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THE LOVE WAVE SCATTERING MATRIX FOR THREE-LAYERED STRUCTURES CONSISTING OF WELDED LAYERED QUARTER-SPACES WITH A PLANE SURFACE

by

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ABSTRACT

In this paper we use spectral representation of the Love wave operator for a three-layer model comprising two homogeneous, infinite strips overlying a uniform half-space, along with a method based on an integral equation formulation and Schwinger-Levine variational principle to describe, by means of a scattering matrix, the diffraction of plane, harmonic, monochromatic Love waves, incident normally (from either side) upon the vertical plane of discontinuity in the three-layered structure consisting of welded layered quarter-spaces with a plane surface. Approximate expressions for the elements of the scattering matrix are obtained through the plane-wave approximation and their variational improvement is sought through the Schwinger-Levine variational principle in such a way as to incorporate the contributions caused by body-wave conversion. Complex reflection and transmission coefficients can be obtained through a transmission matrix related to the scattering matrix. We obtain the form of the transmission matrix (under both approximations) in some simple cases.

INTRODUCTION

In our previous work [Kazi (1978a,b), Niazy and Kazi (1980,1982)] we used a method based on an integral equation formulation and Schwinger-Levine variational principle to describe, by means of a scattering matrix, the diffraction of plane, harmonic, monochromatic Love waves, incident normally upon the vertical planes of discontinuity in laterally discontinuous structures such as a half-space with a surface step and welded layered quarter-spaces (involving single top layers) with a plane surface. The method presupposes the existence of a complete set of proper and improper eigenfunctions, in terms of which the displacement fields on either side of the vertical plane of discontinuity may be expressed. Such a set of functions for the two-dimensional Love wave operator, associated with the propagation of monochromatic SH waves in a half-space overlain by a single layer, has been given in Kazi (1976). In order to be able to extend the method to laterally varying structures involving two layers over a half space, we need explicit spectral representation of the Love wave operator associated with monochromatic SH waves for a three-layer model comprising two homogeneous, infinite strips overlying a uniform half-space. Such a representation has been found in Kazi and Abu-Safiya (1982). In this paper we use this spectral representation to extend the method of integral representation and Schwinger-Levine variational principle to investigate the two-dimensional diffraction problem of plane harmonic Love waves, incident normally (from either side) upon the plane of discontinuity in the three-layered structure consisting of welded layered quarter spaces with a plane surface. The wave field is described by means of a scattering matrix, and approximate expressions for its elements are obtained through the plane-wave approximation and their variational improvement is sought through the variational principle of Schwinger and Levine. Complex reflection and transmission coefficients are obtainable through a transmission matrix related to the scattering matrix. The form of the transmission matrix in some simple special cases under the variational approximation indicates that the variational procedure incorporates the effects of propagated and non-propagated modes arising out of the continuous spectrum, which corresponds to body waves, and is, therefore, of considerable importance. Numerical computation of the reflection and transmission coefficients for backward as well as forward transmission in the welded quarter-spaces problem and other related problems will be given in another paper.

Equations of Motion

Let us suppose that a quarter-space consisting of a material of rigidity μ_3, shear velocity β_3, and density ρ_3, overlain by a layer of uniform depth H_2, density ρ_2, rigidity $\mu_2(<\mu_3)$ and shear velocity $\beta_2(<\beta_3)$ and another layer of uniform depth $H_1(<H_2)$, density ρ_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$, is in welded contact with a similar quarter-space of material of rigidity μ_3', shear velocity β_3' and density ρ_3', overlain by a layer of uniform depth H_2, density ρ_2', rigidity $\mu_2'(<\mu_3')$ and shear velocity $\beta_2'(<\beta_3')$ and another layer of uniform depth H_1, density ρ_1', rigidity $\mu_1'(<\mu_2')$ and shear velocity $\beta_1'(<\beta_2')$ (see Figure 1). We take the vertical plane of welded contact between the two structures to be $x = 0$, the plane of welded contact between the upper two layers to be the xy-plane in the coordinate system shown in the figure and regard the top plane surface $z = -H_1$ to be stress free. All materials are considered to be isotropic and homogeneous.

We consider two-dimensional problems of the diffraction of time-harmonic Love waves normally incident upon the vertical plane of contact (from either side). Again, the wave motion is entirely SH in character. The y-components of the seismic displacement fields in the regions $I(x<0)$ and $II(x>0)$ (see Figure 1) are denoted by $e^{-i\omega t}v(x,z)$ and $e^{-i\omega t}v'(x,z)$, respectively, where

$$e^{-i\omega t}v(x,z) = e^{-i\omega t}v_1(x,z),\ -H_1 \le z \le 0,\ x < 0,$$

$$= e^{-i\omega t}v_2(x,z),\ 0 < z \le H_2,\ x < 0,$$

$$= \ e^{-i\omega t}v_3(x,z),\ H_2 < z,\ x < 0,$$

and $$e^{-i\omega t}v'(x,z) = e^{-i\omega t}v_1'(x,z),\ -H_1 \leq z \leq 0,\ x > 0$$

$$= \ e^{-i\omega t}v_2'(x,z),\ 0 < z \leq H_2,\ x > 0$$

$$= \ e^{-i\omega t}v_3'(x,z),\ H_2 < z,\ x > 0,$$

(ω being the angular frequency) are the solutions of the Love wave differential equation

$$\rho(z)\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left[\mu(z)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu(z)\frac{\partial v}{\partial z}\right]$$

in the two regions on either side of the vertical plane $x = 0$.

The conditions at the free surface $z = -H_1$ and the plane of welded contact $x = 0$ imply

$$\frac{\partial v_1}{\partial z} = 0 \ \text{ and } \ \frac{\partial v_1'}{\partial z} = 0 \ \text{ at } \ z = -H_1, \tag{1a}$$

$$v = v' \ \text{ at } \ x = 0,\ \ z \geq -H_1, \tag{1b}$$

$$\mu(z)\frac{\partial v}{\partial x} = \mu'(z)\frac{\partial v'}{\partial x} \ \text{ at } \ x = 0,\ z \geq -H_1, \tag{1c}$$

where

$$\mu(z) = \mu_1,\ -H_1 \leq z < 0,\ x < 0,$$

$$= \mu_2,\ 0 < z < H_2,\ x < 0, \tag{2}$$

$$= \mu_3,\ H_2 < z,$$

and

$$\mu'(z) = \mu_1', \ -H_1 \leq z < 0, \ x > 0,$$

$$= \mu_2', \ 0 < z < H \ , \ x > 0, \qquad (3)$$

$$= \mu_3', \ H_2 < z, \ x > 0.$$

The complete solution for the displacement $v(x,z)$ in domain I (see Figure 1) can be expressed in terms of proper and improper eigenfunctions of the Love wave operator for a homogeneous half-space of rigidity μ_3 and shear velocity β_3, overlaid by two infinite strips consisting of a layer of depth H_2, rigidity $\mu_2(<\mu_3)$, and shear velocity $\beta_2(<\beta_3)$, and another layer of depth H_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$. Kazi and Abu-Safiya (1982) have found explicit formulas for these proper and improper eigenfunction and have shown that the spectrum of the corresponding two-dimensional Love wave operator is the disjoint union of the discrete spectrum, which corresponds to the ordinary Love modes, and a continuous spectrum (corresponding to body waves) which is the interval $(-\infty,\omega^2/\beta_3^2)$ on the real axis of the complex λ-plane, where $\lambda = k^2$, k being the wave number and ω the angular frequency. Likewise, we can write the complete solution for the displacement $v'(x,z)$ in domain II in terms of proper and improper eigenfunction of the Love wave operator for a homogeneous half-space of rigidity μ_3' and shear velocity β_3', overlaid by two infinite strips consisting of a layer of depth H_2, rigidity $\mu_2'(<\mu_3')$, and shear velocity $\beta_2'(<\beta_3')$ and another layer of depth H_1, rigidity $\mu_1'(<\mu_2')$ and shear velocity $\beta_1'(<\beta_2')$. Using the formulas derived in Kazi and Abu-Safiya(1982)

