero Corson compact is hereditarily metacompact.

Proof. A universally measure zero compact space is scattered, that is, it contains no non-empty perfect subset. Proofs can be found in [PS] or [Kn]. Now K. Alster ([A]) has shown that a scattered Corson-compact is hereditarily metacompact (in fact, an Eberlein compact). |||

In fact (again using [A]) Theorem 4.4 remains true for any compact hereditarily metalindelöf space, not just Corson compacts.

The idea for the next example is to use the construction of Theorem 4.2, but replace ω_1 by the universally measure zero space Y used in Theorem 4.3. The problem is to show that the resulting space is not hereditarily weakly θ-refinable. The argument of [GG], 2.3, must be substantially modified. We need a lemma which has escaped standard texts in measure theory. It was proved for Lebesgue measure by Marczewski in 1945; a much stronger version is ascribed to Argyros and Kalamidas in [CN], Theorem 6.15. Our simple proof uses the partition calculus.

Lemma 4.5. Let μ be a Borel measure in X and $\{B_\alpha\}_{\alpha<\omega_1}$ an uncountable family of Borel sets with $\mu(B_\alpha) > 0$ for $\alpha > \omega_1$. Then there is an uncountable set $A \subset \omega_1$ such that $B_\alpha \cap B_{\alpha'} \neq \phi$ for all α, α' in A.

Proof. There is an uncountable $A' \subset \omega_1$ and $r > 0$ such that $\mu(B_\alpha) \geq r$ for all $\alpha \in A'$. Now if $\alpha_n \in A'$ for $n < \omega$, there are m, m' such that $B_{\alpha_m} \cap B_{\alpha_{m'}} \neq \phi$, since X has finite μ-measure. Partition the set of pairs (α, α') with α, α' in A' by letting $P_1 = \{(\alpha,\alpha'):B_\alpha \cap B_{\alpha'} \neq \phi\}$ and $P_2 = \{(\alpha, \alpha'):B_\alpha \cap B_{\alpha'} = \phi\}$. Then Erdös's theorem $\omega_1 \to (\omega_1,\omega)^2$ of the partition calculus (see, for example, [Ru], p.8) says that either there is a subset A of A', of cardinality ω_1, such that all pairs in A belong to P_1, or else there is a subset E of A', of cardinality ω, such that all pairs in E belong to P_2. We have noted the latter is impossible, and the former is exactly what we want. ||

THEOREM 4.6.(CH) There is a Corson compact Radon space which is not hereditarily weakly θ-refinable.

Proof. Let Y be the locally compact, locally countable hereditarily separable and universally measure zero space from the proof of Theorem 4.3, and let $\{U_\alpha\}_{\alpha<\omega_1}$ be a base of countable open sets for Y. Let Z be Kunen's space of Proposition 3.1, and ν its associated measure, and let $\{V_\alpha\}_{\alpha<\omega_1}$ be the point-countable, separating family of open F_σ-sets in Z as in Theorem 4.2. Define $X_0 = \cup_{\alpha<\omega_1}(U_\alpha \times V_\alpha)$, and X to be the one-point compactification of X_0. That X is a Corson compact follows as in Theorem 4.2.

If μ is any Borel measure in X_0, then the measure m defined by $m(B) = \mu((B \times Z) \cap X_0)$ for Borel sets B in Y is a Borel measure in Y. Since Y is universally measure zero, m is identically zero outside a countable set D. Therefore, μ is identically zero outside $(D \times Z) \cap X_0$, which is hereditarily Lindelöf since Z is. From this and Theorem A one easily deduces that μ is Radon in X_0. The one-point compactification of a Radon space is Radon, so X is a Radon space.

We now show that X is not weakly θ-refinable. Suppose it is; then (see [P], 18.30) the open cover $\{U_\alpha \times V_\alpha\}_{\alpha<\omega_1}$ has a weak θ-refinement $\bigcup_{n=1}^{\infty} \mathcal{G}_n$, so that if

$$X_n = \{x \in X_0 : |st(x, \mathcal{G}_n)| = 1\},$$

then $X_0 = \bigcup_{n=1}^{\infty} X_n$.