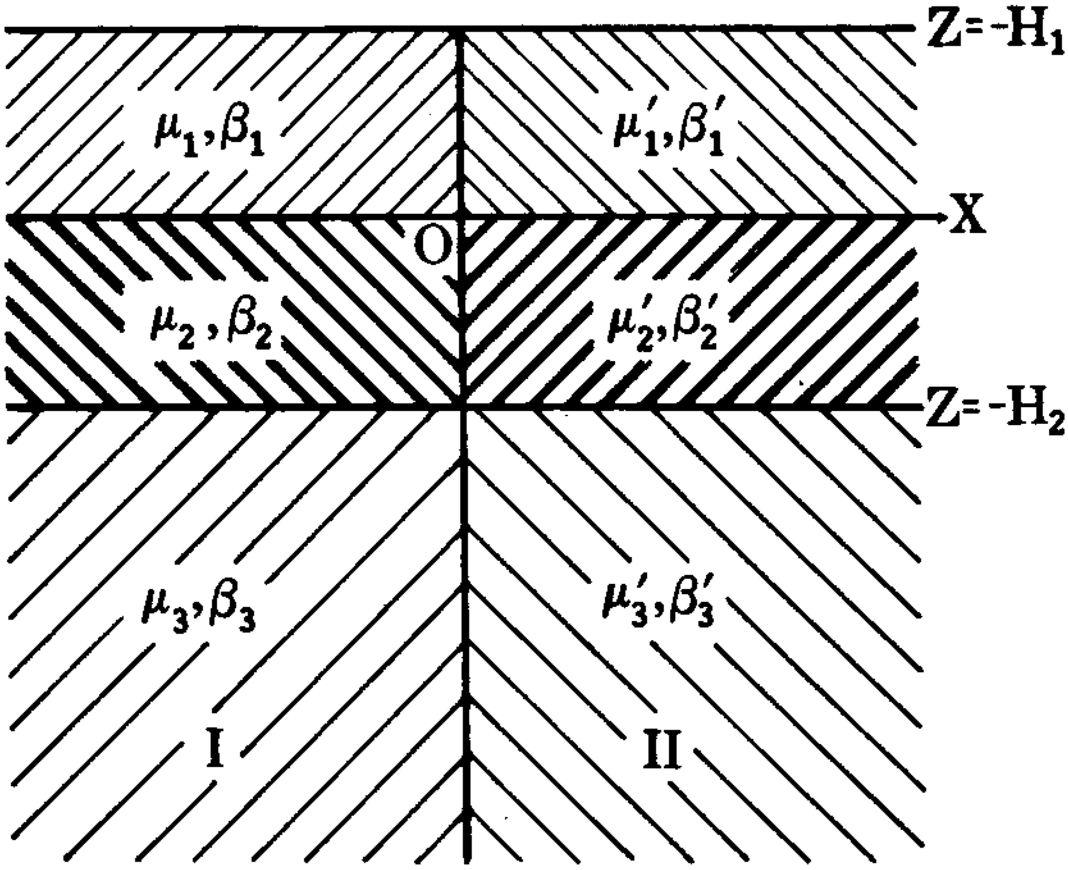

Figure 1 : The Geometry of the Problem

we have in Domain I ($x < 0$, $z \geq -H_1$)

$$v(x,z) = -\Big(\sum_{m=1}^{r} (A_m e^{-ik_m|x|} + B_m e^{ik_m|x|})\chi_m(z)$$

$$+ \int_0^{\omega/\beta_3} \{C(k)e^{-ik|x|} + D(k)e^{ik|x|}\}\phi(z,k)dk$$

$$+ \int_0^{\infty} E(k)e^{-k|x|}\psi(z,k)dk\Big) \tag{4}$$

and in Domain II ($x > 0$, $z \geq -H_1$)

$$v'(x,z) = \Big(\sum_{m=1}^{s} (A'_m e^{-ik'_m x} + B'_m e^{ik'_m x})\chi'_m(z)$$

$$+ \int_0^{\omega/\beta'_3} \{C'(k')e^{-ik'x} + D'(k')e^{ik'x}\}\phi'(z,k')dk'$$

$$+ \int_0^{\infty} E'(k')e^{-k'x} \psi'(z,k')dk' \tag{5}$$

where

$$\chi_m(z) = \phi_1^m(z), \quad -H_1 \leq z \leq 0$$

$$= \phi_2^m(z), \quad 0 \leq z \leq H_2$$

$$= \phi_3^m(z), \quad H_2 \leq z \tag{6}$$

$$\phi_1^m(z) = F_m \frac{\cos\{\sigma_1^m(z+H_1)\}}{\cos(\sigma_1^m H_1)}, \quad -H_1 \leq z \leq 0, \tag{7a}$$

$$\phi_2^m(z) = G_m \frac{\mu_2\sigma_2^m\cos\{\sigma_2^m(z-H_2)\}-\mu_3\sigma_3^m\sin\{\sigma_2^m(z-H_2)\}}{\cos(\sigma_2^m H_2)}, \quad 0 \leq z \leq H_2, \tag{7b}$$

$$\phi_3^m(z) = G_m \frac{\mu_2\sigma_2^m e^{-\sigma_3^m(z-H_2)}}{\cos(\sigma_2^m H_2)}, \quad z \geq H_2 \tag{7c}$$

$$F_m = \left[\left\{ \frac{M}{\frac{\partial}{\partial\lambda}(-\Delta)_{\lambda=\lambda_m}} \right\} \right]^{1/2} \tag{8}$$

$$G_m = \frac{1}{M} F_m , \tag{9}$$

$$M = \mu_2\sigma_2 + \mu_3\sigma_3\tan(\sigma_2 H_2), \tag{10}$$

$$\Delta = \mu_1\sigma_1\mu_2\sigma_2\tan(\sigma_1 H_1) + \mu_1\sigma_1\mu_3\sigma_3\tan(\sigma_2 H_2)\tan(\sigma_1 H_1) - \mu_3\sigma_3\mu_2\sigma_2 + (\mu_2\sigma_2)^2\tan(\sigma_2 H_2) \tag{11}$$

$$\sigma_1(\lambda) = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right)^{1/2}, \quad \sigma_2(\lambda) = \left(\frac{\omega^2}{\beta_2^2} - \lambda\right)^{1/2}, \quad \sigma_3(\lambda) = \left(\lambda - \frac{\omega^2}{\beta_3^2}\right)^{1/2} \tag{12}$$

$$\sigma_i(\lambda_m) = \sigma_i^m , \quad i = 1,2,3 \tag{13}$$

and $\lambda_n = k_n^2$, $k_n > 0$ are the roots of $\Delta = 0$, (14)

which is the dispersion equation for Love wave propagation in two layers over a half-space (see Ewing et al., 1957, p.229), and where

$$\psi(z,\lambda) = \psi_1(z,\lambda) = G^k\mu_2\sigma_2^k \frac{\cos\{\sigma_1^k(z+H_1)\}}{\cos(\sigma_1^k H_1)\cos(\sigma_2^k H_2)}, \quad -H_1 \leq z \leq 0 \tag{15a}$$

$$= \psi_2(z,\lambda) = \frac{G^k}{\cos(\sigma_2^k H_2)} \{\mu_2\sigma_2^k\cos(\sigma_2^k z) - \mu_1\sigma_1^k\sin(\sigma_2^k z)\tan(\sigma_1^k H_1)\},$$

$$0 \leq z \leq H_2, \qquad (15b)$$

$$= \psi_3(z,\lambda) = -\frac{\sin\{\Theta^k + s_3^k(z-H_2)\}}{\sqrt{\pi\mu_3 s_3^k}}, \quad H_2 \leq z, \qquad (15c)$$

where

$$G^k = \frac{\sqrt{2k}\mu_3 s_3^k\cos\theta^k}{p\sqrt{\pi\mu_3 s_3^k}}, \qquad (16)$$

$$s_3^k = \left(\frac{\omega^2}{\beta_3^2} - \lambda\right)^{\frac{1}{2}} \text{ (real and positive)} \qquad (17)$$

$$\Theta^k = \tan^{-1}\left(\frac{q}{p}\right), \qquad (18)$$

$$p = \mu_1\sigma_1^k\mu_2\sigma_2^k\tan(\sigma_1^k H_1) + \mu_2^2\,(\sigma_2^k)^2\tan(\sigma_2^k H_2), \qquad (19)$$

$$q = \mu_1\sigma_1^k\mu_2 s_3^k\tan(\sigma_2^k H_2)\tan(\sigma_1^k H_1) - \mu_2\sigma_2^k\mu_3 s_3^k, \qquad (20)$$

$$\sigma_1^k = \left(\frac{\omega^2}{\beta_1^2} - k^2\right)^{\frac{1}{2}}, \quad \sigma_2^k = \left(\frac{\omega^2}{\beta_2^2} - k^2\right)^{\frac{1}{2}} \qquad (21)$$

Owing to the factor $e^{-k|x|}$ in the integral containing ψ, these represent nonpropagated modes.

$\phi(z,k)$, the improper eigenfunctions belonging to the improper eigenvalues $\lambda = k^2$, $0 < k \leq \omega/\beta_3$, have expressions similar to those for $\psi(z,k)$. Owing to the form of x-dependence in the integral containing ϕ, these represent waves traveling in the x direction.

The orthonormality relations amongst various proper and improper eigenfunctions are given by (cf. Kazi and Abu-Safiya (1982))

$$\int_{-H_1}^{\infty} \mu(z)\chi_m(z)\chi_n(z)dz = \delta_{mn},\ 1 \leq m,\ n \leq r, \tag{22a}$$

$$\int_{-H_1}^{\infty} \mu(z)\chi_m(z)\phi(z,k)dz = 0,\quad 1 \leq m \leq r,\ 0 < k \leq \omega/\beta_3 \tag{22b}$$

$$\int_{-H_1}^{\infty} \mu(z)\chi_m(z)\psi(z,k)dz = 0,\quad 1 \leq m \leq r,\ 0 < k < \infty \tag{22c}$$

$$\int_{-H_1}^{\infty} \mu(z)\psi(z,k)\psi(z,\ell)dz = \delta(k-\ell),\ 0 < k,\ \ell < \infty \tag{22d}$$

$$\int_{-H_1}^{\infty} \mu(z)\phi(z,k)\psi(z,k)dz = 0 \tag{22e}$$

$$\int_{-H_1}^{\infty} \mu(z)\phi(z,k)\phi(z,\ell)dz = \delta(k-\ell),\ 0 \leq k,\ \ell \leq \omega/\beta_3 \tag{22f}$$

The corresponding expressions for $\chi_m'(z)$, $\psi'(z,k')$ and $\phi'(z,k')$ and the orthonormality relations amongst these are the same as for $\chi_m(z)$, $\psi(z,k)$ and $\phi(z,k)$, given above, but in primed notation.

Integral Equation Formulation

Let $\tau(z)$ denote the component of stress at any point of the vertical plane $x = 0$:

$$\tau(z) = \tau_{xy}\Big/_{x=0} = \mu(z)\,\frac{\partial v}{\partial x}\Big/_{x=0^-} = \mu'(z)\frac{\partial v'}{\partial x}\Big/_{x=0^+}\ , \ z > -H_1 \qquad (23)$$

we have both

$$\tau(z) = \mu(z)\frac{\partial v}{\partial x}\Big/_{x\to 0^-} = -\mu(z)\Big[\sum_{m=1}^{r} ik_m(A_m-B_m)\chi_m(z)$$

$$+\int_0^{\omega/\beta_3} ik\{C(k)-D(k)\}\phi(z,k)dk+\int_0^{\infty} kE(k)\psi(z,k)dk\Big] \qquad (24)$$

and

$$\tau(z) = \mu'(z)\,\frac{\partial v'}{\partial x}\Big/_{x\to 0^+} = -\mu'(z)\Big[\sum_{m=1}^{s} ik'_m(A'_m-B'_m)\chi'_m(z)$$

$$+\int_0^{\omega/\beta'_3} ik'\{C'(k')-D'(k')\}\phi'(z,k')dk'+\int_0^{\infty} k'E'(k')\psi'(z,k')dk'\Big] \qquad (25)$$

On multiplying equation (23) separately by $\chi_m(z)$ $(m=1,2,\ldots r)$, $\phi(z,k)$ $(0 < k < \frac{\omega}{\beta_3})$ and $\psi(z,k)$ $(0 < k \lesssim \infty)$, and integrating with respect to z from $-H_1$ to ∞, we obtain [using orthonormality relations (22a) to (22f)]

$$-ik_m(A_m-B_m) = \int_{-H_1}^{\infty} \tau(\eta)\chi_m(\eta)d\eta,\quad m = 1,2,\ldots r, \qquad (26a)$$

$$-ik\{C(k)-D(k)\} = \int_{-H_1}^{\infty} \tau(\eta)\phi(\eta,k)d\eta, \qquad (26b)$$

and

$$-kE(k) = \int_{-H_1}^{\infty} \tau(\eta)\psi(\eta,k)d\eta. \tag{26c}$$

Proceeding similarly, equation (25) leads to the following

$$-ik'_m(A'_m-B'_m) = \int_{-H_1}^{\infty} \tau(\eta)\chi'_m(\eta)d\eta, \quad m = 1,2,\ldots s, \tag{27a}$$

$$-ik'\{C'(k')-D'(k')\} = \int_{-H_1}^{\infty} \tau(\eta)\phi'(\eta,k')d\eta, \tag{27b}$$

and

$$-k'E'(k') = \int_{-H_1}^{\infty} \tau(\eta)\psi'(\eta,k')d\eta. \tag{27c}$$

Eliminating $D(k)$, $D'(k')$, $E(k)$, $E'(k')$ [assuming $C(k) = C'(k') = 0$ and applying the matching condition (1c)], we obtain

$$\sum_{m=1}^{r}(A_m+B_m)\chi_m(z)+\sum_{m=1}^{s}(A'_m+B'_m)\chi'_m(z) = \int_{-H_1}^{\infty} \tau(\eta)G^*(z,\eta)d\eta \tag{28}$$

where

$$G^*(z,\eta) = G(z,\eta)+ ig(z,\eta), \tag{29}$$

$$G(z,\eta) = \int_0^{\infty}\frac{\psi(z,k)\psi(\eta,k)}{k}\,dk + \int_0^{\infty}\frac{\psi'(z,k')\psi'(\eta,k')}{k'}\,dk' \tag{30}$$

and

$$g(z,\eta) = \int_0^{\omega/\beta_3}\frac{\phi(z,k)\phi(\eta,k)}{k}\,dk + \int_0^{\omega/\beta'_3}\frac{\phi'(z,k')\phi'(\eta,k')}{k'}\,dk' \tag{31}$$

It may be noted that $G^*(z,\eta)$ is a Green's function type symmetric kernel, whose real and imaginary parts correspond to non-propagated and propagated modes (respectively) arising out of the continuous part of the spectrum.