Let $Y = \cup_{\alpha<\omega_1}\{y_\alpha\}$, and for each $\alpha < \omega_1$, define

$$X_0^\alpha = \{(y_\alpha, z) : z \in Z\} \cap X_0.$$

Note that in X_n, the collection of sets $\{X_n \cap G : G \in \mathcal{G}_n\}$ is discrete. As Z satisfies the CCC, for each $\alpha < \omega_1$ only countably many of the sets $\{X_n \cap G : G \in \mathcal{G}_n\}$ meet X_0^α. Therefore X_0^α is covered by a countable subfamily of

$\{X_n \cap G : G \in G_n;\ n \in \omega\}$. Nowhere dense subsets of Z have ν-measure zero, while $\mathrm{proj}_Z X_0^\alpha$ is open in Z and therefore has positive ν-measure. It follows that there is an n and a $G_\alpha \in G_n$ such that if

$$E(n, \alpha) = \{z : (y_\alpha, z) \in X_n \cap G_\alpha\},$$

then $Z(n, \alpha) = \mathrm{int}(c\ell\, E(n,\alpha)) \neq \phi$.

For some $n_0 \in \omega$, there is an uncountable set $S \subset \omega_1$ such that for all $\alpha \in S$, $Z_\alpha = Z(n_0, \alpha) \neq \phi$. Suppose α_0 is the least element of S, and choose α_γ for $1 \leq \gamma < \omega_1$ inductively as follows. For each γ, Let $Y_\gamma = \mathrm{proj}_Y(\cup_{\beta<\gamma} G_{\alpha_\beta})$ For each β, $\mathrm{proj}_Y\, G_{\alpha_\beta}$ is a countable set, since G_{α_β} is a member of the refinement of $\{U_\alpha \times V_\alpha\}_{\alpha<\omega_1}$. Consequently Y_γ is also countable, and we can choose α_γ in S larger than each element of Y_γ. Let $S' = \{\alpha_\gamma : \gamma < \omega_1\} \subset S$. Note that if $\alpha < \alpha'$ are in S' then $\alpha' \notin \mathrm{proj}_Y\, G_\alpha$.

For each $\alpha \in S'$, choose a point $x_\alpha = (y_\alpha, z_\alpha) \in X_{n_0} \cap G_\alpha$. Then $z_\alpha \in Z_\alpha$, so there is an H_α open in Y and a J_α open in Z with $z_\alpha \in J_\alpha \subset Z_\alpha$ and $H_\alpha \times J_\alpha \subset G_\alpha$.

Now $\nu(J_\alpha) > 0$ for each $\alpha \in S'$, so by Lemma 4.5 there is an uncountable set $T \subset S'$ such that $J_\alpha \cap J_{\alpha'} \neq \phi$ for all α, α' in T. The set $\{y_\alpha : \alpha \in T\}$ is separable, so there are $\alpha < \alpha'$ in T such that $y_\alpha \in H_{\alpha'}$. Since $J_\alpha \cap J_{\alpha'} \neq \phi$, we can choose a point z in $J_\alpha \cap J_{\alpha'} \cap E(n_0,\alpha)$. Then $x = (y_\alpha, z) \in G_\alpha$, and also $x = (y_\alpha, z) \in H_{\alpha'} \times J_{\alpha'} \subset G_{\alpha'}$. But this implies that $G_\alpha = G_{\alpha'}$ (since $\{X_{n_0} \cap G : G \in \mathcal{G}_{n_0}\}$ is discrete in X_0^α), so $\alpha' \in \mathrm{proj}_Y G_\alpha$. This is a contradiction, since α and α' are also in S'. |||

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Department of Mathematical Sciences
University of Petroleum & Minerals
Dhahran, Saudi Arabia.