The integral equation formulation of the problem is given by equations (26a), (27a) and (28). If the amplitudes $(A_m,\ m = 1,2,\ldots,r,\ A'_m,\ m = 1,2,\ldots,s)$ of incident Love waves are specified, we have to find the amplitudes of the transmitted and reflected waves from the above mentioned $(r+s+1)$ integral equations. Using matrix formalism, we recast the problem in terms of a 'scattering matrix' in the next section.

The Scattering Matrix Formulation

Introducing $n \times 1$ vectors

$$\underline{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \\ A'_1 \\ \vdots \\ A'_s \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \\ B'_1 \\ \vdots \\ B'_s \end{pmatrix}, \quad \underline{\chi}(z) = \begin{pmatrix} \chi_1(z) \\ \chi_2(z) \\ \vdots \\ \chi_r(z) \\ \chi'_1(z) \\ \vdots \\ \chi'_s(z) \end{pmatrix} \tag{32}$$

and the $n \times n$ diagonal matrix

$$\underline{K} = \begin{pmatrix} k_1 & & & & & 0 \\ & k_2 & & & & \\ & & \ddots & & & \\ & & & k_r & & \\ & & & & k'_1 & \\ 0 & & & & & \ddots \ k'_s \end{pmatrix}, \quad n = r + s, \tag{33}$$

equations (26a), (27a) can be rewritten

$$-i\underline{K}\cdot(\underline{A}-\underline{B}) = \int_{-H_1}^{\infty} \underline{\chi}(\eta)\tau(\eta)d\eta \tag{34}$$

and equation (28) as

$$(\underline{A}^T+\underline{B}^T)\cdot\underline{\chi}(z) = \int_{-H_1}^{\infty} G^*(z,\eta)\tau(\eta)d\eta, \; z > -H_1 \tag{35}$$

where the superscript T denotes the transpose.

Equations (34) and (35) imply that both $\underline{A}-\underline{B}$ and the unknown stress $\tau(z)$ on the vertical plane $x = 0$ must be linearly related to $\underline{A}+\underline{B}$. Consequently there exists an $n\times n$ matrix $\underline{S} = \|s_{ij}\|$ and an $n\times 1$ vector

$$\underline{\tau}(z) = \begin{pmatrix} \tau_1(z) \\ \vdots \\ \tau_r(z) \\ \tau_1'(z) \\ \vdots \\ \tau_s'(z) \end{pmatrix} , \tag{36}$$

such that

$$\underline{K}\cdot(\underline{A}-\underline{B}) = i\underline{S}\cdot(\underline{A}+\underline{B}), \tag{37}$$

and

$$\tau(z) = (\underline{A}^T+\underline{B}^T)\cdot\underline{\tau}(z). \tag{38}$$

The matrix $\underline{S} = \|s_{ij}\|$ is the SCATTERING MATRIX. Equation (37) can be rewritten

$$\underline{B} = \underline{T}\cdot\underline{A} \tag{39}$$

where

$$\underline{T} = (\underline{K}+i\underline{S})^{-1}\cdot(\underline{K}-i\underline{S}) \tag{40}$$

provided $\underline{K}+i\underline{S}$ is non-singular. Substituting (38) into (35), we get

$$(\underline{A}^T+\underline{B}^T)\cdot\{\underline{\chi}(z)-\int_{-H_1}^{\infty} G^*(z,\eta)\underline{\tau}(\eta)d\eta\} = 0, \quad -H_1 < z, \tag{41}$$

whence

$$\underline{\chi}(z) = \int_{-H_1}^{\infty} G^*(z,\eta)\underline{\tau}(\eta)d\eta, \quad -H_1 < z, \tag{42}$$

on account of the arbitrary choice and linear independence of the components of $\underline{A}+\underline{B}$. From (42), we obtain the following n uncoupled integral equations for the determination of $\underline{\tau}(\eta)$:

$$\chi_m(z) = \int_0^{\infty} G^*(z,\eta)\tau_m(\eta)d\eta, \quad m = 1,2,\ldots,r, \quad z > -H_1, \tag{43}$$

$$\chi_m'(z) = \int_0^{\infty} G^*(z,\eta)\tau_m'(\eta)d\eta, \quad m = 1,2,\ldots,s, \quad z > -H_1. \tag{44}$$

Substituting (38) into (34), we get

$$\underline{K}\cdot(\underline{A}-\underline{B}) = i\int_0^{\infty} \underline{\chi}(\eta)\ \{(\underline{A}^T+\underline{B}^T)\cdot\underline{\tau}(\eta)\}d\eta,$$

whence from (37)

$$\underline{S}\cdot(\underline{A}+\underline{B}) = \int_0^{\infty}\underline{\chi}(\eta)\ \{(\underline{A}^T+\underline{B}^T)\cdot\underline{\tau}(\eta)\}d\eta,$$

and so

$$s_{ij} = \int_0^\infty \chi_i(\eta)\tau_j(\eta)d\eta, \quad (i,j = 1,2,\ldots,r,r+1,\ldots,n = r+s), \tag{45}$$

where

$$\chi_{r+t} = \chi_t' \qquad \text{and} \quad \tau_{r+t} = \tau_t', \quad 1 \leq t \leq s.$$

The problem has thus been reduced to the solution of the integral equations (43) and (44) and the subsequent determination of the scattering matrix $\underline{S}$ from (45) and the related transmission matrix $\underline{T}$ in (40) which yields the required complex reflection and transmission coefficients (after appropriate normalization) through equation (39). The formulation of the problem is exact at this stage. Unfortunately, it is not possible to solve the problem exactly and we must resort to construct approximate solutions. In the next section we shall proceed to the plane wave approximation neglecting the propagated and non-propagated modes arising out of the continuous spectrum. In the subsequent section, we shall construct expressions for the elements s_{ij} (of the scattering matrix) to which the variational principle of Schwinger and Levine applies and then improve the earlier approximation in such a way as to incorporate the effects of propagated modes (which correspond to body waves) and non-propagated modes indirectly.

Plane Wave Approximation

If we neglect the propagated modes $\phi(z,k)$, $\phi'(z,k')$ and the non-propagated modes $\psi(z,k)$, $\psi'(z,k')$ corresponding to the continuous part of the spectrum, then we can set $G^*(z,\eta) = 0$ [see equations (29) to (31)] in the preceding formulation and assume the following expansion for $\tau(z)$ in terms of the whole set of propagated discrete modes in the left-hand domain:

$$\tau(z) = \mu(z)\{\sum_{m=1}^{r} D_m\, \chi_m(z)\} \tag{46}$$

Substituting this into equation (34) we obtain

$$-i\underline{K}\cdot(\underline{A}-\underline{B}) = \int_{-H_1}^{\infty} \begin{pmatrix} \chi_1(\eta) \\ \chi_2(\eta) \\ \vdots \\ \chi_r(\eta) \\ \chi_1'(\eta) \\ \vdots \\ \chi_s'(\eta) \end{pmatrix} \mu(\eta)\{\sum_{m=1}^{r} D_m\, \chi_m(\eta)\}\, d\eta, \tag{47}$$

or

$$-i \begin{pmatrix} k_1(A_1-B_1) \\ k_2(A_2-B_2) \\ \vdots \\ k_r(A_r-B_r) \\ k_1'(A_1'-B_1') \\ \vdots \\ k_s'(A_s'-B_s') \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \\ \sum_{m=1}^{r} D_m\, P_{1m}\, \lambda_{1m} \\ \vdots \\ \sum_{m=1}^{r} D_m\, P_{sm}\, \lambda_{sm} \end{pmatrix} \tag{48}$$

because of the orthonormality relation (22a), where

$$\lambda_{im} = \left(\frac{k_i'}{k_m} \right)^{\frac{1}{2}} , \quad i = 1,2,\ldots,s; \quad m = 1,2,\ldots,r, \tag{49}$$

and $$\lambda_{im} P_{im} = \int_{-H_1}^{\infty} \mu(\eta)\chi_i'(\eta)\chi_m(\eta)d\eta , \quad i = 1,2,\ldots s; \; m = 1,2,\ldots,t. \tag{50}$$

Substituting $G^* = 0$ in (35), we get

$$(\underline{A}^T + \underline{B}^T)\cdot\underline{\chi}(z) = 0, \quad z > -H_1. \tag{51}$$

Eliminating $D_1, D_2, \ldots, D_r$ from (48) and simplifying, we obtain:

$$\underline{R}\cdot(\underline{A}-\underline{B}) = 0, \tag{52}$$

where the $s \times n$ matrix $\underline{R}$ is given by :

$$\underline{R} = \left(\underbrace{\begin{matrix} \frac{P_{11}}{\lambda_{11}} & \frac{P_{12}}{\lambda_{12}} & \cdots & \frac{P_{1r}}{\lambda_{1r}} \\ \frac{P_{21}}{\lambda_{21}} & \frac{P_{22}}{\lambda_{22}} & \cdots & \frac{P_{2r}}{\lambda_{2r}} \\ \vdots & & & \\ \frac{P_{s1}}{\lambda_{s1}} & \frac{P_{s2}}{\lambda_{s2}} & \cdots & \frac{P_{sr}}{\lambda_{sr}} \end{matrix}}_{s \times r} \; \vdots \; \underbrace{\begin{matrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{matrix}}_{s \times s} \right) \tag{53}$$

Calculating the first moments of (51) with respect to $\mu(z)\chi_i(z)$, $i = 1,2,\ldots,r$, we obtain a set of r simultaneous, linear, algebraic equations equivalent to the matrix equation :

$$\underline{Q}\cdot(\underline{A}+\underline{B}) = 0, \tag{54}$$

where the $r \times n$ matrix $\underline{Q}$ is given by

$$\underline{Q} = \left[\begin{array}{c:c} \underbrace{\begin{matrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{matrix}}_{r\times r} & \underbrace{\begin{matrix} \lambda_{11}P_{11} & \lambda_{21}P_{21} \ldots \lambda_{s1}P_{s1} \\ \lambda_{12}P_{12} & \lambda_{21}P_{22} \ldots \lambda_{s2}P_{s2} \\ \vdots & \\ \lambda_{1r}P_{1r} & \lambda_{2r}P_{2r} \ldots \lambda_{sr}P_{sr} \end{matrix}}_{r\times s} \end{array}\right] \tag{55}$$

Combining (52) and (54) into a single matrix equation, we get

$$\begin{pmatrix} \underline{Q} \\ \underline{R} \end{pmatrix}\cdot\underline{A} = \begin{pmatrix} -\underline{Q} \\ \underline{R} \end{pmatrix}\cdot\underline{B} \tag{56}$$

or

$$\underline{B} = \underline{T}\cdot\underline{A} \quad \text{where} \quad \underline{T} = \begin{pmatrix} -\underline{Q} \\ \underline{R} \end{pmatrix}^{-1}\cdot\begin{pmatrix} \underline{Q} \\ \underline{R} \end{pmatrix} \tag{57}$$

The matrix gives the reflection and transmission coefficients. We now proceed to compute the integral in equation (50) to find $\lambda_{im}\, P_{im}$

Rewrite equation (50) as

$$I = \lambda_{im}\, P_{im} = \int_{-H_1}^{\infty} \mu(z)\chi_i'(z)\chi_m(z)dz,$$

$$I = \mu_1 \int_{-H_1}^{0} \phi_1'^{\,i}(z)\phi_1^m(z)dz + \mu_2 \int_{0}^{H_2} \phi_2'^{\,i}(z)\phi_2^m(z)dz + \mu_3 \int_{H_2}^{\infty} \phi_3'^{\,i}(z)\phi_3^m(z)dz$$

$$= I_1 + I_2 + I_3 \;, \tag{58}$$

with

$$I_1 = \mu_1 \int_{-H_1}^{0} \phi_1'^{\,i}(z)\phi_1^m(z)dz$$

$$= \frac{\mu_1 F_i' F_m}{\cos(\sigma_1'^{\,i}H_1)\cos(\sigma_1^m H_1)} \int_{-H_1}^{0} \cos\{\sigma_1'^{\,i}(z+H_1)\}\cos\{\sigma_1^m(z+H_1)\}dz$$

[using (7a)]

$$= \frac{\mu_1 F_i' F_m}{(\sigma_1'^{\,i})^2 - (\sigma_1^m)^2}[\sigma_1'^{\,i}\tan(\sigma_1'^{\,i}H_1) - \sigma_1^m\tan(\sigma_1^m H_1)], \tag{59}$$

$$I_2 = \mu_2 \int_{0}^{H_2} \phi_2'^{\,i}(z)\phi_2^m(z)dz$$

$$= \frac{\mu_2 G_i' G_m}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^{\,i}H_2)} \int_{0}^{H_2} [\mu_2\sigma_2^m\cos\{\sigma_2^m(z-H_2)\} - \mu_3\sigma_3^m\sin\{\sigma_2^m(z-H_2)\}]\cdot$$

$$\mu_2'\sigma_2'^{\,i}\cos\{\sigma_2'^{\,i}(z-H_2)\} - \mu_3'\sigma_3'^{\,i}\sin\{\sigma_2'^{\,i}(z-H_2)\}]dz$$

[using (7b)]

which yields after somewhat lengthy but straightforward calculation:

$$I_2 = \frac{\mu_2 G_i' G_m}{(\sigma_2^m)^2 - (\sigma_2'^{\,i})^2}[\sigma_2'^{\,i}\{\mu_2\mu_2'(\sigma_2^m)^2 + \mu_3\mu_3'\sigma_3^m\sigma_3'^{\,i}\}\tan(\sigma_2^m H_2)$$

$$- \sigma_2^m\{\mu_2\mu_2'(\sigma_2'^{\,i})^2 + \mu_3\mu_3'\sigma_3^m\sigma_3'^{\,i}\}\tan(\sigma_2'^{\,i}H_2)$$

$$+ \{\mu_2\mu_3'\sigma_3'^{i}(\sigma_2^{m})^2 - \mu_3\mu_2'\sigma_3^{m}(\sigma_2'^{i})^2\}\tan(\sigma_2^{m}H_2)\tan(\sigma_2'^{i}H_2)$$

$$+ \frac{\sigma_2^{m}\sigma_2'^{i}(\mu_3\mu_2'\sigma_3^{m} - \mu_2\mu_3'\sigma_3'^{i})}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2'^{i}H_2)}$$

$$+ \sigma_2^{m}\sigma_2'^{i}\{-\mu_2'\mu_3\sigma_3^{m} + \mu_2\mu_3'\sigma_3'^{i}\}], \qquad (60)$$

$$I_3 = \mu_3\int_{H_2}^{\infty}\phi_3'^{i}(z)\phi_3^{m}(z)dz$$

$$= \mu_3\int_{H_2}^{\infty}\frac{G_m G_i'}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2'^{i}H_2)}\,\mu_2\mu_2'\sigma_2^{m}\sigma_2'^{i}\; e^{-(\sigma_3^{m}+\sigma_3'^{i})(z-H_2)}.dz$$

[using (7c)]

$$= \frac{G_m G_i'\mu_2\mu_2'\mu_3\sigma_2^{m}\sigma_2'^{i}(\sigma_3^{m} - \sigma_3'^{i})}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2'^{i}H_2)\{(\sigma_3^{m})^2 - (\sigma_3'^{i})^2\}}, \qquad (61)$$

whence from equations (58) to (61) we obtain

$$\lambda_{im}P_{im} = \frac{\mu_1 F_i' F_m}{(k_m^2 - k_i'^2) + \omega^2(\frac{1}{\beta_1'^2} - \frac{1}{\beta_1^2})}[\sigma_1'^{i}\tan(\sigma_1'H_1) - \sigma_1^{m}\tan(\sigma_1^{m}H_1)]$$

$$+ \frac{\mu_2 G_i' G_m}{(k_m^2 - k_i'^2) + \omega^2(\frac{1}{\beta_2'^2} - \frac{1}{\beta_2^2})}[\sigma_2^{m}\{\mu_2\mu_2'(\sigma_2'^{i})^2 + \mu_3\mu_3'\sigma_3^{m}\sigma_3'^{i}\}\tan(\sigma_2'^{i}H_2)$$

$$- \sigma_2'^{i}\{\mu_2\mu_2'(\sigma_2^{m})^2 + \mu_3\mu_3'\sigma_3^{m}\sigma_3'^{i}\}\tan(\sigma_2^{m}H_2)$$

$$+ \{\mu_3\mu_2'\sigma_3^{m}(\sigma_2'^{i})^2 - \mu_2\mu_3'\sigma_3'^{i}(\sigma_2^{m})^2\}\tan(\sigma_2^{m}H_2)\tan(\sigma_2'^{i}H_2)$$

$$+ \frac{\sigma_2^{m}\sigma_2'^{i}}{\cos(\sigma_2^{m}H_2)\cos(\sigma_2'^{i}H_2)}(\mu_3'\mu_2\sigma_3'^{i} - \mu_3\mu_2'\sigma_3^{m}) - \sigma_2^{m}\sigma_2'^{i}(\mu_2\mu_3'\sigma_3'^{i} - \mu_2'\mu_3\sigma_3^{m})]$$

$$+ \frac{\mu_2\mu_2'\mu_3\sigma_2\sigma_2'^{\,i}G_m G_i'(\sigma_3'^{\,i}-\sigma_3^m)}{\cos(\sigma_2^m H_2)\cos(\sigma_2'^{\,i}H_2)[(k_i'^2-k_m^2)+\omega^2(\frac{1}{\beta_3^2}-\frac{1}{\beta_3'^2})]} \tag{62}$$

A check on the validity of these formulae is provided by the fact that in the limit as $H_2 \to 0$, $\mu_3 \to \mu_2$, $\mu_3' \to \mu_2'$, $\sigma_2 \to \sigma_3$, $\sigma_2' \to \sigma_3'$, the expression for $\lambda_{im}P_{im}$ in (62) reduces to the corresponding expression found in Niazy and Kazi ((1980)[eq.(41)]) for the welded quarter-spaces problem involving single upper layers.

The form of the transmission matrix $\underline{T}$ in (57) in the following special cases can be shown to be:

(I) $r = 1$, $s = 1$:

$$\underline{T} = \frac{1}{1+P_{11}^2}\begin{pmatrix} -1+P_{11}^2 & -2\lambda_{11}P_{11} \\ \dfrac{-2P_{11}}{\lambda_{11}} & 1-P_{11}^2 \end{pmatrix} \tag{63}$$

(II) $r = 1$, $s = 2$:

$$\underline{T} = \frac{1}{1+P_{11}^2+P_{21}^2}\begin{pmatrix} -1+P_{11}^2+P_{21}^2 & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \\ \dfrac{-2P_{11}}{\lambda_{11}} & -P_{11}^2+1+P_{21}^2 & \dfrac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \\ \dfrac{-2P_{21}}{\lambda_{21}} & \dfrac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P_{21}^2+1+P_{11}^2 \end{pmatrix} \tag{64}$$

Variational Formulation and Direction Approximation

Returning to the scattering matrix formulation of the problem, we shall construct expressions for the elements of the matrix in such a way that the variational principle of Schwinger and Levine becomes applicable.

Multiplying the equation

$$\chi_i(z) = \int_{-H_1}^{\infty} G^*(z,\eta)\tau_i(\eta)d\eta, \quad i = 1,2,\ldots, \quad n = r + s$$

[(43),(44)]

by $\tau_j(z)$, $j = 1,2,\ldots,n$ and integrating with respect to z over the interval $(-H_1,\infty)$, we obtain

$$s_{ij} = \int_{-H_1}^{\infty} \chi_i(\eta)\tau_j(\eta)d\eta = \int_{-H_1}^{\infty}\int_{-H_1}^{\infty} \tau_i(z)G^*(z,\eta)\tau_j(\eta)dzd\eta \tag{65}$$

by equation (45). Since the kernel $G^*(z,\eta)$ is symmetric [see equations (29)-(31)] it follows from (65) that $s_{ij} = s_{ji}$ and so the scattering matrix $\underline{S} = \|s_{ij}\|$ is symmetric. Thus we may write

$$s_{ij} = \frac{\int_{-H_1}^{\infty} \chi_i(z)\tau_j(z)dz \cdot \int_{-H_1}^{\infty} \chi_j(\eta)\tau_i(\eta)d\eta}{\int_{-H_1}^{\infty}\int_{-H_1}^{\infty} \tau_i(z)G^*(z,\eta)\tau_j(\eta)dzd\eta} \tag{66}$$

If we introduce the notations

$$\langle f,u \rangle = \int_{-H_1}^{\infty} fu\, dz, \quad G^*u = \int_{-H_1}^{\infty} G^*(z,\eta)u(\eta)d\eta,$$

then

$$<G^*u,v> = <u,G^*v> \quad \forall \; u,v$$

and we may rewrite (66) as

$$s_{ij} = (<\chi_i,\tau_j>\cdot<\chi_j,\tau_i>)/(<G^*\tau_i,\tau_j>) \tag{67}$$

As in Kazi (1978), we have the following

Theorem: Let $F(u,v) = <\chi_i,v> + <\chi_j,u> - <G^*u,v>$.
Then F is stationary for variations of u,v about $u = \tau_i$, $v = \tau_j$ where τ_i,τ_j are the solutions of the integral equations

$$\chi_i(z) = G^*\tau_i = \int_{-H_1}^{\infty} G^*(z,\eta)\tau_i(\eta)d\eta$$

and

$$\chi_j(z) = G^*\tau_j = \int_{-H_1}^{\infty} G^*(z,\eta)\tau_j(\eta)d\eta,$$

respectively. Moreover, the stationary value of F is $s_{ij}(\tau_i,\tau_j)$

Corollary (Schwinger-Levine Variational Principle): Let $R(u,v) = (<\chi_j,u><\chi_i,v>)/(<G^*u,v>)$. Then R is staionary about $u = \alpha\tau_i$, $v = \beta\tau_j$ where α,β are arbitrary non-zero constants. Moreover, $R(\alpha\tau_i,\beta\tau_j) = s_{ij}(\tau_i,\tau_j)$.

By invoking the above theorem we obtain variational improvement of the plane wave approximation used in the previous section by assuming the expansions for $\tau_i(z)$:

$$\tau_i(z) = \sum_{p=1}^{r} D_{ip}\mu(z)\chi_p(z), \quad i = 1,2,\ldots,n \tag{68}$$

and considering

$$F(\tau_i,\tau_j) = <\chi_j, \sum_{p=1}^{r} D_{ip}\mu(z)\chi_p(z)>+<\chi_i, \sum_{q=1}^{r} D_{jq}\mu(z)\chi_q(z)>-<G^*\tau_i,\tau_j>$$

$$= \sum_{p=1}^{r} D_{ip}<\chi_j,\mu(z)\chi_p(z)>+\sum_{q=1}^{r} D_{jq}<\chi_i,\mu(z)\chi_q(z)>$$

$$- \sum_{q=1}^{r}\sum_{p=1}^{r} D_{ip}D_{jq}I_{pq} , \tag{69}$$

where

$$I_{pq} = \int_{-H_1}^{\infty} \{ \int_{-H_1}^{\infty} G^*(z,\eta)\chi_p(\eta)\mu(\eta)d\eta\}\mu(z)\chi_q(z)dz, \tag{70}$$

The requirement that the coefficients D_{ip} and D_{jq} in (69) make $F(\tau_i,\tau_j)$ stationary implies

$$\frac{\partial F}{\partial D_{ip}} = 0, \quad p = 1,2,\ldots,r,$$

and

$$\frac{\partial F}{\partial D_{jq}} = 0, \quad q = 1,2,\ldots,r,$$

which lead to a set of r linear algebraic equations for D_{ip}, $p = 1,2,\ldots,r$ and another set of r linear algebraic equations for D_{jq}, $q = 1,2,\ldots,r$. Solving for D_{ip}'s and D_{jq}'s and substituting in (69), we get the entry s_{ij} of the scattering matrix. Suitable expressions for the integrals I_{pq} are constructed in the appendix.

In the special cases

(i) when $r = 1, s = 1$:

$$\underline{S} = \begin{bmatrix} \dfrac{1}{I_{11}} & \dfrac{\lambda_{11}P_{11}}{I_{11}} \\[2ex] \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{11}^2 P_{11}^2}{I_{11}} \end{bmatrix} \tag{71}$$

and

$$\underline{T} = (\underline{K} + i\underline{S})^{-1} \cdot (\underline{K} - i\underline{S})$$

$$= \frac{1}{(1+P_{11}^2 - iI_{11}')} \begin{bmatrix} P_{11}^2 - 1 - iI_{11}' & -2P_{11}\lambda_{11} \\[3ex] \dfrac{-2P_{11}}{\lambda_{11}} & 1 - P_{11}^2 - iI_{11}' \end{bmatrix} \tag{72}$$

where $I_{11}' = k_1 I_{11}$ and I_{11} is given by (A10) when $m = n = 1$.

(ii) when $r = 1, s = 2$:

The scattering matrix $\underline{S}$ and the transmission matrix $\underline{T}$ are given by

$$\underline{S} = \begin{bmatrix} \dfrac{1}{I_{11}} & \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{21}P_{21}}{I_{11}} \\[2ex] \dfrac{\lambda_{11}P_{11}}{I_{11}} & \dfrac{\lambda_{11}^2 P_{11}^2}{I_{11}} & \dfrac{\lambda_{11}P_{11}\lambda_{21}P_{21}}{I_{11}} \\[2ex] \dfrac{\lambda_{21}P_{21}}{I_{11}} & \dfrac{\lambda_{11}P_{11}\lambda_{21}P_{21}}{I_{11}} & \dfrac{\lambda_{21}^2 P_{21}^2}{I_{11}} \end{bmatrix}$$

and

$$\underline{T} = \frac{1}{1+P_{11}^2+P_{21}^2-iI_{11}'}\begin{pmatrix} -1+P_{11}^2+P_{21}^2-iI_{11}' & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} \\ \dfrac{-2P_{11}}{\lambda_{11}} & -P_{11}^2+1+P_{21}^2-iI_{11}' & \dfrac{-2P_{11}\lambda_{21}P_{21}}{\lambda_{11}} \\ \dfrac{-2P_{11}}{\lambda_{21}} & \dfrac{-2P_{21}P_{11}\lambda_{11}}{\lambda_{21}} & -P_{21}^2+1+P_{11}^2-iI_{11}' \end{pmatrix} \quad (73)$$

On comparing the forms of the transmission matrix $\underline{T}$ for the special cases discussed above [see equations (72) and (73)] with those under the plane wave approximation [see equations (63) and (64)], we find that the latter can be recovered from the former on substituting $I_{11}' = 0$ and so it follows that the parameter I_{11}' incorporates the effects of propagated and non-propagated modes which arise out of the continuous part of the spectrum.

Numerical computation of our results under both approximations, and for several special laterally discontinuous structures involving double surface layers will be presented in another paper.

APPENDIX

Substituting

$$G^*(z,\eta) = G(z,\eta)+ig(z,\eta) \tag{29}$$

$$G(z,\eta) = \int_0^\infty [\psi(z,k)\psi(\eta,k)dk]/k+\int_0^\infty \frac{[\psi'(z,k')\psi'(\eta,k')]dk'}{k'} \tag{30}$$

$$g(z,\eta) = \int_0^{\omega/\beta_3} \frac{[\phi(z,k)\phi(\eta,k)]dk}{k} + \int_0^{\omega/\beta_3'} \frac{\phi'(z,k')\phi'(\eta,k')dk'}{k'} \tag{31}$$

in (70) and using the orthonormality relations [see (22a)-(22f)] we obtain

$$I_{mn} = \int_0^\infty \frac{dk'}{k'} \int_{-H_1}^\infty \mu(\eta)\psi'(\eta,k')\chi_m(\eta)d\eta \int_{-H_1}^\infty \mu(z)\psi'(z,k')\chi_n(z)dz$$

$$+ i \int_0^{\omega/\beta_3'} \frac{dk'}{k'} \int_{-H_1}^\infty \mu(\eta)\phi'(\eta,k')\chi_m(\eta)d\eta \int_{-H_1}^\infty \mu(z)\phi'(z,k')\chi_n(z)dz \tag{A1}$$

Next, we evaluate integrals of the form

$$I(k',m) = \int_{-H_1}^\infty \mu(z)\chi_m(z)\phi'(z,k')dz$$

and

$$I'(k',m) = \int_{-H_1}^\infty \mu(z)\chi_m(z)\psi'(z,k')dz, \quad m = 1,2,\ldots,r,$$

which occur in (A1).

Let $I(k',m) = \int_{-H_1}^{\infty} \mu(z)\chi_m(z)\phi'(z,k')dz$

$$= \mu_1 \int_{-H_1}^{0} \phi_1'(z,k')\phi_1^m(z)dz + \mu_2\int_{0}^{H_2}\phi_2'(z,k')\phi_2^m(z)dz + \mu_3\int_{H_2}^{\infty}\phi_3'(z,k')\phi_3^m(z)dz$$

$$= I_1+I_2+I_3\ , \qquad (\lambda'=k'^2, \quad 0<k'<\omega/\beta_3') \tag{A2}$$

where

$$I_1 = \mu_1 \int_{-H_1}^{0} \phi_1'(z,k')\phi_1^m(z)dz$$

$$= \frac{\mu_1 F_m}{\cos(\sigma_1^m H_1)} \cdot \frac{\overset{k'}{G'}\mu_2'\overset{k'}{\sigma_2'}}{\cos(\overset{k'}{\sigma_1'}H_1)\cos(\overset{k'}{\sigma_2'}H_2)} \int_{-H_1}^{0} \cos\{\sigma_1^m(z+H_1)\}\cdot\cos\{\overset{k'}{\sigma_1'}(z+H_1)\}dz$$

[using expressions for $\phi_1^m(z)$ from equation (7a) and for $\phi_1'(z,k')$ similar to $\psi_1(z,\lambda)$ in equation (15a)]

$$= \frac{\mu_1 F_m \overset{k'}{G'}\mu_2'\overset{k'}{\sigma_2'}}{\cos(\overset{k'}{\sigma_2'}H_2)} \cdot \frac{1}{(\overset{k'}{\sigma_1'})^2-(\sigma_1^m)^2}[\overset{k'}{\sigma_1'}\tan(\overset{k'}{\sigma_1'}H_1)-\sigma_1^m\tan(\sigma_1^m H_1)], \tag{A3}$$

$$I_2 = \mu_2\int_{0}^{H_2}\phi_2'(z,k')\phi_2^m(z)dz$$

$$= \frac{\mu_2 \overset{k'}{G'} G_m}{\cos(\sigma_2^m H_2)\cos(\overset{k'}{\sigma_2'}H_2)} \int_{0}^{H_2} [\mu_2\sigma_2^m\cos\{\sigma_2^m(z-H_2)\}-\mu_3\sigma_3^m\sin\{\sigma_2^m(z-H_2)\}]\cdot$$

$$[\mu_2'\overset{k'}{\sigma_2'}\cos(\overset{k'}{\sigma_2'}z)-\mu_1'\overset{k'}{\sigma_1'}\sin(\overset{k'}{\sigma_2'}z)\tan(\overset{k'}{\sigma_1'}H_1)]dz$$

[using expressions for $\phi_2^m(z)$ from (7b) and for $\phi_2'(z,k')$ similar to $\psi_2(z,\lambda)$ in (15b)]

$$= \frac{\mu_2 \overset{k'}{G'} G_m}{\cos(\sigma_2^m H_2)\cos({\sigma'}_2^{k'} H_2)} \cdot \frac{1}{(\sigma_2^m)^2 - ({\sigma'}_2^{k'})^2} [\{\mu_2 \mu_2' (\sigma_2^m)^2 {\sigma'}_2^{k'} + \mu_3 \sigma_3^m \mu_1' {\sigma'}_2^{k'} {\sigma'}_1^{k'} \tan({\sigma'}_1^{k'} H_1)$$

$$\cdot \sin(\sigma_2^m H_2) - \{\mu_2 \mu_2' \sigma_2^m ({\sigma'}_2^{k'})^2 + \mu_3 \sigma_3^m \mu_1' {\sigma'}_1^{k'} \sigma_2^m \tan({\sigma'}_1^{k'} H_1)\} \sin({\sigma'}_2^{k'} H_2)$$

$$+ \{\sigma_2^m \mu_3 \mu_2' \sigma_3^m {\sigma'}_2^{k'} - \mu_1' \mu_2 \sigma_2^m {\sigma'}_1^{k'} {\sigma'}_2^{k'} \tan({\sigma'}_1^{k'} H_1)\}\{\cos({\sigma'}_2^{k'} H_2) - \cos(\sigma_2^m H_2)\}]$$

(obtained after considerable simplification), (A4)

$$I_3 = \mu_3 \int_{H_2}^{\infty} \phi_3'(z,k') \phi_3^m(z)\, dz$$

$$= \frac{-\mu_3 G_m \mu_2 \sigma_2^m}{\cos(\sigma_2^m H_2)} \cdot \frac{\int_{H_2}^{\infty} \sin\{\overset{k'}{\theta'} + {s'}_3^{k'}(z-H_2)\} . e^{-\sigma_3^m (z-H_2)} . dz}{\sqrt{\pi \mu_3 {s'}_3^{k'}}}$$

[using expressions for $\phi_3^m(z)$ from (7c) and for $\phi_3'(z,k')$ similar to $\psi_3(z,\lambda)$ in (15c)]

$$= \frac{-\mu_3 G_m \sigma_2^m}{\cos(\sigma_2^m H_2)} \cdot \frac{1}{\sqrt{\pi \mu_3' {s'}_3^{k'}}} \cdot \frac{1}{(\sigma_3^m)^2 + ({s'}_3^{k'})^2} [(\cos\overset{k'}{\theta'}) {s'}_3^{k'} + (\sin\overset{k'}{\theta'}) \sigma_3^m]$$

$$= \frac{-\mu_3 G_m G'_{k'}}{\cos(\sigma_2^m H_2)} \cdot \frac{p'}{\mu_3' {s'}_3^{k'}} \cdot \frac{\sigma_2^m}{(\sigma_3^m)^2 + ({s'}_3^{k'})^2} [{s'}_3^{k'} + (\tan\overset{k'}{\theta'}) \sigma_3^m], \qquad \text{(A5)}$$

obtained on using the relation

$$G'_{k'}\, p'/(\mu_3' {s'}_3^{k'}) \cos\overset{k'}{\theta'} = \frac{1}{\sqrt{\pi {s'}_3^{k'} \mu_3'}} \qquad \text{[see(16)]}$$

From (A2)-(A5) we get

$$I(k',m) = \int_{-H_1}^{\infty} \mu(z)\chi_m(z)\phi'(z,k')dz$$

$$= \frac{\mu_1 F_m G^{k'} \mu_2' \sigma_2^{k'}}{[(k_m^2-k'^2)+\omega^2(\frac{1}{\beta_1'^2}-\frac{1}{\beta_1^2})]\cos(\sigma_2^{k'}H_2)}[\sigma_1^{k'}\tan(\sigma_1^{k'}H_1)-\sigma_1^m\tan(\sigma_1^m H_1)]$$

$$+ \frac{\mu_2 G^{k'} G_m}{\cos(\sigma_2^m H_2)\cos(\sigma_2^{k'}H_2)} \cdot \frac{1}{(k'^2-k_m^2)+\omega^2(\frac{1}{\beta_2^2}-\frac{1}{\beta_2'^2})} \cdot$$

$$[\{\mu_2\mu_2'(\sigma_2^m)^2\sigma_2^{k'}+\mu_3\sigma_3^m\mu_1'\sigma_2^{k'}\sigma_1^{k'} \cdot \tan(\sigma_1^{k'}H_1)\}\sin(\sigma_2^m H_2)$$

$$-\{\mu_2\mu_2'\sigma_2^m(\sigma_2^{k'})^2+\mu_3\sigma_3^m\mu_1'\sigma_1^{k'}\sigma_2^m\tan(\sigma_1^{k'}H_1)\}\sin(\sigma_2^{k'}H_2)$$

$$+\{\mu_3\sigma_2^m\mu_2'\sigma_3^{mk'}\sigma_2 - \mu_1'\mu_2\sigma_2^{mk'}\sigma_1^{k'}\sigma_2^{k'}\tan(\sigma_1^{k'}H_1)\}\cdot$$

$$\{\cos(\sigma_2^{k'}H_2)-\cos(\sigma_2^m H_2)\}]$$

$$\frac{-\mu_3 G_m G'_{k'}}{\cos(\sigma_2^m H_2)} \cdot \frac{p'}{\mu_3' s_3^{k'}} \cdot \frac{\sigma_2^m}{(k_m^2-k'^2)+\omega^2(\frac{1}{\beta_3'^2}-\frac{1}{\beta_3^2})}[s_3^{k'}+(\tan\theta'^{k'})\sigma_3^m] \tag{A6}$$

[on using relations of the type given in equations (12), (13), (17) and (21)]

Thus

$$\int_{-H_1}^{\infty}\mu(\eta)\phi'(\eta,k')\chi_m(\eta)d\eta \int_{-H_1}^{\infty}\mu(z)\phi'(z,k')\chi_n(z)dz = I(k',m)I(k',n) \tag{A7}$$

Likewise

$$\int_{-H_1}^{\infty}\mu(\eta)\psi'(\eta,k')\chi_m(\eta)d\eta \int_{-H_1}^{\infty}\mu(z)\psi'(z,k')\chi_n(z)dz = I'(k',m)I'(k',n) \tag{A8}$$

where the expression for $I'(k',m)$ can be obtained from (A6) on replacing k'^2 by $-k'^2$ i.e.,

$$I'(k',m) = I(ik',m) \tag{A9}$$

From (A1), (A7) and (A8), we finally obtain :

$$I_{mn} = \int_0^{\infty}\frac{I'(k',m)I'(k,n)}{k'}\,dk' + i\int_0^{\omega/\beta_3'}\frac{I(k',m)(I(k',n)}{k'}\,dk', \tag{A10}$$

where $I(k',m)$ and $I'(k',m)$ are given by (A6) and (A(9) respectively. The real and imaginary parts of I_{mn} correspond to the non-propagated and propagated modes arising from the continuous part of the spectrum. The integrands in the integrals occuring in (A10) are regular. These integrals are convergent. However, the integrals will have to be evaluated numerically because of the complicated forms of the integrands.

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